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REMOTEST POINTS AND APPROXIMATE REMOTES POINTS IN $G-$METRIC SPACE

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Abstract. The aim of this paper is to define the concepts of remotest points and approximate remotest points in $G-$metric spaces and obtain some existence results on these concepts. In particular, we define $G-$remotest points and $G-\epsilon-$approximate remotest points by considering a cyclic map and prove some results in $G-$metric spaces.

Keywords: $G-$remotest points, $G-\epsilon-$approximate remotest points, Cyclic maps, $G-$metric spaces.
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1. Introduction

Recently, Mustafa and Sims [10] introduced a new generalized metric space structure and called it, $G-$metric space, in which every triplet of elements is assigned to a non-negative real number. Physically, this is a measure of mutual distance between three elements taken together. A metric space is a special case of a $G-$metric space. Analysis of the structure of this space was done in some detail in Mustafa and Sims [10]. Several studies relevant to metric spaces have been and are being extended to $G-$metric spaces (see, for instances, [7, 8, 9, 10]).

On the other hand, the problem of characterizing remotest points is an interesting problem, it has its applications in approximation theory and geometry of Banach spaces. We can find some results about remotest points in [1, 2, 3, 4, 5, 6, 11, 12]. Motivated by the growing interest in $G-$metric spaces as the context of studies originally related to ordinary metric spaces, here we formulate the remotest point problem in the context of $G-$metric spaces.

Let (X,G) be a $G-$metric space and A, B and C non-empty bounded subsets of X. If there is a triplet $(x_0,y_0,z_0) \in A \times B \times C$ for which $d(x_0,y_0,z_0) = \delta(A,B,C)$, that $\delta(A,B,C)$ is remotest distance of A, B and C, define by

$$\delta(A,B,C) = \sup\{G(x,y,z) : x \in A, y \in B, z \in C\}.$$

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The triplet (x_0, y_0, z_0) is called a $G-$remotest triplet for A, B and C and put

$$F(A,B,C) = \{(x,y,z) \in A \times B \times C : \; G(x,y,z) = \delta(A,B,C)\}$$

as the set of all $G-$remotest triplet (A,B,C). In this paper, we obtain the $G-$remotest points of the non-empty bounded subsets A, B and C of a $G-$metric space X, by considering a cyclic map $T : A \cup B \cup C \longrightarrow A \cup B \cup C$ i.e. $T(A) \subseteq B$, $T(B) \subseteq C$ and $T(C) \subseteq A$. We also define $G - \epsilon-$approximate remotes points for such maps. In particular, we introduce a concept of $G - \epsilon-$approximate remotest triplet that is stronger than $G-$remotest triplet. Furthermore, we obtain existence and convergence results of $G-$remotest points and $G - \epsilon-$approximate remotest points in a $G-$metric space X. We recall some definitions and preliminaries that are needed for main results as follows.

Definition 1.1. [10] Let X be a non-empty set and $G : X \times X \times X \longrightarrow \mathbb{R}^+$ a function satisfying the following properties:
(G1) $G(x,y,z) = 0$ if and only if $x = y = z$;
(G2) $0 < G(x,x,y) for\ all\ x,y \in X$ with $x \neq y$;
(G3) $G(x,x,y) \leq G(x,y,z)$ for all $x,y,z \in X$with $z \neq y$;
(G4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \cdots$ (symmetry in all three variables);
(G5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$ (rectangle inequality).
Then, the function G is called a $G-$metric on X and the pair (X,G) is called a $G-$metric space.

Definition 1.2. [10] Let (X,G) be a $G-$metric space and $\{x_n\}$ a sequence of points of X. A point $x \in X$ is said to be the limit of the sequence $\{x_n\}$ if

$$lim_{n,m\to\infty} G(x,x_n,x_m) = 0$$

and one says that the sequence $\{x_n\}$ is $G-$ *convergent* to x. Thus, if $x_n \to x$ in a $G-$metric space (X,G), then for any $\epsilon > 0$, there exists a positive integer N such that $G(x,x_n,x_m) < \epsilon$ for all $n,m \geq N$.

