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A LIKELIHOOD CONTROL CHART FOR MONITORING BIVARIATE LIFETIME PROCESSES

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Abstract. In this survey, two new control charts CCLR and CCALR for bivariate exponential variables by dependence structure based on Farlie-Gumbel-Morgenstern copula model are introduced. Simulation study is done to make a comparison between two proposed control charts in terms of average run length (ARL). Results show that the CCALR performs better than CCLR. A numerical example is provided to fortify the theoretical findings.

Keywords: Control chart, Bivariate exponential distribution, Farlie-Gumbel-Morgenstern copula, Likelihood ratio test, Average run length.
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1. Introduction

Statistical process control (SPC) chart techniques can be classified in two groups: multivariate and univariate. Multivariate control charts are used when two or more related quality characteristics need to be monitored, such as the inner and outer diameters of roller bearing [6] .

In the literature, there have been many studies on the multivariate control charts, which proposed approaches based on parametric or non-parametric mehods. But, rare researches could be found employing the joint distribution of the related variables. The control chart proposed in this paper is based on the copula modelling, which is a very usefull tool for multivariate modelling.

Hotelling T^2 control chart is the most used rule in industry for the multivariate fault detection. This rule relies on the assumption that the observations under control are normal. When this method is applied on non-normal multivariate observations, it can lead to a lot of false alarms and non-detections.

Individual or seperate control of related variables will result in errors of "over"and "under"control. These errors become more pronounced if the variables are correlated. In acturial science, when two lives are subject to failure, such as under a joint life insurance or annuity policity, it is concerned with joint distribution of lifetimes. In the present paper, it is assumed that the lifetime of the products counts on two related characterisctics in which their dependence

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structure is according to Farlie-Gumbel-Morgenstern (FGM) copula model and marginal distributions are exponential.

For these prosesses, in the case of the simulataneous use of two seperate univariate control chars, correlation between two characteristics is ignored and so, the type 1 error will be increased. Therefore, control chart based on the likelihood ratio test is developed that employs the correlation structure.

In the literature, there have been researches on the control charts concerned to the likelihood ratio test statistic. Apley and Shi [2] presented an on-line statistical process control (SPC) technique, based on a generalized likelihood ratio (GLR) test, for detecting and estimating mean shifts in autocorrelated processes that follow a normally distributed autoregressive integrated moving average (ARIMA) model. Cappizi and Masarotto [4] introduced a practical approach to implementate GLR charts for monitoring an autoregressive moving average process assuming that only a phase I sample is available. Their proposed approach, based on automatic time series identifications, estimates the GLR control limits through stochastic approximation using bootstrap resampling and thus is able to take into account the uncertainty about the underlying model.

Zhang et al. [14] proposed a control chart based on the likelihood ratio for monitoring the linear profiles, that integrates the exponentially weighted moving average (EWMA) procedure to detect shifts in either the intercept or the slope or the integratede standard deviation, or simultaneously by a single chart. Zhang et al. [15] introduced a control chart that integrate the EWMA procedure with the GLR test statistic for jointly monitoring both the process mean and variance. Zhou et al. [16] presnted a control chart which integrates the EWMA procedure with the GLR test statistic to minitor the process with patterned mean and variance shifts, which has reference-free proporty.

Xu et al. [13] considered the problem of monitoring a normally distributed process variable when a special cause may produce a time-varying linear drift in the mean and designed a GLR control chart for evaluating drift detection. Xu et al. [12] developed a GLR control chart for detecting sustained changes in the parameters of linear profiles when individual observations are sampled. There have been other reseaches in this suject that for more information one can see Zhang et al. [14,15]; Zhou et al. [17]; Qi et al. [7,8]; Wu et al [11]. But there has been little attempt to study the control charts aggregating likelihood ratio test statistic in lifetimes.

The structure of the rest of this paper is as follows. In the subsequent section, some basic definitions of FGM copula model as well as the bivariate exponential distribution are presented. Section 3 provides maximum likelihood estimations of two parameters of the mentioned distribution. Two new control charts are introduced in Section 4. In Section 5, simulation study is carried out to investigate the performance of the proposed control charts in terms of the ARL. Section 6 discusses an illustrative example to show the use of the proposed control charts. Finally, some conclusions are presented in Section 7.

