Journal of Mahani Mathematical Research Center
Shahid Bahonar University of Kerman
Print ISSN: 2251-7952 Online ISSN: 2645-4505

ON THE STABILITY OF 2-DIMENSIONAL PEXIDER QUADRATIC FUNCTIONAL EQUATION IN NON-ARCHIMEDEAN SPACES

M.A. Abolfathi*

Article type: Research Article
(Received: 24 December 2021, Revised: 11 February 2022, Accepted: 13 February 2022)
(Available Online: 27 February 2022)
(Communicated by F. Abdollahi)

Abstract. In this papers we investigate the Hyers-Ulam stability of the following 2-dimensional Pexider quadratic functional equation

$$f(x+y,z+w)+f(x-y,z-w)=2g(x,z)+2g(y,w)$$

in non-Archimedean normed spaces.

Keywords: Hyers-Ulam stability, 2-dimensional quadratic functional equation, Pexider quadratic functional equation, non-Archimedean normed space, p-adic field.
2020 MSC: 39B82, 39B52, 46S10, 26E30.

1. Introduction

In 1897, Hensel [13] discovered the p-adic numbers as a number theoretical analogue of power series in complex analysis. The most important examples of non-Archimedean spaces are p-adic numbers. A key property of p-adic numbers is that they done not satisfy the Archimedean axiom: for all $x, y > 0$, there exists an integer n such that $x < ny$.

During the last three decades theory of non-Archimedean spaces has gained the interest of physicists for their research, in particular the problems that emerge in quantum physics, p-adic strings and superstrings. Although many results in the classical normed space theory have a non-Archimedean counterpart, their proofs are essentially different and require an entirely new kind of intuition. One may note that for $|n| \leq 1$ in each valuation field, every triangle is isosceles and there many be no unit vector in a non-Archimedean normed space. These facts show that the non-Archimedean framework is of special interest. It turned out that non-Archimedean spaces have many nice applications [12, 25, 27].

One of the most interesting questions in the theory of functional analysis concerning the Ulam stability problem of functional equations is as follows: If the problem accepts a solution, we say that the equation is stable. The first

*Corresponding author, ORCID: 0000-0002-1298-003X
E-mail: m.abolfathi@urmia.ac.ir
DOI: 10.22103/jmmrc.2022.18746.1188

How to cite: M.A. Abolfathi, *On the stability of 2-dimensional Pexider quadratic functional equation in non-Archimedean spaces*, J. Mahani Math. Res. Cent. 2022; 11(2): 31-43.

problem concerning group homomorphisms was raised by Ulam [29] in 1940. In the next year Hyers [14] gave a first affirmative answer to the question of Ulam in context of Banach spaces. Later, Bourgin [5] and Aoki [3] treated this problem for approximate additive mappings by allowing the Cauchy difference to be unbounded, that is, controlled by the sum of two powered. In 1978, Rassias [22] proved a generalization of the Hyers' theorem by proving the existence of unique linear mappings near approximate linear mappings (see also [10]). Also, Trif [28] studied the Cauchy-Rassias stability of the Jensen type functional equation. The result of Rassias has provided a lot of influence during the last three decades in the development of generalization of Hyers-Ulam stability concept. Furthermore, in 1994, Găvruţa [11] provided a further generalization of Rassias' theorem in which he replaced the bound $\varepsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x, y)$. Usually the stability problem for functional equations is solved by direct method in which the exact solution of the functional equation is explicitly constructed as a limit of a (Hyers) sequence, starting from the given approximate solution of function f (see [4, 10, 17, 19, 24, 26]).

Functional equations find a lot of application in information theory, information science, measure of information, coding theory, computer graphics, spatial filtering in image processing, behavioral and social sciences, astronomy, number theory, fuzzy system models, economics, stochastic processes, mechanics, cryptography and physics.

Recently several stability results have been obtained for various equations and mappings with more general domains and ranges have been investigated by a number of authors and there are many interesting results concerning this problem (see [1, 2, 6, 7, 9, 15, 16, 18, 21, 23, 30, 31]).

Fix a prime number p. For any nonzero rational number x, there exists a unique integer n_x such that $x = \frac{a}{b}p^{n_x}$, where a and b are integers not divisible by p. Then $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(x, y) = |x - y|_p$ is denoted by $\mathbb{Q}_p$, and it is called the p-adic number field. In fact, $\mathbb{Q}_p$ is the set of all formal series $x = \sum_{k\geq n}^{\infty} a_k p_k$, where $|a_k| \leq p - 1$ are integers. The addition and multiplication between any two elements of $\mathbb{Q}_p$ are defined naturally. The norm $|\sum_{k\geq n}^{\infty} a^k p_k|_p = p^{-n_x}$ is a non-Archimedean norm on $\mathbb{Q}_p$ and it makes $\mathbb{Q}_p$ a locally compact field. Note that if $p \geq 3$, then $|2^n|_p = 1$ for each integer n.