Proposition 1.3. *[10] Let (X,G) be a $G-$metric space. Then the followings are equivalent:*
(a) $\{x_n\}$ is $G-$convergent to x;
(b) $G(x_n,x_n,x) \longrightarrow 0$ as $n \to \infty$;
(c) $G(x_n,x,x) \longrightarrow 0$ as $n \to \infty$;
(d) $G(x_n,x_m,x) \longrightarrow 0$ as $n,m \to \infty$.

Proposition 1.4. *[10] Let (X,G) be a $G-$metric space. Then the function $G(x,y,z)$ is jointly continuous in all three of its variables.*

Definition 1.5. [10] Let (X,G) and (X',G') be $G-$metric spaces. A map $f : (X,G) \longrightarrow (X',G')$ is said to be $G-$continuous at a point $a \in X$ if and only if, given $\epsilon > 0$, there exists $\delta > 0$ such that $x, y \in X$ and $G(a, x, y) < \delta$ implies $G^{'}(f(a), f(x), f(y)) < \epsilon$. A map f is $G-$continuous on X if and only if it is $G-$continuous at all $a \in X$.

2. $G-$remotest Points

In this section, we obtain existence and convergence results of the $G-$remotest points by considering the cyclic map $T : A \cup B \cup C \longrightarrow A \cup B \cup C$ on subsets of a $G-$metric space.

Definition 2.1. Let A, B and C be non-empty bounded subsets of a $G-$metric space X and $T : A \cup B \cup C \longrightarrow A \cup B \cup C$ be a cyclic map. The point $x \in A \cup B \cup C$ is a $G-$remotest point of the map T, if $G(x, Tx, T^2x) = \delta(A, B, C)$.

Example 2.2. *Let (X, d) be a metric space. The function $G : X \times X \times X \to \mathbb{R}^+$, defined by $G(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$, for all $x, y, z \in X$, is a $G-$metric on X (see [10]). Suppose $A = [0, 1]$, $B = [2, 3]$ and $C = [4, 5]$ are subsets of a $G-$metric space $X := \mathbb{R}$, define $T : A \cup B \cup C \longrightarrow A \cup B \cup C$, by*

$$T(x) = \begin{cases} 3 & \textit{if } x = 0 \\ x+2 & \textit{if } 0 < x \leq 1 \\ x+2 & \textit{if } 2 \leq x \leq 3 \\ x-4 & \textit{if } 4 \leq x < 5 \\ 0 & \textit{if } x = 5. \end{cases}$$

Clearly, T is cyclic map. Then $\delta(A, B, C) = 5$ and $G(0, 3, 5) = G(0, T0, T^20) = 5$. Hence $x = 0$ is a $G-$remotest point of the map T.

Theorem 2.3. *Let A, B and C be non-empty bounded subsets of a $G-$metric space X. Suppose that the $G-$continuous cyclic mapping $T : A \cup B \cup C \longrightarrow A \cup B \cup C$ satisfying*

$$\begin{aligned} G(Tx, Ty, Tz) &\geq \alpha G(x, y, z) + \beta[G(x, Tx, T^2x) + G(y, Ty, T^2y) \\ &+ G(z, Tz, T^2z)] + \gamma\delta(A, B, C) \end{aligned} \tag{1}$$

for all $x, y, z \in A \cup B \cup C$, where $\alpha, \beta, \gamma > 0$ and $\alpha + 2\beta + \gamma = 1$. For $x_0 \in A$, define $x_{n+1} = Tx_n$ for every $n \in \mathbb{N} \cup \{0\}$. If $\{x_{3n}\}$ has a convergent subsequence in A, then there exists $x \in A$ with $G(x, Tx, T^2x) = \delta(A, B, C)$.