2. Copula

Copulas are used to combine marginal distributions to create bivariate/multivariate distributions. They contain information from the joint distribution that is not contained in the marginal distributions. The concept of copula was introduced by Sklar [9], and has for a long time been recognized as a powerful tool for modelling dependence between random variables. Some basic information in this subject are presented in [1]

The joint cumulative distribution function (cdf) of two random variables X_1 and X_2 based of FGM copula model is as following;

$$F_{X_1,X_2}(x_1,x_2) = F_{X_1}(x_1)F_{X_2}(x_2)\big[1+\theta(1-F_{X_1}(x_1))(1-F_{X_2}(x_2))\big], \tag{1}$$

and the joint probability density function (pdf) is as

$$f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1)f_{X_2}(x_2)\big[1+\theta(2f_{X_1}(x_1)-1)(2f_{X_2}(x_2)-1)\big]. \tag{2}$$

The scalar θ is dependence parameter, ranges from -1 to 1. It is noted that the independence structure is reached when $\theta = 0$.

For FGM copula family, the relation between Kendall's tau and the dependence parameter θ is $\tau_{X,Y} = 2\theta/9$. Accordingly, for the processes with unknown dependence parameter, first the Kendall's tau for the sample is estimated as $\widehat{\tau}_{X,Y} = \tau$ and then $\widehat{\theta} = 9\tau/2$. More explanation of this subject is presented in [1].

Let X_1 be the lifetime of first characteristic and X_2 is the lifetime of another one. These two variables are distributed as exponential, in which $X_1 \sim E(\lambda_1)$ and $X_2 \sim E(\lambda_2)$. Then, by using Eq. (1), we have

$$F_{X_1,X_2}(x_1,x_2) = (1-e^{-\frac{x_1}{\lambda_1}})(1-e^{-\frac{x_1}{\lambda_1}})\big[1+\theta e^{-\frac{x_1}{\lambda_1}-\frac{x_2}{\lambda_2}}\big]. \tag{3}$$

3. Maximum likelihood estimations of parameters λ_1 and λ_2

Now, we want to find the maximum likelihood estimations (MLEs) of two parameters λ_1 and λ_2.

Since X_1 and X_2 follow exponential distribution, the joint pdf is as follows;

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{\lambda_1\lambda_2}e^{-x_1/\lambda_1-x_2/\lambda_2}\big[1+\theta(2e^{-x_1/\lambda_1}-1)(2e^{-x_2/\lambda_2}-1)\big]. \tag{4}$$

Consequently, the likelihood function is given by

$$L(\lambda_1,\lambda_2) = \frac{1}{\lambda_1\lambda_2}e^{-x_1/\lambda_1-x_2/\lambda_2}\big[1+\theta(2e^{-x_1/\lambda_1}-1)(2e^{-x_2/\lambda_2}-1)\big]. \tag{5}$$

Then, the log-likelihood fuction can be obtained as what follows;

$$l(\lambda_1,\lambda_2) = -\ln\lambda_1-\ln\lambda_2-\frac{x_1}{\lambda_1}-\frac{x_2}{\lambda_2}+\ln\big[1+\theta(2e^{-x_1/\lambda_1}-1)(2e^{-x_2/\lambda_2}-1)\big]. \tag{6}$$

Based on a random sample of size n, as (x_{11},x_{21}), (x_{12},x_{22}),..., (x_{1n},x_{2n}), the sample joint pdf is as

$$f_{(\mathbf{X}_1,\mathbf{X}_2)}(\mathbf{x}_1,\mathbf{x}_2) = \frac{1}{\lambda_1^n\lambda_2^n} e^{-\sum_{i=1}^n x_{1i}/\lambda_1 - \sum_{i=1}^n x_{2i}/\lambda_2} \prod_{i=1}^n \Big[1+\theta(2e^{-x_{1i}/\lambda_1}-1) \times (2e^{-x_{2i}/\lambda_2}-1)\Big]. \tag{7}$$

