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NEW GENERALIZATION OF MANIFOLDS AND ORBIFOLDS USING OF GENERALIZED GROUPS

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Abstract. Our ultimate goal in this paper is to introduce a special type of topological spaces including manifolds and also, orbifolds. Because of using of generalized groups, we call them *GG-spaces*. We will study their properties, and then we will introduce a special *GG*-space that is not manifold and orbifold. Finally we obtain conditions that cause a *GG*-space to become manifold.

Keywords: Generalized group,T-Space, Quotient space, Orbifold, *GG*-Space.
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1. Introduction

One of the interesting problems in geometry is to extend our definitions in order to add more objects to a certain category. We know geometric objects like torus and spheres are manifolds, but cones aren't. Extending the notion of manifolds one can define a new structure called orbifold to include cones and some other objects as well. Intuitively, a manifold is a topological space locally modeled on Euclidean space $\mathbb{R}^n$. Manifolds have origins in Carl Friedrich Gauss's works and Bernhard Riemann's lecture in Gottingen in 1854 laid the foundations of higher-dimensional differential geometry [17]. As an extension of manifolds, an orbifold is a topological space locally modeled on a quotient of $\mathbb{R}^n$ by the action of a finite group. The simplest examples of orbifolds are cones, lens spaces and $\mathbb{Z}_p$-teardrops. Orbifolds lie at the intersection of many different areas of mathematics, including algebraic and differential geometry, topology, algebra and string theory [16]. *GG-spaces* are a fascinating extension of orbifolds and manifolds. We can be roughly described a *GG*-space as a topological space that is locally modeled on a quotient of $\mathbb{R}^n$ by the *generalized action* of a *topological generalized group*. *GG*-spaces will yield a geometrical and algebraic device useful for showing the existence of structures that are not a manifold or an orbifold such as Example (3).

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Let us recall the definition of orbifolds. They were first introduced into topology and differential geometry by Satake [15], who called them *V-manifolds*. Satake described them as topological spaces generalizing smooth manifolds and generalized concepts such as de Rham cohomology and the Gauss-Bonnet theorem to orbifolds. The late 1970s, orbifolds were used by Thurston in his work on three-manifolds [16]. The name *V-manifold* was replaced by the word *orbifold* by Thurston. An orbifold $\mathbb{O}$, consists of a paracompact, Hausdorff topological space $\mathbb{X}_{\mathbb{O}}$ called the *underlying space*, such that for each $x \in \mathbb{X}_{\mathbb{O}}$ and neighborhood U of x, there exists a neighborhood $U_x \subseteq U$, an open set $\tilde{U}_x \cong \mathbb{R}^n$, a finite group G_x acting continuously and effectively on $\tilde{U}_x$ which fixes $0 \in \tilde{U}_x$, and a homeomorphism $\phi_x : \tilde{U}_x G_x \to U_x$ with $\phi_x(0) = x$ [3].

In this paper, we first recall some important preliminaries about topological generalized groups and their generalized action on a topological space. In section 3, we introduce and study GG-spaces using generalized action of topological generalized groups. Then we will show that there is the GG-space that is not manifold.

2. **Preliminaries**

Generalized groups or completely simple semi-groups [2, 9] are an extension of groups. This notion has been studied first in 1999 [8, 10, 12]. Topological generalized groups have been applied in geometry, dynamical systems and also genetic [1, 11, 13]. The notion of generalized action [8] is an extension of the notion of group actions. It has been applied by the other researchers [4]. Furthermore, the notion of T-spaces have been introduced and studies as an extentionof the notion of G-spaces using of topological generalized groups [7]. We refer to [7, 12, 14] for more details. We start by recalling the notions of topological generalized groups and their generalized action on a topological space. [10] A *topological generalized group* is a Hausdorff topological space T which is endowed with a semigroup structure such that the following conditions hold:

(a) For each $t \in T$, there is a unique $e(t) \in T$ such that $t \cdot e(t) = e(t) \cdot t = t$,
(b) For each $t \in T$, there is $s \in T$ such that $s \cdot t = t \cdot s = e(t)$,
(c) For each $s, t \in T$, $e(s \cdot t) = e(s) \cdot e(t)$,
(d) The generalized group operations $m_1 : T \to T$ defined by $m_1(t) = t^{-1}$ and $m_2 : T \times T \to T$ defined by $m_2((s, t)) = s \cdot t$ are continuous maps, where $t^{-1} \in T$ with $t \cdot t^{-1} = t^{-1} \cdot t = e(t)$.