The following example shows that the same results for stability of functional equations in (Archimedean) normed spaces always can not be true in non-Archimedean normed spaces.

Let $p \geq 3$ and $f : \mathbb{Q}_p \to \mathbb{Q}_p$ be defined by $f(x) = 2$. Then for $\varepsilon = 1$,

$$|f(x + y) - f(x) - f(y)| = 1 \leq \varepsilon$$

for all $x, y \in \mathbb{Q}_p$. By using the fact $|2| = 1$, we have

$$|\frac{f(2^n x)}{2^n} - \frac{f(2^{n+1}x)}{2^{n+1}}| = |\frac{2}{2^n} - \frac{2}{2^{(n+1)}}| = |2^{-n}| = 1,$$

and

$$|2^n f(\frac{x}{2^n}) - 2^{n+1} f(\frac{x}{2^{n+1}})| = |2^n(2) - 2^{(n+1)}(2)| = |2^{(n+1)}| = 1,$$

for all $x, y \in \mathbb{Q}_p$ and $n \in \mathbb{N}$. Therefore the sequences $\{\frac{f(2^n x)}{2^n}\}_{n=1}^{\infty}$ and $\{2^n f(\frac{x}{2^n})\}_{n=1}^{\infty}$ are not Cauchy. In fact these sequences are not convergent in $\mathbb{Q}_p$.

A valuation is a function $|.|$ from a field $\mathbb{K}$ into $[0,\infty)$ such that 0 is the unique element having the 0 , $|ab| = |a||b|$, and the triangle inequality holds, that is, for all $a, b \in \mathbb{K}$, we have $|a+b| \leq |a| + |b|$. A field $\mathbb{K}$ is called a valued field if $\mathbb{K}$ carries a valuation. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuations. Let us consider a valuation which satisfies a stronger condition than the triangle inequality.

Note that $|1| = |-1| = 1$ and $|n| \leq 1$ for each integer n. A trivial example of a non-Archimedean valuation is the functional $|.|$ taking everything except for 0 into 1 and $|0| = 0$. We always assume, in addition, that $|.|$ is non-trivial, i.e., there exists an $a_0 \in \mathbb{K}$ such that $|a_0| \notin \{0, 1\}$.

Definition 1.1. Let X be a linear space over a scaler field $\mathbb{K}$ with a non-Archimedean nontrivial valuation $|.|$. A $\|.\| : X \to \mathbb{R}$ is a non-Archimedean norm (valuation) if it satisfies the following conditions:

(N1) $\|x\| = 0$ if and only if $x = 0$,

(N2) $\|rx\| = |r|\|x\|$,

(N3) $\|x+y\| \leq max\{\|x\|, \|y\|\}$ (the strict triangle inequality (ultrametric)) for all $x, y \in X$.

Then $(X, \|.\|)$ is called a non-Archimedean space.
It follows from (N3) that

$$\|x_n - x_m\| \leq \max\{\|x_{i+1} - x_i\| : m \leq i \leq n-1\} \quad (n > m).$$

Definition 1.2. Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X.

(1) The sequence $\{x_n\}$ is called a Cauchy sequence if for any $\varepsilon > 0$, there is a positive integer N such that $\|x_n - x_m\| < \varepsilon$ for all $n, m \geq N$.
(2) The sequence $\{x_n\}$ is said to be convergent if for any $\varepsilon > 0$, there is a positive integer N such that $\|x_n - x\| < \varepsilon$ for all $n \geq N$. Then the point $x \in X$ is called the limit of the sequence $\{x_n\}$, which is denote by $\lim_{n\to\infty} x_n = x$.
(3) If every Cauchy sequence in X converges, then the non-Archimedean normed space X is called a non-Archimedean Banach space.

The functional equation $f(x+y)+f(x-y) = 2f(x)+2f(y)$ is called quadratic functional equation. Obviously, the mapping f between two real vector spaces X and Y is a solution of this equation, if and only if there exists a unique symmetric bi-additive mapping $B_1 : X \times X \to Y$ such that $f(x) = B_1(x, x)$ for all $x \in X$. The bi-additive mapping B_1 is given by

$$B_1(x,y) = \frac{1}{4}(f(x+y) - f(x-y)).$$

The function $f(x) = cx^2$ satisfies the functional equation, which is called a quadratic functional equation. The Hyers- Ulam stability problem for the quadratic functional equation was solved by Skof [**?**] and, independently, by Cholewa [8]. The stability for the bi-quadratic functional equation

$$f(x+y,z+w) + f(x+y,z-w) + f(x-y,z+w) + f(x-y,z-w) \\ = 4[f(x,z) + f(x,w) + f(y,z) + f(y,w)]$$

was proved by Bae and Park [20] for $f : X \times X \longrightarrow Y$, where X is a real normed space and Y is a Banach space.

Consider the 2-dimensional quadratic functional equation

(1) $$f(x+y,z+w) + f(x-y,z-w) = 2f(x,z) + 2f(y,w)$$

which has quadratic form

$$f(x,y) = ax^2 + bxy + cy^2$$

as solutions.