It is noted that $(\mathbf{X}_1, \mathbf{X}_2) = \big((X_{11}, X_{21}), (X_{12}, X_{22}), ..., (X_{1n}, X_{2n})\big)$, and similarly, $(\mathbf{x}_1, \mathbf{x}_2) = \big((x_{11}, x_{21}), (x_{12}, x_{22}), ..., (x_{1n}, x_{2n})\big)$. Then, the MLEs of λ_1 and λ_2, noted by $\widehat{\lambda}_1$ and $\widehat{\lambda}_2$, are obtained by solving the following system of nonlinear equations;

$$\begin{aligned} \sum_{i=1}^n \frac{x_{1i}}{\lambda_1^2} - \frac{n}{\lambda_1} + 2\theta \sum_{i=1}^n \frac{x_{1i}e^{-x_{1i}/\lambda_1}(2e^{-x_{2i}/\lambda_2}-1)}{\lambda_1^2\big[1+\theta(2e^{-x_{1i}/\lambda_1}-1)(2e^{-x_{2i}/\lambda_2}-1)\big]} &= 0, \\ \sum_{i=1}^n \frac{x_{2i}}{\lambda_2^2} - \frac{n}{\lambda_2} + 2\theta \sum_{i=1}^n \frac{x_{2i}e^{-x_{2i}/\lambda_2}(2e^{-x_{1i}/\lambda_1}-1)}{\lambda_2^2\big[1+\theta(2e^{-x_{1i}/\lambda_1}-1)(2e^{-x_{2i}/\lambda_2}-1)\big]} &= 0. \end{aligned} \tag{8}$$

In this paper, Newton's iterative method is used to solve the above system of nonlinear equations and the start point for λ_1 and λ_2 are $\sum_{i=1}^n x_{1i}/n$ and $\sum_{i=1}^n x_{2i}/n$, respectively.

It is noted that for the processes with unknown parameter θ, first it should be estimated.

4. **Control Charts**

A product is considered to be conforming if the lifetime of its first characteristic exceeds L_1 and of the other one exceeds L_2, that is, $X_1 > L_1$ and $X_2 > L_2$, so the following hypotheses on the parameters are applied;

$$\begin{cases} H_0 : \lambda_1 > l_1 \ \ \wedge \ \ \lambda_2 > l_2, \\ H_1 : \lambda_1 \le l_1 \ \vee \ \lambda_2 \le l_2. \end{cases} \tag{9}$$

In fact, l_1 and l_2 are the out-of-control detectable values that the control chart is expected to give an alarm.

Set $\lambda_1^0 = l_1 + h_1$ and $\lambda_2^0 = l_2 + h_2$, which $h_1, h_2 \longrightarrow 0$. Then, the above hypotheses are equivalent to the following one;

$$\begin{cases} H_0 : \lambda_1 \ge \lambda_1^0 \ \ \wedge \ \ \lambda_2 \ge \lambda_2^0, \\ H_1 : \lambda_1 < \lambda_1^0 \ \vee \ \lambda_1 < \lambda_2^0. \end{cases} \tag{10}$$

Under the null hypothesis, the parameters of the lifetime variables are at least λ_1^0 and λ_2^0, respectively, and under the alternative one, at least for one of the parameters λ_1 and λ_2, the above situation does not hold. Hence, the likelihood ratio statistic is as what follows;

$$\lambda(\mathbf{X}_1, \mathbf{X}_2) = \frac{l(\lambda_1^0, \lambda_2^0)}{\max\{l(\lambda_1^0, \lambda_2^0), l(\widehat{\lambda}_1, \widehat{\lambda}_2)\}}. \tag{11}$$

$l(\lambda_1^0, \lambda_2^0)$ is the likelihood function under the null hypothesis, and $l(\widehat{\lambda}_1, \widehat{\lambda}_2)$ is the likelihood function with respect to λ_1 and λ_2.

The goal of this paper is to detect whether or not a new manufacturing product has the lifetime generated from the discussed distribution with parameters under the null hypothesis. Here, two control charts are introduced.