Note that in condition (b), we can easily prove that each t in T, has a unique inverse in T, denoted by t^{-1}. So the mapping m_1 is well-defined. Moreover; for given $t \in T$, $e(t) = e(t^{-1})$ and $e(e(t)) = e(t)$. One can show that the condition (c) implies that $e(s) \cdot e(t) \cdot e(s) = e(s)$. We can also show that $(s \cdot t)^{-1} = e(s) \cdot t^{-1} \cdot s^{-1} \cdot e(t)$. As shown in [12], the mapping $e : T \to T$ defined by $t \mapsto e(t)$, is a continuous map. Let T and S be two generalized groups.

A map $f : T \to S$ is called a *homomorphism* if $f(st) = f(s) \cdot f(t)$ for every $s, t \in T$.

Every topological group is a topological generalized group.

Let T be the topological space $\mathbb{R} \setminus \{0\}$. We can see that T with the multiplication $x \cdot y = x|y|$ is a topological generalized group. The identity set $e(T)$ is $\{-1, 1\}$.

If T is the topological space

$$\mathbb{R}^2 - \{(0,0)\} = \{re^{i\theta} | \quad r > 0 \quad \text{and} \quad 0 \leqslant \theta < 2\pi\}$$

with the Euclidean metric, then T with the multiplication

$$(r_1 e^{i\theta_1}) \cdot (r_2 e^{i\theta_2}) = r_1 r_2 e^{i\theta_2} \tag{1}$$

is a topological generalized group. We have $e(re^{i\theta}) = e^{i\theta}$ and $(re^{i\theta})^{-1} = \frac{1}{r} e^{i\theta}$. So we can see the identity set $e(T)$ is the unit circle S^1. However, T is not a topological group.

In the following we introduce a new method for constructing a generalized group from a group.

Theorem 2.1. *Let G be a group. Then the set $T := G \times G$ by the multiplication*

$$(s_1, t_1)(s_2, t_2) = (s_1, t_1 s_2 t_2)$$

is a generalized group. Moreover, if G is a topological group then T is also a topological generalized group.

Proof.

$$(s,t).(s,s^{-1}) = (s, tss^{-1}) = (s, te) = (s,t) = (s, et) = (s, s^{-1}st) = (s, s^{-1}).(s,t)$$

thus $e((s,t)) = (s, s^{-1})$. Moreover, we have

$$\begin{aligned}(s,t).(s, s^{-1}t^{-1}s^{-1}) &= (s, tss^{-1}t^{-1}s^{-1}) \\ &= (s, tet^{-1}s^{-1}) \\ &= (s, tt^{-1}s^{-1}) \\ &= (s, es^{-1}) \\ &= (s, s^{-1}) \\ &= e((s,t)).\end{aligned}$$

And also

$$\begin{aligned}(s, s^{-1}t^{-1}s^{-1}).(s,t) &= (s, s^{-1}t^{-1}s^{-1}st) \\ &= (s, s^{-1}t^{-1}et) \\ &= (s, s^{-1}t^{-1}t) \\ &= (s, s^{-1}e) \\ &= (s, s^{-1}) \\ &= e((s,t)).\end{aligned}$$

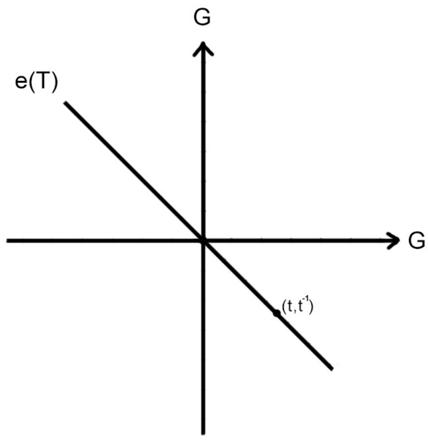

FIGURE 1. The Induced Topological Generalized Group

So $(s,t)^{-1} = (s, s^{-1}t^{-1}s^{-1})$ and $G \times G$ is a generalized group. If G is a topological group, then the generalized group operations m_1 and m_2 for $G \times G$ are also continuous. So $G \times G$ is a topological generalized group.