In this paper, we prove the generalized Hyers-Ulam stability for the 2-dimensional vector variable Pexider quadratic functional equation

$$f(x+y,z+w) + f(x-y,z-w) = 2g(x,z) + 2g(y,w)$$

in non-Archimedean spaces.

2. Main results

We assume that X is an additive semigroup and Y is a complete non-Archimedean spaces. Also, let $|4| < 1$ and we assume that $4 \neq 0$ in $\mathbb{K}$ (i.e., the characteristic of $\mathbb{K}$ is not 4).

Lemma 2.1. *Let $\varphi : X \times X \times X \times X \to [0,\infty)$ and $f, g : X \times X \to Y$ are mappings satisfying $f(0,0) = g(0,0) = 0$ such that*

(2) $$\|f(x+y,z+w) + f(x-y,z-w) - 2g(x,z) - 2g(y,w)\| \leq \varphi(x,y,z,w)$$

for all $x, y, z, w \in X$, then

$$\|f(x+y,z+w) + f(x-y,z-w) - 2f(x,z) - 2f(y,w)\|$$

(3) $$\leq \max\{\varphi(x,y,z,w), \varphi(x,0,z,0), \varphi(y,0,w,0)\}$$

and

(4)

$$\|g(x+y,z+w) + g(x-y,z-w) - 2g(x,z) - 2g(y,w)\| \\ \leq \max\{\varphi(x,y,z,w), \frac{\varphi(x+y,0,z+w,0)}{|2|}, \frac{\varphi(x-y,0,z-w,0)}{|2|}\}$$

for all $x, y, z, w \in X$

Theorem 2.2. *Let* $\varphi : X \times X \times X \times X \to [0, \infty)$ *be a function such that*

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y, 2^n z, 2^n w)}{|4|^n} = 0 \tag{5}$$

for all $x, y, z, w \in X$ *and let for each* $x, z \in X$ *the limit*

$$\tilde{\varphi}(x,z) := \lim_{n\to\infty} \max\{\max\{\frac{|2|\varphi(2^j x, 0, 2^j z, 0)}{|4|^j}, \frac{\varphi(2^j x, 2^j x, 2^j z, 2^j z)}{|4|^j}\} : 0 \le j < n\} \tag{6}$$

exist. Suppose that $f, g : X \times X \to Y$ *are mappings satisfying* $f(0,0) = g(0,0) = 0$ *and*

$$\|f(x+y, z+w) + f(x-y, z-w) - 2g(x,z) - 2g(y,w)\| \le \varphi(x,y,z,w) \tag{7}$$

for all $x, y, z, w \in X$. *Then there exists a unique 2-dimensional quadratic mapping* $\mathcal{Q} : X \times X \to Y$ *such that*

$$\|\mathcal{Q}(x,z) - f(x,z)\| \le \frac{1}{|4|}\tilde{\varphi}(x,z) \tag{8}$$

and

$$\|\mathcal{Q}(x,z) - g(x,z)\| \le \max\{\frac{1}{|4|}\tilde{\varphi}(x,z), \frac{1}{|2|}\varphi(x,0,z,0)\} \tag{9}$$

for all $x, z \in X$. *Moreover, if*

$$\lim_{i\to\infty}\lim_{n\to\infty} \max\{\max\{\frac{|2|\varphi(2^j x, 0, 2^j z, 0)}{|4|^j}, \frac{\varphi(2^j x, 2^j x, 2^j z, 2^j z)}{|4|^j}\} : 0 \le i \le j < n+i\} = 0 \tag{10}$$

then $\mathcal{Q}$ *is the unique 2-dimensional quadratic mapping satisfying (8) and (9).*

Proof. Letting $y = 0$ and $w = 0$ in (7), we get

$$\|f(x,z) - g(x,z)\| \le \frac{\varphi(x,0,z,0)}{|2|} \tag{11}$$

Putting $y = x$ and $w = z$ in (7) and dividing both sides by 4, we have

$$\|\frac{1}{4}f(2x,2z) - g(x,z)\| \le \frac{\varphi(x,x,z,z)}{|4|} \tag{12}$$

and so

$$\|\frac{1}{4}f(2x,2z) - f(x,z)\| \le \max\{\frac{|2|\varphi(x,0,z,0)}{|4|}, \frac{\varphi(x,x,z,z)}{|4|}\} \tag{13}$$

for all $x, z \in X$. Replacing x and z by $2^{n-1}x$ and $2^{n-1}z$ in (13) respectively and dividing both sides by 4^{n-1}, we get

$$\begin{aligned}(14)\qquad &\|\frac{1}{4^n}f(2^n x, 2^n z) - \frac{1}{4^{n-1}}f(2^{n-1}x, 2^{n-1}z)\| \\ &\leq \frac{1}{|4|}\max\{\frac{|2|\varphi(2^{n-1}x, 0, 2^{n-1}z, 0)}{|4|^{n-1}}, \\ &\frac{\varphi(2^{n-1}x, 2^{n-1}x, 2^{n-1}z, 2^{n-1}z)}{|4|^{n-1}}\}.\end{aligned}$$