Proof. By inequality (1),

$$\begin{aligned}
G(x_n,x_{n+1},x_{n+2}) &\geq \alpha G(x_{n-1},x_n,x_{n+1})+\beta[G(x_{n-1},x_n,x_{n+1})\\
&+ G(x_n,x_{n+1},x_{n+2})+G(x_{n+1},x_{n+2},x_{n+3})]\\
&+ \gamma\delta(A,B,C)\\
&\geq (\alpha+\beta)G(x_{n-1},x_n,x_{n+1})+\beta[G(x_{n+2},x_n,x_{n+2})\\
&+ G(x_{n+2},x_{n+1},x_{n+2})]\\
&+ \gamma\delta(A,B,C)\\
&\geq (\alpha+\beta)G(x_{n-1},x_n,x_{n+1})+\beta G(x_{n+1},x_n,x_{n+2})\\
&+ \gamma\delta(A,B,C),
\end{aligned}$$

which implies that

$$\begin{aligned}
(1-\beta)G(x_n,x_{n+1},x_{n+2}) &\geq (\alpha+\beta)G(x_{n-1},x_n,x_{n+1})+\gamma\delta(A,B,C)\\
&= (1-\beta)G(x_{n-1},x_n,x_{n+1}).
\end{aligned}$$

Put $G_n=G(x_{n-1},x_n,x_{n+1})$. So $G_{n+1}\geq G_n$. The sequence $\{G_n\}$ is increasing and above bounded, hence G_n is a convergent sequence. Let $\lim_{n\to\infty}G_n=t_0$. Hence

$$(1-\beta)\lim_{n\to\infty}G_{n+1}\geq(\alpha+\beta)\lim_{n\to\infty}G_n+\gamma\delta(A,B,C).$$

So

$$\begin{aligned}
(1-\beta)t_0 &\geq (\alpha+\beta)t_0+\gamma\delta(A,B,C)\\
&= (\alpha+\beta)t_0+(1-\alpha-2\beta)\delta(A,B,C).
\end{aligned}$$

Therefore $t_0\geq\delta(A,B,C)$. Since $G_n\leq\delta(A,B,C)$, hence $t_0\leq\delta(A,B,C)$. So $t_0=\delta(A,B,C)$. Let $\{x_{3n_k}\}$ be a subsequence of $\{x_{3n}\}$ with $x_{3n_k}\to x$ for all $k\geq 1$. Since $\lim_{k\to\infty}G(x_{3n_k},Tx_{3n_k},T^2x_{3n_k})=\delta(A,B,C)$ and T is a $G-$continuous map, we have $G(x,Tx,T^2x)=\delta(A,B,C)$. □

Theorem 2.4. *Let A, B and C be non-empty bounded subsets of a $G-$metric space X. Suppose that the $G-$continuous cyclic mapping $T:A\cup B\cup C\longrightarrow A\cup B\cup C$ satisfying*

$$\begin{aligned}
G(Tx,Ty,Tz) &\geq \alpha\max\{G(x,y,z),(1/2)[G(x,Tx,T^2x)+G(y,Ty,T^2y)\\
&+ G(z,Tz,T^2z)]\}+\gamma\delta(A,B,C)
\end{aligned}$$

for all $x,y,z\in A\cup B\cup C$, where $\gamma>0,\alpha>0$ and $\alpha+\gamma=1$. For $x_0\in A$, define $x_{n+1}=Tx_n$ for for every $n\in\mathbb{N}\cup\{0\}$. If $\{x_{3n}\}$ has a convergent subsequence in A, then there exists a $x\in A$ with $G(x,Tx,T^2x)=\delta(A,B,C)$.

Proof. Assume

$$\begin{aligned}
&\max\{G(x,y,z),(1/2)[G(x,Tx,T^2x)+G(y,Ty,T^2y)+G(z,Tz,T^2z)]\}\\
&\quad=G(x,y,z).
\end{aligned}$$

Then

$$G(Tx,Ty,Tz)\geq\alpha G(x,y,z)+\gamma\delta(A,B,C).$$

Let $G_n = G(x_{n-1}, x_n, x_{n+1})$. So

$$\begin{aligned} G_{n+1} &\geq \alpha G_n + \gamma\delta(A,B,C) \\ &\geq \alpha G_n + (1-\alpha)\delta(A,B,C) \\ &\geq G_n. \end{aligned}$$