□

The above topological generalized group T is called the *induced topological generalized group* of the topological group G. We can see that $e(T) = \{(t, t^{-1})|\ t \in G\}$, (See Figure 1). We are able to construct generalized groups from groups with this method.

For any positive integer n, there is a topological generalized group T such that $\mathrm{Card}(e(T)) = n$.

Proof. In Theorem 2.1, if consider $G := \mathbb{Z}_n$, then $T = G \times G$ is a generalized group that $\mathrm{Card}(e(T)) = n$. □

Let G be the group $\{0, \theta, \theta^2, \theta^3\}$ where θ is the 90 degrees counterclockwise rotation of xy-plane. So the induced topological generalized group of G has 16 members and $\mathrm{Card}(e(T)) = 4$.

Let X be a topological space and let T be a topological generalized group. A *generalized action* of T on X is a continuous map $\lambda : T \times X \longrightarrow X$ such that the following conditions hold:

(a) $\lambda(s, \lambda(t, x)) = \lambda(s \cdot t, x)$, for $s, t \in T$ and $x \in X$;
(b) If $x \in X$, there is $e(t) \in T$ such that $\lambda(e(t), x) = x$.

A T-*space* is a triple (X, T, λ) where X is a Hausdorff topological space, T is a topological generalized group and $\lambda : T \times X \longrightarrow X$ is a generalized action of T on X. Moreover, for each $x \in X$:

(i) $T_x = \{t \in T|\quad tx = x\}$ is called the *stabilizer* of x in T;

(ii) $T(x) = \{tx| \quad t \in T\}$ is called T-*orbit* of x in X.

For $x \in X$, T_x is a generalized subgroup of T [5]. For $x \in X$, if the stabilizer T_x is trivial (i.e. $T_x = \{e(t)\}$ for some $t \in T$), we say that x is a *regular* point, otherwise, it is called a *singular* point. The *singular* set of X, denoted by $\sum_X$, is the set of singular points of X. We define two maps $\theta_t : X \to X$ and $\rho_x : T \to X$, by $\theta_t(x) = tx$ and $\rho_x(t) = tx$, respectively, where $t \in T$ and $x \in X$. We can see that θ_t and ρ_x are continuous maps. Clearly, $T(x) = \rho_x(T)$ and $T_x = (\rho_x)^{-1}(x)$. So T_x is a closed subset of T and then we can say it is a closed generalized subgroup of T. Now for continuing we need the definition of *top spaces* that are introduced in [14]. The topological generalized group T is said to be a *top space* if T is a manifold and the generalized group operations m_1 and m_2 are smooth [14]. As shown in [5] if X is a manifold and $\text{Card}(e(T)) < \infty$, then we can show that for $x \in X$, T_x is a generalized subtop space of T and also a topological generalized subgroup of T.

By the action λ of a T on X, we can define the following equivalence relation on X:

$$x \sim y \text{ if and only if there is } t \in T \text{ such that } tx = y.$$

Now, we can consider the quotient space $X\sim$. By the projection map $\pi : X \to X\sim$, we can define a natural topology on $X\sim$ such that π is a continuous map. In fact $U \subseteq X\sim$ is open if $\pi^{-1}(U)$ is open in X. We will use of the notation XT for the topological quotient space $X\sim$.

Let T be $\mathbb{R} - \{0\}$ with the Euclidean metric. T with the multiplication $x \cdot y = x$ is a topological generalized group that if $x \in T$, then $e(x) = x^{-1} = x$. T acts on itself with this multiplication.

We recall that if T is a generalized group, X is a set and $S = \{\varphi^t \mid \quad \varphi^t : X \to X \quad \text{is a mapping and } t \in T\}$, then the triple (X, S, T) is called a *complete semidynamical system* if:

(i) $\varphi^{t_1} \circ \varphi^{t_2} = \varphi^{t_1 \cdot t_2}$, for all $t_1, t_2 \in T$;
(ii) For given $x \in X$, there is $\varphi^t \in D$ such that x is a fixed point of φ^t.

We see that each T-space (X, T, λ) generates a complete semidynamical system (X, S, T) where

$$S = \{\theta_t : X \to X \mid \theta_t(x) = tx, \quad \text{for } x \in X \text{ and } t \in T\}.$$

As shown in [5], if $e(T) \subseteq T_x$, for each $x \in X$, then S with the multiplication $\theta_s \circ \theta_t = \theta_{st}$ is a topological generalized group. In this case, for each $\theta_t \in S, e(\theta_t) = \theta_{e(t)}$ and $(\theta_t)^{-1} = \theta_{t^{-1}}$.