Therefore by (5) and (14) $\{\frac{1}{4^n}f(2^n x, 2^n z)\}$ is a Cauchy sequence in Y. Since Y is complete, the sequence $\{\frac{1}{4^n}f(2^n x, 2^n z)\}$ is convergent for all $x, z \in X$.
Set

$$\mathcal{Q}(x, z) = \lim_{n\to\infty}\frac{1}{4^n}f(2^n x, 2^n z).$$

By induction we have

(15)

$$\begin{aligned}&\|\frac{1}{4^n}f(2^n x, 2^n z) - f(x, z)\| \\ &\leq \frac{1}{|4|}\max\{\max\{\frac{|2|\varphi(2^j x, 0, 2^j z, 0)}{|4|^j}, \frac{\varphi(2^j x, 2^j x, 2^j z, 2^j z)}{|4|^j}\} : 0 \leq j < n\}\end{aligned}$$

for all $x, z \in X$. For $n = 1$ we get (13), obviously. Now, if (15) holds for every $0 \leq j < n-1$, we have

(16)

$$\begin{aligned}&\|\frac{1}{4^n}f(2^n x, 2^n z) - f(x, z)\| \\ &\leq \max\{\|\frac{1}{4^n}f(2^n x, 2^n z) - \frac{1}{4^{n-1}}f(2^{n-1}x, 2^{n-1}z)\|, \|\frac{1}{4^{n-1}}f(2^{n-1}x, 2^{n-1}z) - f(x, z)\|\} \\ &\leq \frac{1}{|4|}\max\{\max\{\frac{|2|\varphi(2^{n-1}x, 0, 2^{n-1}z, 0)}{|4|^{n-1}}, \frac{\varphi(2^{n-1}x, 2^{n-1}x, 2^{n-1}z, 2^{n-1}z)}{|4|^{n-1}}\}, \\ &\frac{1}{|4|}\max\{\max\{\frac{|2|\varphi(2^j x, 0, 2^j z, 0)}{|4|^j}, \frac{\varphi(2^j x, 2^j x, 2^j z, 2^j z)}{|4|^j}\} : 0 \leq j < n-1\}\} \\ &\leq \frac{1}{|4|}\max\{\max\{\frac{|2|\varphi(2^j x, 0, 2^j z, 0)}{|4|^j}, \frac{\varphi(2^j x, 2^j x, 2^j z, 2^j z)}{|4|^j}\} : 0 \leq j < n\}\}.\end{aligned}$$

Therefore for all $x, z \in X$ and all $n \in \mathbb{N}$, (16) holds. By taking n approach to infinity in (16) and using (6) ,we have

$$\|\mathcal{Q}(x, z) - f(x, z)\| \leq \frac{1}{|4|}\tilde{\varphi}(x, z).$$

On the other hand, by (11), we obtain

$$\begin{aligned}\|\mathcal{Q}(x, z) - g(x, z)\| &\leq \max\{\|\mathcal{Q}(x, z) - f(x, z)\|, \|f(x, z) - g(x, z)\|\} \\ &\leq \max\{\frac{1}{|4|}\tilde{\varphi}(x, z), \frac{1}{|2|}\varphi(x, 0, z, 0)\}.\end{aligned}$$

By (3) and (5), we have

$$\begin{aligned}
&\|\mathcal{Q}(x+y,z+w)+\mathcal{Q}(x-y,z-w)-2\mathcal{Q}(x,z)-2\mathcal{Q}(y,z)\| \\
&\quad = \lim_{n\to\infty}\|\frac{1}{4^n}f(2^n(x+y),2^n(z+w))+\frac{1}{4^n}f(2^n(x-y),2^n(z-w)) \\
&\quad - 2\frac{1}{4^n}f(2^nx,2^nz)-2\frac{1}{4^n}f(2^ny,2^nw)\| \\
&\quad \le \max\{\frac{\varphi(2^nx,2^ny,2^nz,2^nw)}{|4|^n},\frac{\varphi(2^nx,0,2^nz,0)}{|4|^n},\frac{\varphi(2^ny,0,2^nw,0)}{|4|^n}\}=0. \qquad (17)
\end{aligned}$$

Hence $\mathcal{Q}$ fulfills (1). If $\mathcal{Q}'$ is another mapping satisfying (8) and (9), then for all $x,z\in X$, we have

$$\begin{aligned}
\|\mathcal{Q}(x,z)-\mathcal{Q}'(x,z)\| &= \lim_{i\to\infty}\|\frac{\mathcal{Q}(2^ix,2^iz)}{4^i}-\frac{\mathcal{Q}'(2^ix,2^iz)}{4^i}\| \\
&\le \lim_{i\to\infty}\max\{\|\frac{\mathcal{Q}(2^ix,2^iz)}{4^i}-\frac{f(2^ix,2^iz)}{4^i}\|,\|\frac{f(2^ix,2^iz)}{4^i}-\frac{\mathcal{Q}'(2^ix,2^iz)}{4^i}\|\} \\
&\le \lim_{i\to\infty}\lim_{n\to\infty}\frac{1}{|4|}\max\{\max\{\frac{|2|\varphi(2^jx,0,2^jz,0)}{|4|^j},\frac{\varphi(2^jx,2^jx,2^jz,2^jz)}{|4|^j}\}: \\
&\qquad 0\le i\le j<n+i\}=0.
\end{aligned}$$