4.1. **Control chart based on likelihood ratio statistic(CCLR).** Consider a sample of size n as $(\mathbf{X}_1, \mathbf{X}_2)$ and the problem of testing the null hypothesis $H_0 : \lambda_1 \geq \lambda_1^0 \ \wedge \ \lambda_2 \geq \lambda_2^0$ versus the alternative hypothesis $H_1 : \lambda_1 < \lambda_1^0 \vee \lambda_1 < \lambda_2^0$. The likelihood functions are as what follows;

$$l(\lambda_1^0, \lambda_2^0) = (\frac{1}{\lambda_1^0 \lambda_2^0})^n e^{-\sum_{i=1}^n x_{1i}/\lambda_1^0 - \sum_{i=1}^n x_{2i}/\lambda_2^0} \prod_{i=1}^n \Big[1 + \theta(2e^{-x_{1i}/\lambda_1^0} - 1) \times (2e^{-x_{2i}/\lambda_2^0} - 1)\Big], \tag{12}$$

and

$$l(\widehat{\lambda}_1, \widehat{\lambda}_2) = (\frac{1}{\widehat{\lambda}_1 \widehat{\lambda}_2})^n e^{-\sum_{i=1}^n x_{1i}/\widehat{\lambda}_1 - \sum_{i=1}^n x_{2i}/\widehat{\lambda}_2} \prod_{i=1}^n \Big[1 + \theta(2e^{-x_{1i}/\widehat{\lambda}_1} - 1) \times (2e^{-x_{2i}/\widehat{\lambda}_2} - 1)\Big], \tag{13}$$

where $\widehat{\lambda}_1$ and $\widehat{\lambda}_2$ are MLEs of λ_1 and λ_2, respectively.

This control chart has only LCL, which is constructed such as the probability of false alarm (to consider an observation drawn from the in-control process as an out-of-control) is equal to a special level α. That is,

$$P\big(\lambda(\mathbf{X}_1, \mathbf{X}_2) < k_\alpha\big) = \alpha, \tag{14}$$

and so, $LCL = k_\alpha$. It is noted that k_α is a constant which is less than 1 $(k_\alpha < 1)$.

Calculation of the exact distribution function of $\lambda(\mathbf{X}_1, \mathbf{X}_2)$ is so complicated that k_α is estimated by an empirical quantile computed from a large number of simulated samples, that is, $\widehat{k}_\alpha = k_{\alpha,n}$ is the α-quantile of $\lambda_n(\mathbf{X}_1, \mathbf{X}_2)$ in simulated samples of size n.

Suppose N samples of size n from the bivariate exponential distribution by dependence structure based on FGM copula model are generated and the likelihood ratio statistic for each sample is gain as $\lambda_n(\mathbf{x}_{1i}, \mathbf{x}_{2i}, N)$; $\quad i = 1, 2, ..., N$. The new sample is noted as $\lambda_n(\mathbf{x}_{11}, \mathbf{x}_{21}, N)$, $\lambda_n(\mathbf{x}_{12}, \mathbf{x}_{22}, N)$, ..., $\lambda_n(\mathbf{x}_{1N}, \mathbf{x}_{2N}, N)$. Then the empirical distribution function is as what follows;

$$\widehat{F}_{\lambda(\mathbf{x}_1,\mathbf{x}_2)}(t) = F_{\lambda_n(\mathbf{x}_1,\mathbf{x}_2)}(t) = \frac{1}{N}\sum_{i=1}^{N} I\big(\lambda_n(\mathbf{x}_{1i},\mathbf{x}_{2i},N) \le t\big), \tag{15}$$

where

$$I\big(\lambda_n(\mathbf{x}_{1i},\mathbf{x}_{2i},N)) \le t\big) = \begin{cases} 1 & \lambda_n(\mathbf{x}_{1i},\mathbf{x}_{2i},N)) \le t, \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

is the indicator function.

Therefore, $k_{\alpha,n}$ is obtained from the following equation;

$$k_{\alpha,n} = \inf\{t \in \mathbb{R}; \quad \widehat{F}_{\lambda(\mathbf{x}_1,\mathbf{x}_2)}(t) \ge \alpha\}. \tag{17}$$

It should be noted that $k_{\alpha,n} < 1$. The simplicity of this chart is that unlike some other control charts, it is not need to treat trail lower control limit.