The sequence $\{G_n\}$ is increasing and above bounded. Put $\lim_{n\to\infty} G_n = t_0$. Now,

$$\lim_{n\to\infty} G_{n+1} \geq \alpha \lim_{n\to\infty} G_n + \gamma\delta(A,B,C).$$

So

$$\begin{aligned} t_0 &\geq \alpha t_0 + \gamma\delta(A,B,C) \\ &= \alpha t_0 + (1-\alpha)\delta(A,B,C). \end{aligned}$$

Therefore $t_0 \geq \delta(A,B,C)$. Since $G_n \leq \delta(A,B,C)$, hence $t_0 \leq \delta(A,B,C)$. So $t_0 = \delta(A,B,C)$. Now, we assume,

$$\begin{aligned} &\max\{G(x,y,z), (1/2)[G(x,Tx,T^2x) + G(y,Ty,T^2y) + G(z,Tz,T^2z)]\} \\ &\quad = (1/2)[G(x,Tx,T^2x) + G(y,Ty,T^2y) + G(z,Tz,T^2z)]. \end{aligned}$$

Then

$$\begin{aligned} G(Tx,Ty,Tz) &\geq (\alpha/2)[G(x,Tx,T^2x) + G(y,Ty,T^2y) \\ &+ G(z,Tz,T^2z)] + \gamma\delta(A,B,C). \end{aligned}$$

Which implies that

$$\begin{aligned} G(x_n,x_{n+1},x_{n+2}) &\geq (\alpha/2)[G(x_{n-1},x_n,x_{n+1}) + G(x_n,x_{n+1},x_{n+2}) \\ &+ G(x_{n+1},x_{n+2},x_{n+3})] + \gamma\delta(A,B,C) \\ &\geq (\alpha/2)[G(x_{n-1},x_n,x_{n+1}) + G(x_n,x_{n+2},x_{n+2}) \\ &+ G(x_{n+1},x_{n+2},x_{n+2})] + \gamma\delta(A,B,C) \\ &\geq (\alpha/2)[G(x_{n-1},x_n,x_{n+1}) + G(x_n,x_{n+1},x_{n+2})] \\ &+ \gamma\delta(A,B,C). \end{aligned}$$

Hence

$$\begin{aligned} (1-(\alpha/2))G(x_n,x_{n+1},x_{n+2}) &\geq (\alpha/2)G(x_{n-1},x_n,x_{n+1}) + \gamma\delta(A,B,C) \\ &= (\alpha/2)G(x_{n-1},x_n,x_{n+1}) + (1-\alpha)\delta(A,B,C) \\ &= (1-(\alpha/2))G(x_{n-1},x_n,x_{n+1}). \end{aligned}$$

Put $G_n = G(x_{n-1}, x_n, x_{n+1})$. So $G_{n+1} \geq G_n$. The sequence $\{G_n\}$ is increasing and above bounded, hence G_n is a convergent sequence. Let $\lim_{n\to\infty} G_n = t_0$. Hence

$$(1-(\alpha/2)) \lim_{n\to\infty} G_{n+1} \geq (\alpha/2) \lim_{n\to\infty} G_n + \gamma\delta(A,B,C).$$

So

$$\begin{aligned} (1-(\alpha/2))t_0 &\geq (\alpha/2)t_0 + \gamma\delta(A,B,C) \\ &= (\alpha/2)t_0 + (1-\alpha)\delta(A,B,C). \end{aligned}$$

Therefore $t_0 \geq \delta(A,B,C)$. Since $G_n \leq \delta(A,B,C)$, hence $t_0 \leq \delta(A,B,C)$. So $t_0 = \delta(A,B,C)$.
Let $\{x_{3n_k}\}$ be a subsequence of $\{x_{3n}\}$ with $x_{3n_k} \to x$ for all $k \geq 1$. Since $\lim_{k\to\infty} G(x_{3n_k}, Tx_{3n_k}, T^2x_{3n_k}) = \delta(A,B,C)$ and T is a $G-$continuous map, we have $G(x,Tx,T^2x) = \delta(A,B,C)$. □