Therefore

$$\mathcal{Q}(x,z)=\mathcal{Q}'(x,z),$$

for all $x,z\in X$. This completes the proof of the theorem. □

Corollary 2.3. *Let $\zeta:[0,\infty)\longrightarrow[0,\infty)$ be a function satisfying*

$$\xi(|2|t)\le\xi(|2|)\xi(t), \qquad \xi(|2|)<|4|$$

for all $t\ge 0$. Let $\kappa>0$, X be a normed space and let $f,g:X\times X\to Y$ are mappings with $f(0,0)=g(0,0)=0$, such that

$$\begin{aligned}
&\|f(x+y,z+w)+f(x-y,z-w)-2g(x,z)-2g(y,z)\| \\
&\qquad\qquad \le \kappa(\xi(\|x\|)+\xi(\|y\|)+\xi(\|z\|)+\xi(\|w\|))
\end{aligned}$$

for all $x,y,z,w\in X$. Then there exists a unique 2-dimensional quadratic mapping $\mathcal{Q}:X\times X\longrightarrow Y$ such that

$$\|\mathcal{Q}(x,z)-f(x,z)\|\le\frac{2\kappa(\xi(\|x\|)+\xi(\|z\|))}{|4|}$$

and

$$\|\mathcal{Q}(x,z)-g(x,z)\|\le\max\{\frac{2\kappa(\xi(\|x\|+\xi(\|z\|))}{|4|},\frac{\kappa(\xi(\|x\|+\xi(\|z\|))}{|2|}\}$$

for all $x,z\in X$.

Proof. We define $\varphi : X \times X \times X \times X \to [0,\infty)$ by

$$\varphi(x,y,z,w) = \kappa(\xi(\|x\|) + \xi(\|y\|) + \xi(\|z\|) + \xi(\|w\|))$$

then we have

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y, 2^n z, 2^n w)}{|4|^n} \leq \lim_{n\to\infty} (\frac{\xi(|2|)}{|4|})^n \varphi(x,y,z,w) = 0$$

for all $x,y,z,w \in X$. Also

$$\begin{aligned}\tilde{\varphi}(x,z) :&= \lim_{n\to\infty} \max\{\max\{\frac{|2|\varphi(2^j x, 0, 2^j z, 0)}{|4|^j}, \frac{\varphi(2^j x, 2^j x, 2^j z, 2^j z)}{|4|^j}\} : 0 \leq j < n\} \\ &= \max\{|2|\varphi(x,0,z,0), \varphi(x,x,z,z)\}\end{aligned}$$

for all $x,z \in X$. On the other hand

$$\lim_{i\to\infty} \lim_{n\to\infty} \max\{\max\{\frac{|2|\varphi(2^j x, 0, 2^j z, 0)}{|4|^j}, \frac{\varphi(2^j x, 2^j x, 2^j z, 2^j z)}{|4|^j}\} : 0 \leq i \leq j < n+i\} = 0.$$

Applying Theorem 2.2, we conclude the desired result. □

Corollary 2.4. *Let $\zeta : [0,\infty) \longrightarrow [0,\infty)$ be a function satisfying*

$$\xi(|2|t) \leq \xi(|2|)\xi(t) \qquad \xi(|2|) < |4|$$

for all $t \geq 0$. Let $\kappa > 0$, X be a normed space and let $f, g : X \times X \to Y$ are mappings with $f(0,0) = g(0,0) = 0$, such that

$$\|f(x+y, z+w) + f(x-y, z-w) - 2g(x,z) - 2g(y,z)\| \leq \kappa(\xi(\|x\|)\xi(\|y\|)\xi(\|z\|)\xi(\|w\|))$$

for all $x,y,z,w \in Y$. Then there exists a unique 2-dimensional quadratic mapping $\mathcal{Q} : X \times X \longrightarrow Y$ such that

$$\|\mathcal{Q}(x,z) - f(x,z)\| \leq \frac{\kappa(\xi(\|x\|)\xi(\|z\|))^2}{|4|}$$

and

$$\|\mathcal{Q}(x,z) - g(x,z)\| \leq \frac{2\kappa(\xi(\|x\|)\xi(\|z\|))^2}{|4|}$$

for all $x,z \in X$.