Theorem 2.2. *If $e(T) \subseteq T_x$, for each $x \in X$, then each θ_t is a homeomorphism.*

Proof. Every θ_t is a continuous map on X. Now we claim that each θ_t is one to one, onto and has the inverse $\theta_{t^{-1}}$. If $\theta_t(x) = \theta_t(y)$, then $tx = ty$, and so $t^{-1}tx = e(t)x = x = y = e(t)y = t^{-1}ty$. Hence θ_t is one to one. On the other hand, for $x \in X$, there exists $t^{-1}x \in X$ such that $\theta_t(t^{-1}x) = tt^{-1}x = e(t)x = x$. Thus θ_t is also onto. If $t \in T$ and $x \in X$, then

$$\theta_t \circ \theta_{t^{-1}}(x) = \theta_{tt^{-1}}(x) = \theta_{e(t)}(x) = e(t)x = x.$$

The last equity follows from the fact $e(T) \subseteq T_x$. In the same way

$$\theta_{t_{-1}} \circ \theta_t(x) = x.$$

Therefore, every θ_t is a homeomorphism. □

Theorem 2.3. *Let (X, T, λ) be a T-space. If T is compact and $e(T) \subseteq T_x$ for each $x \in X$, then $\lambda : T \times X \to X$ is a closed map.*

Proof. Assume that C is a closed subset of $T \times X$ and $x \in X$ is a limit point of $\lambda(C)$. So there exists a sequence $\{(t_i, x_i)\}$ in C such that $\lambda(t_i, x_i) = t_i x_i$ converges to x. As T is compact, then there is a subsequence of $\{t_i\}$ such that converges to a t in T. We rename that subsequence be $\{t_i\}$. T is a topological generalized group, so the map $m_1 : T \to T$, defined by $t \mapsto t^{-1}$ is continuous. This implies that $\{t_i^{-1}\}$ converges to t^{-1}. λ is also continuous, $\lambda(t_i^{-1}, t_i x_i)$ converges to $\lambda(t^{-1}, x)$, so $\{e(t_i)x_i\}$ converges to $t^{-1}x$. But for each $x \in X$, $e(T) \subseteq T_x$, thus $e(t_i)x_i = x_i$ and consequently, x_i converges to $t^{-1}x$. So the sequence $\{(t_i, x_i)\}$ in C converges to $(t, t^{-1}x)$. Since C is a closed subset of $T \times X$, then $(t, t^{-1}x) \in C$. Therefore, $\lambda(t, t^{-1}x) = e(t)x \in \lambda(C)$. According to the assumption, $e(t)x = x$. So $x \in \lambda(C)$ which means that $\lambda(C)$ is closed, that is, λ is a closed map. □

A generalized action λ of T on X is called *perfect* if $e(T) \subseteq T_x$ for each $x \in X$. Moreover, λ is called *super perfect* if for each $x \in X$, $e(T) = T_x$.

Theorem 2.4. *Suppose that (X, T, λ) is a T-space and λ is perfect. If X is locally connected, then $Y := XT$ is also locally connected.*

Proof. We know that X is locally connected if and only if every open subset $U \subseteq X$ can be decomposed into a disjoint union of open connected subspaces of X [17]. Now we consider the projection map $\pi : X \to XT$. Let V be an open subset of $Y := XT$ and consider the inverse image $\pi^{-1}(V)$ that is open in X. Since X is locally connected so $\pi^{-1}(V)$ is decomposed into a disjoint union of connected components of X. As λ is a perfect generalized action of T on X, we can define a natural generalized action of T on the set of connected components of X. The image of each component of X under π is the same within an orbit in Y. So $\pi(\pi^{-1}(V)) = V$ is decomposed into a disjoint union of open connected subsets of Y. Hence, Y is locally connected.