For each sample or subgroup, based on Eqs. (12) and (13), the value of likelihood statistic $\lambda(\mathbf{x}_1,\mathbf{x}_2)$ (Eq. (11)) is plotted on the chart. The process is declared as out-of-control (the null hypothesis in Eq. (10) will be rejected) if and only if $\lambda(\mathbf{x}_1,\mathbf{x}_2) < k_{\alpha,n}$.

Tables 1, 2, and 3 show values of $k_{\alpha,n}$ based on the simulation scheme, for various values of the parameters λ_1 and λ_2. Throughout this simlation study, all $k_{\alpha,n}$ values are obtained from 10000 replications, using programs written in Mathematica software. More extensive tables of $k_{\alpha,n}$ for some other values of the parameters λ_1 and λ_2 are available from the authors on request.

TABLE 1. $k_{\alpha,n}$ values of CCLR for $\alpha = 0.0027$, $n = 5$, $\theta = -0.6$ and various values of λ_1 and λ_2.

λ_2	λ_1									
	1	2	3	4	5	6	7	8	9	10
1	0.00279	0.00227	0.00142	0.00189	0.00191	0.00311	0.00198	0.00296	0.00213	0.00225
2	0.00178	0.00153	0.00196	0.00211	0.00278	0.00271	0.00238	0.00205	0.00211	0.00282
3	0.00201	0.00259	0.00224	0.00231	0.00236	0.00181	0.00175	0.00254	0.00252	0.00240
4	0.00225	0.00291	0.00195	0.00248	0.00184	0.00203	0.00300	0.00307	0.00271	0.00281
5	0.00348	0.00243	0.00229	0.00235	0.00171	0.00243	0.00253	0.00244	0.00232	0.00199
6	0.00242	0.00239	0.00281	0.00255	0.00211	0.00224	0.00277	0.00219	0.00241	0.00194
7	0.00146	0.00191	0.00172	0.00204	0.00227	0.00233	0.00273	0.00217	0.00319	0.00196
8	0.00282	0.00245	0.00271	0.00319	0.00180	0.00157	0.00231	0.00184	0.00169	0.00173
9	0.00172	0.00221	0.00215	0.00246	0.00260	0.00200	0.00308	0.00351	0.00309	0.00194
10	0.00186	0.00288	0.00245	0.00225	0.00274	0.00212	0.00229	0.00223	0.00217	0.00253

TABLE 2. $k_{\alpha,n}$ values of CCLR for $\alpha = 0.0027$, $n = 10$, $\theta = 0.9$ and various values of λ_1 and λ_2.

λ_2	λ_1									
	1	2	3	4	5	6	7	8	9	10
1	0.00162	0.00294	0.00248	0.00213	0.00225	0.00282	0.00223	0.00274	0.00250	0.00254
2	0.00278	0.00171	0.00301	0.00278	0.00208	0.00228	0.00179	0.00298	0.00222	0.00185
3	0.00215	0.00253	0.00183	0.00353	0.00258	0.00240	0.00210	0.00239	0.00181	0.00238
4	0.00302	0.00305	0.00200	0.00258	0.00231	0.00167	0.00298	0.00158	0.00193	0.00225
5	0.00235	0.00262	0.00223	0.00203	0.00250	0.00223	0.00353	0.00232	0.00234	0.00218
6	0.00220	0.00218	0.00256	0.00289	0.00285	0.00168	0.00208	0.00293	0.00244	0.00227
7	0.00291	0.00201	0.00205	0.00266	0.00215	0.00270	0.00206	0.00233	0.00182	0.00243
8	0.00299	0.00220	0.00256	0.00261	0.00173	0.00302	0.00228	0.00245	0.00214	0.00229
9	0.00265	0.00180	0.00216	0.00279	0.00189	0.00222	0.00183	0.00168	0.00292	0.00282
10	0.00309	0.00238	0.00233	0.00212	0.00251	0.00278	0.00205	0.00254	0.00258	0.00239

TABLE 3. $k_{\alpha,n}$ values of CCLR for $\alpha = 0.0027$, $n = 50$, $\theta = 0.3$ and various values of λ_1 and λ_2.