Proof. If we define
$\varphi : X \times X \times X \times X \to [0,+\infty)$ by

$$\varphi(x,y,z,w) = \kappa(\xi(\|x\|)\xi(\|y\|)\xi(\|z\|)\xi(\|w\|))$$

and apply Theorem 2.2 then we get the conclusion. □

Theorem 2.5. *Let $\varphi : X \times X \times X \times X \to [0,\infty)$ be a function such that*

$$\lim_{n\to\infty} |4|^n \varphi(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}, \frac{w}{2^n}) = 0 \tag{18}$$

for all $x, y, z, w \in X$ and let for each $x, z \in X$ the limit

$$\text{(19)}\quad \tilde{\varphi}(x,z) := \lim_{n\to\infty} \max\{\max\{|2||4|^{j+1}\varphi(\frac{x}{2^{j+1}},0,\frac{x}{2^{j+1}},0),\ |4|^{j+1}\varphi(\frac{x}{2^{j+}},\frac{y}{2^{j+1}},\frac{z}{2^{j+1}},\frac{w}{2^{j+1}})\}:0\le j<n\}$$

exist. Suppose that $f, g : X \times X \to Y$ are mappings satisfying $f(0,0) = g(0,0) = 0$ and

$$\text{(20)}\quad \|f(x+y,z+w)+f(x-y,z-w)-2g(x,z)-2g(y,w)\| \le \varphi(x,y,z,w)$$

for all $x, y, z, w \in X$. Then there exists a unique 2-dimensional quadratic mapping $\mathcal{Q} : X \times X \to Y$ such that

$$\text{(21)}\quad \|\mathcal{Q}(x,z)-f(x,z)\| \le \frac{1}{|4|}\tilde{\varphi}(x,z)$$

and

$$\text{(22)}\quad \|\mathcal{Q}(x,z)-g(x,z)\| \le \max\{\frac{1}{|4|}\tilde{\varphi}(x,z),\frac{1}{|2|}\varphi(x,0,z,0)\}$$

for all $x, z \in X$. Moreover, if

$$\text{(23)}\quad \lim_{i\to\infty}\lim_{n\to\infty} \max\{\max\{|2||4|^{j+1}\varphi(\frac{x}{2^{j+1}},0,\frac{x}{2^{j+1}},0),\ |4|^{j+1}\varphi(\frac{x}{2^{j+1}},\frac{y}{2^{j+1}},\frac{z}{2^{j+1}},\frac{w}{2^{j+1}})\}:0\le i\le j<n+i\}=0$$

then $\mathcal{Q}$ is the unique 2-dimensional quadratic mapping satisfying (21) and (22).

Proof. Replacing x and z by $\frac{x}{2}$ and $\frac{y}{2}$ respectively in (13) and multiplying both sides in 4 we get

$$\text{(24)}\quad \|f(x,z)-4f(\frac{x}{2},\frac{z}{2})\| \le \max\{|2|\varphi(\frac{x}{2},0,\frac{z}{2},0),\varphi(\frac{x}{2},\frac{y}{2},\frac{z}{2},\frac{z}{2})\}$$

for all $x, y, z, w \in X$. Replacing x and z by $\frac{x}{2^{n-1}}$ and $\frac{z}{2^{n-1}}$ in (24) respectively and multiplying both sides by 4^{n-1}, we get

$$\text{(25)}\quad \|4^n f(\frac{x}{2^n},\frac{z}{2^n})-4^{n-1}f(\frac{x}{2^{n-1}},\frac{z}{2^{n-1}})\| \le \frac{1}{|4|}\max\{|2||4|^n\varphi(\frac{x}{2^n},0,\frac{z}{2^n},0),|4|^n\varphi(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n},\frac{w}{2^n})\}$$

Therefore by (16) and (25) the sequence $\{4^n f(\frac{x}{2^n},\frac{z}{2^n})\}$ is a Cauchy sequence in Y. Since Y is complete, the sequence $\{4^n f(\frac{x}{2^n},\frac{z}{2^n})\}$ is convergent for all $x, z \in X$. Set

$$\mathcal{Q}(x,z) = \lim_{n\to\infty} 4^n f(\frac{x}{2^n},\frac{z}{2^n}).$$

By induction we have

(26)

$$\|4^n f(\frac{x}{2^n},\frac{z}{2^n}) - f(x,z)\| \\ \le \frac{1}{|4|}\max\{\max\{|2||4|^{j+1}\varphi(\frac{x}{2^{j+1}},0,\frac{x}{2^{j+1}},0), \\ |4|^{j+1}\varphi(\frac{x}{2^{j+1}},\frac{x}{2^{j+1}},\frac{x}{2^{j+1}},\frac{x}{2^{j+1}})\} : 0 \le j < n\}$$

for all $x, z \in X$. For $n = 1$ we get (24), obviously. Now, if (26) holds for every $0 \le j < n-1$, we have