□

3. *GG*-spaces

Now we are ready to define *GG*-spaces. A *GG*-space is a topological space that is locally homeomorphic to a quotient of $\mathbb{R}^n$ by the generalized action of a topological generalized group. First, we need to define charts. Let X be a topological space. Then a *chart* for X is a $(U, \widetilde{U}, \varphi, T)$ where U is an open subset of X, $\widetilde{U}$ is an open subset of $\mathbb{R}^n$, T is a topological generalized group that acts continuously on $\widetilde{U}$ by a generalized action λ and $\varphi : \widetilde{U} \longrightarrow U$ is a continuous map inducing a homeomorphism between $\widetilde{U}T$ and U.

The collection $\{(U_i, \widetilde{U}_i, \varphi_i, T_i) : i \in I\}$ of charts of X is said to be an *atlas* for X if the following properties are satisfied:

(i) $\{U_i : i \in I\}$ is a cover of X that closed under finite intersection;

(ii) whenever $U_i \subset U_j$, there is an injective generalized group homomorphism

$$f_{ij} : T_i \hookrightarrow T_j$$

and an embedding

$$\psi_{ij} : \widetilde{U}_i \hookrightarrow \widetilde{U}_j$$

such that for $t \in T_i$,

$$\psi_{ij}(tx) = f_{ij}(t)\psi_{ij}(x) \tag{2}$$

and also

$$\varphi_j \circ \psi_{ij} = \varphi_i. \tag{3}$$

Note that the equation (2) means that ψ_{ij} is equivariant with respect to f_{ij}.

A *GG-space* is a pair $(X, \mathcal{A})$ where X is a topological space and $\mathcal{A}$ is an atlas for X. We consider that the atlas $\mathcal{A}$ is maximal. An atlas $\mathcal{A}$ for X is *maximal* if it is not contained in any strictly larger atlas. This just means every chart satisfying conditions (i) and (ii) of the definition 3, is already in $\mathcal{A}$.

Theorem 3.1. *Let X be a GG-space. Every atlas for X is contained in a unique maximal atlas.*

Proof. Let $\mathcal{A}$ be an atlas for X and $\overline{\mathcal{A}}$ denote the set of all charts satisfying conditions (i) and (ii) of the definition 3 for every chart in $\mathcal{A}$. To show that $\overline{\mathcal{A}}$ is an atlas for X, let $(U_i, \widetilde{U}_i, \varphi_i, T_i)$ and $(U_j, \widetilde{U}_j, \varphi_j, T_j)$ be two charts in $\overline{\mathcal{A}}$, $U_i \subset U_j$ and $x \in U_i$ be arbitrary. Because the domains of the charts in $\mathcal{A}$ cover X, there is some chart $(U_k, \widetilde{U}_k, \varphi_k, T_k) \in \mathcal{A}$ such that $x \in U_k$. We can consider U_k such that $U_i \subset U_k \subset U_j$. Since every chart in $\overline{\mathcal{A}}$ satisfies conditions (i) and (ii) of the definition 3 for every chart in $\mathcal{A}$, there are two injective generalized group homomorphisms

$$f_{ik} : T_i \hookrightarrow T_k$$
$$f_{kj} : T_k \hookrightarrow T_j$$

and two embeddings

$$\psi_{ik} : \widetilde{U}_i \hookrightarrow \widetilde{U}_k$$
$$\psi_{kj} : \widetilde{U}_k \hookrightarrow \widetilde{U}_j$$

such that for $t \in T_i$ and $s \in T_k$,

$$\psi_{ik}(tx) = f_{ik}(t)\psi_{ik}(x)$$

$$\psi_{kj}(sx) = f_{kj}(s)\psi_{kj}(x)$$

and also

$$\varphi_k \circ \psi_{ik} = \varphi_i$$
$$\varphi_j \circ \psi_{kj} = \varphi_k.$$

Now, we consider

$$f_{ij} := f_{kj} \circ f_{ik} : T_i \hookrightarrow T_j$$

and

$$\psi_{ij} := \psi_{kj} \circ \psi_{ik} : \widetilde{U}_i \hookrightarrow \widetilde{U}_j.$$

We can see that f_{ij} is an injective generalized group homomorphism because it is a composition of two injective generalized group homomorphisms. Moreover, ψ_{ij} is an embedding because it is a composition of two embeddings and we have

$$\begin{aligned} \psi_{ij}(tx) &= \psi_{kj}(\psi_{ik}(tx)) \\ &= \psi_{kj}(f_{ik}(t)\psi_{ik}(x)) \\ &= f_{kj}(f_{ik}(t))(\psi_{kj}\psi_{ik}(x)) \\ &= f_{ij}(t)\psi_{ij}(x). \end{aligned}$$