Corollary 2.5. *Let A, B and C be non-empty bounded subsets of a $G-$metric space X. Suppose that the $G-$continuous cyclic mapping $T : A \cup B \cup C \longrightarrow A \cup B \cup C$ satisfying*

$$\begin{aligned} G(Tx,Ty,Tz) &\geq \alpha \max\{G(x,y,z), (1/2)[G(x,Tx,T^2x) + G(y,Ty,T^2y) \\ &+ G(z,Tz,T^2z)], (1/3)[G(x,y,z) + G(x,Tx,T^2x) \\ &+ G(y,Ty,T^2y) + G(z,Tz,T^2z)]\} + \gamma\delta(A,B,C) \end{aligned}$$

for all $x,y,z \in A\cup B\cup C$, where $\gamma > 0, \alpha > 0$ and $\alpha+\gamma = 1$. For $x_0 \in A$, define $x_{n+1} = Tx_n$ for for every $n \in \mathbb{N} \cup \{0\}$. If $\{x_{3n}\}$ has a convergent subsequence in A, then there exists a $x \in A$ with $G(x,Tx,T^2x) = \delta(A,B,C)$.

Regarding the analogy, we omit the above corollary proof .

3. $G - \epsilon-$approximate remotest point

In this section we prove the existence and convergence results of $G-$approximate remotest points for the cyclic map $T : A\cup B\cup C \longrightarrow A\cup B\cup C$ in a $G-$metric space.

Definition 3.1. Let A, B and C be non-empty bounded subsets of a $G-$metric space X, $\epsilon > 0$ and $T : A \cup B \cup C \longrightarrow A \cup B \cup C$ be a cyclic map. The point $x \in A \cup B \cup C$ is a $G - \epsilon-$approximate remotest point of the map T, if $G(x,Tx,T^2x) \geq \delta(A,B,C) - \epsilon$. Put

$$F^{\epsilon}_T(A,B,C) = \{x \in A \cup B \cup C : G(x,Tx,T^2x) \geq \delta(A,B,C) - \epsilon\}$$

as the set of all $G - \epsilon-$approximate remotest triplet (A,B,C),

Example 3.2. *Suppose $A = \{(x,y) : (x-2)^2 + (y-2)^2 \leq 1\}$, $B = \{(x,y) : (x+2)^2 + (y-2)^2 \leq 1\}$ and $C = \{(x,y) : (x+2)^2 + (y+2)^2 \leq 1\}$ are subsets of a $G-$metric space X, where $G : X \times X \times X \to \mathbb{R}^+$, defined by $G(x,y,z) = \max\{d(x,y), d(y,z), d(z,x)\}$, for all $x,y,z \in X$, such that (X,d) is a metric space. We define $T : A \cup B \cup C \longrightarrow A \cup B \cup C$, by*

$$T(x,y) = \begin{cases} (-x,y) & \text{if } (x,y) \in A \\ (x,-y) & \text{if } (x,y) \in B \\ (-x,-y) & \text{if } (x,y) \in C \end{cases}$$

Clearly, T is cyclic map. Then $\delta(A,B,C) = 2 + 4\sqrt{2}$. Let $\epsilon = 0.5$,

$$G\Big(\Big(\frac{3.9+\sqrt{2}}{2}, \frac{3.9+\sqrt{2}}{2}\Big), \Big(\frac{-3.9-\sqrt{2}}{2}, \frac{3.9+\sqrt{2}}{2}\Big), \Big(\frac{-3.9-\sqrt{2}}{2}, \frac{-3.9-\sqrt{2}}{2}\Big)\Big)$$

$$= G\Big(\Big(\frac{3.9+\sqrt{2}}{2}, \frac{3.9+\sqrt{2}}{2}\Big), T\Big(\frac{3.9+\sqrt{2}}{2}, \frac{3.9+\sqrt{2}}{2}\Big), T^2\Big(\frac{3.9+\sqrt{2}}{2}, \frac{3.9+\sqrt{2}}{2}\Big)\Big)$$

$$= 2 + 3.9\sqrt{2} \geq 2 + 4\sqrt{2} - 0.5 = \delta(A, B, C) - 0.5,$$

Hence $\left(\frac{3.9+\sqrt{2}}{2}, \frac{3.9+\sqrt{2}}{2}\right)$ *is a* $G - 0.5-$*approximate remotest point of the map* T.