(27)

$$\|4^n f(\frac{x}{2^n},\frac{z}{2^n}) - f(x,z)\| \\ \le \max\{\|4^n f(\frac{x}{2^n},\frac{z}{2^n}) - 4^{n-1}f(\frac{x}{2^{n-1}},\frac{z}{2^{n-1}})\|, \|4^{n-1}f(\frac{x}{2^{n-1}},\frac{z}{2^{n-1}}) - f(x,z)\|\} \\ \le \frac{1}{|4|}\max\{\max\{|2||4|^n\varphi(\frac{x}{2^n},0,\frac{z}{2^n},0), |4|^n\varphi(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n},\frac{w}{2^n})\}, \\ \frac{1}{|4|}\max\{\max\{|2||4|^{j+1}\varphi(\frac{x}{2^{j+1}},0,\frac{x}{2^{j+1}},0), \\ |4|^{j+1}\varphi(\frac{x}{2^{j+1}},\frac{y}{2^{j+1}},\frac{z}{2^{j+1}},\frac{w}{2^{j+1}})\} : 0 \le j < n-1\}\} \\ \le \frac{1}{|4|}\max\{\max\{|2||4|^{j+1}\varphi(\frac{x}{2^{j+1}},0,\frac{x}{2^{j+1}},0), \\ |4|^{j+1}\varphi(\frac{x}{2^{j+1}},\frac{y}{2^{j+1}},\frac{z}{2^{j+1}},\frac{w}{2^{j+1}})\} : 0 \le j < n\}\}.$$

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.6. *Let $\zeta : [0,\infty) \longrightarrow [0,\infty)$ be a function satisfying*

$$\xi(\frac{1}{|2|}t) \le \xi(\frac{1}{|2|})\xi(t), \qquad \xi(\frac{1}{|2|}) < \frac{1}{|4|}$$

for all $t \ge 0$. Let $\kappa > 0$, X be a normed space and let $f, g : X \times X \to Y$ are mappings with $f(0,0) = g(0,0) = 0$, such that

$$\|f(x+y, z+w) + f(x-y, z-w) - 2g(x,z) - 2g(y,z)\| \\ \le \kappa(\xi(\|x\|) + \xi(\|y\|) + \xi(\|z\|) + \xi(\|w\|))$$

for all $x, y, z, w \in X$. Then there exists a unique 2-dimensional quadratic mapping $\mathcal{Q} : X \times X \longrightarrow Y$ such that

$$\|\mathcal{Q}(x,z) - f(x,z)\| \le \frac{2\kappa(\xi(\|x\|) + \xi(\|z\|))}{|4|}$$

and

$$\|\mathcal{Q}(x,z) - g(x,z)\| \le \max\{\frac{2\kappa(\xi(\|x\| + \xi(\|z\|)))}{|4|}, \frac{\kappa(\xi(\|x\| + \xi(\|z\|))}{|2|}\}$$

for all $x, z \in X$.

Proof. We define $\varphi : X \times X \times X \times X \to [0, \infty)$ by

$$\varphi(x, y, z, w) = \kappa(\xi(\|x\|) + \xi(\|y\|) + \xi(\|z\|) + \xi(\|w\|))$$

then

$$\lim_{n\to\infty} |4|^n \varphi(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}, \frac{w}{2^n}) \leq \lim_{n\to\infty} (\xi \frac{1}{|2|} |4|)^n \varphi(x, y, z, w) = 0$$

for all $x, y, z, w \in X$. Also

$$\tilde{\varphi}(x, z) := \lim_{n\to\infty} \max\{\max\{|2||4|^{j+1} \varphi(\frac{x}{2^{j+1}}, 0, \frac{x}{2^{j+1}}, 0), \\ |4|^{j+1} \varphi(\frac{x}{2^{j+}}, \frac{y}{2^{j+1}}, \frac{z}{2^{j+1}}, \frac{w}{2^{j+1}})\} : 0 \leq j < n\} \\ = \max\{|6|\varphi(\frac{x}{2}, 0, \frac{z}{2}, 0), |4|\varphi(\frac{x}{2}, \frac{x}{2}, \frac{z}{2}, \frac{z}{2})\}$$

for all $x, z \in X$. On the other hand

$$\lim_{i\to\infty} \lim_{n\to\infty} \max\{\max\{|2||4|^{j+1} \varphi(\frac{x}{2^{j+1}}, 0, \frac{x}{2^{j+1}}, 0), \\ |4|^{j+1} \varphi(\frac{x}{2^{j+}}, \frac{y}{2^{j+1}}, \frac{z}{2^{j+1}}, \frac{w}{2^{j+1}})\} : 0 \leq i \leq j < n + i\} = 0,$$

applying Theorem 2.5, we conclude the desired result. □

3. Conclusion

In the real world, the approximation theory have very applications. In this work, stability for functional equations in non-Archimedean spaces has been studied by using 2-dimensional Pexider quadratic functional equation. We showed that there is an approximate solution for this functional equation in non-Archimedean spaces.

4. Aknowledgement

I would like to thank the reviewers for their thoughtful comments and efforts towards improving our manuscript.