Also we can show that

$$\varphi_j\psi_{ij} = \varphi_j\psi_{kj}\psi_{ik} = \varphi_k\psi_{ik} = \varphi_i.$$

Therefore $\overline{\mathcal{A}}$ is an atlas. To check that it is maximal, just note that any chart that satisfies conditions (i) and (ii) of the definition 3 for every chart in $\overline{\mathcal{A}}$ must in particular satisfy conditions (i) and (ii) of the definition 3 for every chart in $\mathcal{A}$, so it is already in $\overline{\mathcal{A}}$. This proves the existence of a maximal atlas containing $\mathcal{A}$. If $\mathcal{B}$ is any other maximal atlas containing $\mathcal{A}$, each of its charts satisfy conditions (i) and (ii) of the definition 3 with each chart in $\mathcal{A}$, so $\mathcal{B} \subset \overline{\mathcal{A}}$. By maximality of $\mathcal{B}$, $\mathcal{B} = \overline{\mathcal{A}}$. □

Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be two GG-spaces. The *product* $(X \times Y, \mathcal{A} \times \mathcal{B})$ is the GG-space that for each $(x, y) \in X \times Y$,

$$(U_x \times U_y, \widetilde{U}_x \times \widetilde{U}_y, \varphi_x \times \varphi_y, T_x \times T_y)$$

is a chart, where $(U_x, \widetilde{U}_x, \varphi_x, T_x) \in \mathcal{A}$ and $(U_y, \widetilde{U}_y, \varphi_y, T_y) \in \mathcal{B}$ are charts for x in X and y in Y.

In the following example, the distinction between the geometrical structure of GG-spaces and classical geometrical structures such as Manifolds and orbifolds is well illustrated. In the Manifold theory, no center is considered for the unit circle, but in the concept of GG-spaces we are able to consider the unit circle with its center as a connected geometric structure.

Let $Y = \mathbb{R}^2$ and T be the generalized group of example 2 which acts on Y by

$$(r_1 e^{i\theta_1}).(r_2 e^{i\theta_2}) = r_1 r_2 e^{i\theta_2}.$$

We can see that $T_x = e(T)$, for each $x \in Y$, so the action of T is super perfect. For $x = r_1 e^{i\theta_1}$ and $y = r_2 e^{i\theta_2}$, $[x] = [y]$ if and only if $\theta_1 = \theta_2$. Now suppose

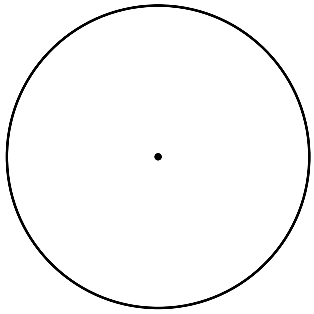

FIGURE 2. The GG-space which is not an orbifold.

$X := YT$. We can see that (X, Y, π, T) is a chart for X where $\pi : Y \to X$ is the projection map. Intuitively, X is $S^1 \bigcup \{(0, 0)\}$ (See Figure 2). Note that X is a connected space with the quotient topology.

Let $\mathbb{T}$ be the product $X \times X$ in Example 3. Based on Definition 3, $\mathbb{T}$ is a GG-spaces that we call it *generalized torus.* Let X be the topological space $[0, \infty)$ with the topology generated by Euclidean Metric. Then $(X, \mathbb{R}, \varphi, T)$ is a chart for X, where the finite topological generalized group $T = \{\pm 1\}$ with the multiplication $s \cdot t = s|t|$ acts on $\mathbb{R}$ by the generalized action $tx = t|x|$, for $t \in T$ and $x \in \mathbb{R}$ and $\varphi : \mathbb{R} \to X$ is defined by $\varphi(x) = |x|$.