Theorem 3.3. *Let* A, B *and* C *be non-empty bounded subsets of a* $G-$*metric space* X. *Suppose* $T : A \cup B \cup C \longrightarrow A \cup B \cup C$ *be a cyclic map and*

$$\lim_{n\to\infty} G(T^n x, T^{n+1}x, T^{n+2}x) = \delta(A, B, C) \text{ for some } x \in A \cup B \cup C.$$

Then the triplet (A, B, C) *is a* $G - \epsilon-$*approximate remotest triplet.*

Proof. Let $\epsilon > 0$ be given and $x \in A \cup B \cup C$ such that

$$\lim_{n\to\infty} G(T^n x, T^{n+1}x, T^{n+2}x) = \delta(A, B, C),$$

then there exists $N_0 > 0$ such that

$$\forall n \geq N_0 : G(T^n x, T^{n+1}x, T^{n+2}x) > \delta(A, B, C) - \epsilon.$$

If $n = N_0$, then $G(T^{N_0}(x), T(T^{N_0}x), T^2(T^{N_0}x)) > \delta(A, B, C) - \epsilon$, then $T^{N_0}(x) \in F_T^{\epsilon}(A, B, C)$. So (A, B, C) is a $G - \epsilon-$approximate remotest triplet. □

Theorem 3.4. *Let* A, B *and* C *be non-empty bounded subsets of a* $G-$*metric space* X. *Suppose* $T : A \cup B \cup C \longrightarrow A \cup B \cup C$ *be a cyclic map and satisfying* (1)*, for all* $x, y, z \in A \cup B \cup C$. *Then the triplet* (A, B, C) *is a* $G - \epsilon-$*approximate remotest triplet.*

Proof. Let $G_n = G(T^{n-1}x, T^n x, T^{n+1}x)$. By Theorem 2.3, the sequence G_n is a convergent sequence. Let $\lim_{n\to\infty} G_n = t_0$. So

$$(1 - \beta) \lim_{n\to\infty} G_{n+1} \geq (\alpha + \beta) \lim_{n\to\infty} G_n + \gamma\delta(A, B, C).$$

So

$$\begin{aligned} (1 - \beta)t_0 &\geq (\alpha + \beta)t_0 + \gamma\delta(A, B, C) \\ &= (\alpha + \beta)t_0 + (1 - \alpha - 2\beta)\delta(A, B, C). \end{aligned}$$

Therefore $t_0 \geq \delta(A, B, C)$. Since $G_n \leq \delta(A, B, C)$, hence $t_0 \leq \delta(A, B, C)$. So $t_0 = \delta(A, B, C)$. By Theorem 3.3, (A, B, C) is a $G - \epsilon-$approximate remotest triplet. □

Definition 3.5. Let A, B and C be non-empty bounded subsets of a $G-$metric space X. Suppose $T : A \cup B \cup C \longrightarrow A \cup B \cup C$ be a cyclic map. We say that the sequence $\{z_n\} \subseteq A \cup B \cup C$ is $T - G-$maximizing if

$$\lim_{n\to\infty} G(z_n, Tz_n, T^2 z_n) = \delta(A, B, C).$$

Theorem 3.6. *Let* A, B *and* C *be non-empty bounded subsets of a* $G-$*metric space* X. *Suppose* $T : A \cup B \cup C \longrightarrow A \cup B \cup C$ *be a cyclic map. If* $\{T^n x\}$ *is a* $T - G-$*maximizing, then* (A, B, C) *is a* $G - \epsilon-$*approximate remotest triplet.*

Proof. Since

$$\lim_{n\to\infty} G(T^n x, T^{n+1}x, T^{n+2}x) = \delta(A,B,C) \text{ for some } x \in A\cup B\cup C.$$

By Theorem 3.3, the triplet (A,B,C) is a $G-\epsilon-$approximate remotest triplet. □

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