References

[1] M. R. Abdollahpour and M. Th. Rassias, *Hyers-Ulam stability of hypergeometric differential equation*, Aequationes Mathematicae, 93 (4) (2019), 691—698.

[2] M. A. Abolfathi, A. Ebadian and R. Aghalary, *Stability of mixed additive-quadratic Jensen type functional equation in non-Archimedean ℓ-fuzzy normed spaces*, Ann. Univ. Ferrara Sez. VII Sci. Mat., 60 (2) (2014), 307—319.

[3] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan, 2 (1950), 245–251.

[4] J. Aczel and J. Dhombres, *Functional Equations in Several Variables*, Cambridge University Press, Vol.31, 1989.

[5] D. G. Bourgin, *Classes of transformations and bordering transformations*, Bull. Amer. Soc., 57 (1951), 223–237.

[6] N. Brillouet-Belluot, J. Brzdek, and K. Cieplinski, *On some recent developments in Ulam's type stability*, Abstract and Applied Analysis, vol. 2012, Article ID 716936, 41 pages, 2012.

[7] J. Brzdek and K. Cieplinski, *A fixed point theorem and the Hyers-Ulam stability in non-Archimedean spaces*, Journal of Mathematical Analysis and Applications, 400 (1) (2013), 68–75.

[8] P. W. Cholewa, *Remarks on the stability of functional equations*, Aequ. Math., **27** (1984), 76–86.

[9] K. Cieplinski and T. Z. Xu, *Approximate multi-Jensen and multi-quadratic mappings in 2-Banach spaces*, Carpathian Journal of Mathematics, 29 (2013), 159–166.

[10] S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific, 2002.

[10] Z. Gajda, *On the stability of additive mappings*, Internat. J. Math. Math. Sci., 14 (1991), 431–434.

[11] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of the approximately additive mappings*, J. Math. Anal. Appl., 184 (1994), 431–436.

[12] F. Q. Gouvea, *p-Adic Numbers. An Introduction, Universitext*, Springer-Verlag, Berlin, 1997.

[13] K. Hensel, *Über eine neue Begundung der Theorie der algebraischen Zahlen. Jahres*, Deutschen Mathematiker-Vereinigung, 6 (1897), 83–88.

[14] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A., 27 (1941), 222–224.

[15] D. H. Hyers, G. Isac and Th. M. Rssias, *Stability of Functional Equation in Several Variables*, Birkhäuser, Basel, 1998.

[16] G. Isac, Th. M. Rassias, *Stability of ψ-additive mappings: applications to nonlinear analysis*, Internat. J. Math. Math. Sci., 19 (1996), 219–228.

[17] S. M. Jung, *Hyers-Ulam-Rassias Stability of functional Equation in Nonlinear Analysis*, Springer, 2011.

[18] PI Kannappan, *Functional Equations and Inequalitites With Applications*, Springer, 2009.

[19] Y. Lee, S. M. Jung and M. Th. Rassias, *On an n-dimnsional mixed type additive and quadratic functional equation*, Appl. Math. Comput., 228 (2014), 13–16.

[20] W. G. Park and J. H. Bae , *On a bi-quadratic functional equation and its stability*, Nonlinear Anal., 62 (2005), 643–654.

[21] C. Park and M. Th. Rassias, *Additive functional equations and partial multipliers in C*-algebras*, Rev. R. Acad. Cienc. Exactas, Serie A. Matemáticas, 113 (3) (2019), 2261–2275.

[22] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., 72 (1978), 297–300.

[23] Th. M. Rassias, *On the stability of functional equations and a problem of Ulam*, Acta Appl. Math., 62 (1) (2000), 23–130.

[24] Th. M. Rassias, *Functional quations and Inequalities*, Kluwer Academic Publishers, 2000.

[25] A. M. Robert, *A Course in p-Adic Analysis*, Grad. Texts in Math., vol. 198, Springer-Verlag, New York, 2000.

[26] P. K. Sahoo and P. Kannappan, *Introduction to Functional Equations*, CRC Press, 2011.

[27] N. Shilkret, *Non-Archimedian Banach algebras*, Ph. D. thesis, Polytechnic University, 1968.

[28] T. Trif, *On the stability of a functional equation deriving from an inequality of Popoviciu for convex functions*, J. Math. Anal. Appl., 272 (2) (2002), 604–616.

[29] S. M. Ulam, *Problem in Modern Mathematics, Chapter VI, Science Editions*, Wiley, New York, 1964.

[30] T. Z. Xu, *On the stability of multi-Jensen mappings in β-normed spaces*, Applied Mathematics Letters, 25 (11) (2012), 1866–1870.

[31] T. Z. Xu, J. M. Rassias, and Wan Xin Xu, *A fixed point approach to the stability of a general mixed additive-cubic equation on Banach modules*, Acta Mathematica Scientia, 32B (3) (2012), 866–892.

Mohammad Ali Abolfathi
Orcid number:0000-0002-1298-003X
Department of Mathematics
Urmia University
Urmia, Iran
Email address: m.abolfathi@urmia.ac.ir