Theorem 3.2. *For any open connected T-space (X, T, λ) that $X \subseteq \mathbb{R}^n$, the quotient space XT is a GG-space.*

Proof. we can see that $\mathcal{A} = \{(XT, X, \pi, T)\}$ is the atlas of XT, where π is the projection mapping $\pi : X \to XT$. □

Let T be the space of all real 2×2 matrices with product

$$\mathbf{Mat}(a_{11}, a_{12}, a_{21}, a_{22}) \times \mathbf{Mat}(b_{11}, b_{12}, b_{21}, b_{22}) = \mathbf{Mat}(a_{11}, b_{12}, b_{21}, a_{22}).$$

As shown in [5], T is a topological generalized group. Let $X = \mathbb{R}^4$ that T acts on it with $\lambda : T \times X \to X$ defined by

$$\mathbf{Mat}(a_{11}, a_{12}, a_{21}, a_{22}) \times (b, c, d, e) = (a_{11}, c, d, a_{22}).$$

We can see that λ is a generalized action and (X, T, λ) is a connected T-space. For each $x = (b, c, d, e) \in \mathbb{R}^4$, $T(x) = \{(y, c, d, z) | y, z \in \mathbb{R}\}$, so $XT \simeq \mathbb{R}^2$ is a GG-space with the single chart $\{(XT, X, \pi, T)\}$.

However, we can show below theorem.

Theorem 3.3. *The GG-space $(X, \mathcal{A})$ is an orbifold if every topological generalized group T_i is a finite group. Moreover, $(X, \mathcal{A})$ is a manifold if every topological generalized group T_i is trivial.*

Proof. Using the definition of an orbifold [16] and a manifold [6], we can prove this theorem. □

Note. There are GG-spaces that are not a orbifold. (See Example 3).

Theorem 3.4. *Let $(X, \mathcal{A})$ be a GG-space. If every topological generalized group T_i in the atlas of X is finite and its generalized action is super perfect, then X is a manifold.*

Proof. We know that for each $x \in X$ there is a chart $(U, \widetilde{U}, \varphi, T)$ such that $x \in U$ and $\widetilde{U} \subseteq \mathbb{R}^n$ and a continuous map $\varphi : \widetilde{U} \to U$ induces a homeomorphism between $\widetilde{U}T$ and U. We claim that $\widetilde{U}T$ is locally Euclidean ,i.e. U is locally Euclidean and then X is a manifold.

Since the generalized action of T on $\widetilde{U}$ is super perfect, $tz \neq z$ for each $t \notin e(T)$ and for each $z \in \widetilde{U}$. Moreover, T is finite, so we can say that for each $z \in \widetilde{U}$ there is a neighborhood $\widetilde{V} \subseteq \widetilde{U}$ of z such that

$$t\widetilde{V} \bigcap \widetilde{V} = \varnothing, \tag{4}$$

where $t \notin e(T)$.

Now we consider the projection map $\pi : \widetilde{U} \to \widetilde{U}T$. We will show that $\pi(\widetilde{V})$ is a open subset of $\widetilde{U}T$ that is homeomorphic to the open subset $\widetilde{V}$ of $\mathbb{R}^n$. This implies that $\widetilde{U}T$ and also U are locally Euclidean.

We can see that $\pi^{-1}(\pi(\widetilde{V})) = \bigcup t\widetilde{V}$, where $t \in T$. Since the action of T on $\widetilde{U}$ is perfect, so every $\lambda_t : X \to X$ defined by $\lambda_t(x) = tx$, is a homeomorphism and so is an open map. So $t\widetilde{V} = \lambda_t(\widetilde{V})$ is an open subset of $\widetilde{U}$. So $\pi^{-1}(\pi(\widetilde{V}))$ is open in $\widetilde{U}$. According to the quotient topology, $\pi(\widetilde{V})$ is open in $\widetilde{U}T$. Moreover, we knew that $\pi_{|_{\widetilde{V}}} : \widetilde{V} \to \pi(\widetilde{V})$ is an open surjective continuous map. Also using 4, it is injective. So $\pi(\widetilde{V})$ is homeomorphic to $\widetilde{V}$ and $\widetilde{U}T$ is locally Euclidean. Therefore U is locally Euclidean. □

Note that in Theorem 3.4, the finiteness of the topological generalized group T in atlas $\mathcal{A}$ is necessary. For instance, in example 3, in which X is not a manifold.

4. **Conclusion**

In this paper, we introduced and studied new geometric structures called GG-spaces. They are a new generalization of manifolds and orbifolds. We also prove that if each topological generalized group T_i in atlas of X is finite and its generalized action is super perfect, then X is a manifold. GG-spaces are new geometric spaces on which concepts such as GG-subspaces, smooth maps and tangent space can be studied.

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