λ_2	λ_1									
	1	2	3	4	5	6	7	8	9	10
1	0.00343	0.00302	0.00296	0.00276	0.00293	0.00257	0.00317	0.00250	0.00252	0.00177
2	0.00220	0.00248	0.00232	0.00227	0.00222	0.00310	0.00271	0.00257	0.00248	0.00347
3	0.00271	0.00299	0.00315	0.00387	0.00255	0.00264	0.00234	0.00184	0.00278	0.00241
4	0.00236	0.00247	0.00330	0.00382	0.00263	0.00254	0.00235	0.00368	0.00211	0.00431
5	0.00295	0.00303	0.00304	0.00314	0.00188	0.00302	0.00224	0.00224	0.00230	0.00275
6	0.00311	0.00331	0.00211	0.00215	0.00272	0.00257	0.00195	0.00263	0.00251	0.00377
7	0.00165	0.00242	0.00294	0.00278	0.00264	0.00188	0.00221	0.00166	0.00256	0.00303
8	0.00288	0.00257	0.00347	0.00222	0.00257	0.00270	0.00189	0.00276	0.00267	0.00330
9	0.00204	0.00206	0.00256	0.00349	0.00232	0.00229	0.00182	0.00353	0.00215	0.00183
10	0.00312	0.00324	0.00220	0.00337	0.00261	0.00191	0.00203	0.00273	0.00317	0.00180

4.2. **Control chart based on asymptotic distribution of likelihood ratio statistic(CCALR).** For each sample, the likelihood statistic $\lambda(\mathbf{x}_1, \mathbf{x}_2)$ is calculated, as what explained in subsection 4.1. Define

$$S = -2\log\lambda(\mathbf{x}_1, \mathbf{x}_2). \tag{18}$$

S is the statistic put on the control chart. If $S > \chi^2_{r,\alpha}$, then a signal is observed and the process is declared to be out-of-control. Here, it is supposed to $\alpha = 0.0027$.

This control chart has only the upper control limit, $UCL = \chi^2_{r,0.0027}$. Since this paper is working on the bivariate cases, $UCL = 11.829$. If S falls above the upper control limit, then the system is declared out-of-control.

It is trivial that this control limit is fixed and unlike to CCLR, CCALR does not have trial upper control limit.

4.3. **Convergence of CCLR lower control limit in probability.** This subsection deals with the idea of allowing the sample size to approach infinity and shows that $k_{\alpha,n}$ converges in probability to k_α. First, a definition is presented.

Definition 4.1. Suppose $F, G : \mathbb{R} \to [0,1]$. Levy metric (Levy distance) between them is as the following [10];

$$L(F,G) = \inf\big\{\epsilon > 0 \mid F(x-\epsilon) - \epsilon \leq G(x) \leq F(x+\epsilon) + \epsilon; \quad \forall x \in \mathbb{R}\big\}. \tag{19}$$

Let $||\ .\ ||$ be the usual Euclidean norm and $||\ .\ ||_\infty$ be the uniform norm as what follows;

$$|| \lambda ||_\infty = \sup_{(x_1,x_2)\in\mathbb{R}^2} \mid \lambda(x_1,x_2) \mid, \tag{20}$$

and $\mathbb{H}_0$ is the following set;

$$\mathbb{H}_0 = \Big\{ c \in \big(0, \sup_{(x_1,x_2)\in\mathbb{R}^2} \lambda(x_1,x_2)\big); \quad \inf_{\lambda=c} || \nabla\lambda || = 0 \Big\}. \tag{21}$$

The Glivenko-Cantelli theorem implies a strong convergence results on the empirical distribution as the following;

$$|| F_n - F ||_\infty = \sup_{x\in\mathbb{R}} \mid F_n(x) - F(x) \mid \longrightarrow^{a.s} 0. \tag{22}$$

Following is some assumptions that we need to provide our theorem.

Assumptions

1. The likelihood ratio statistic λ is of class C^2 with a bounded Hessian matrix and $\lambda(\mathbf{x}) \longrightarrow 0$ as $|| \mathbf{x} || \longrightarrow \infty$.
2. $\mathbb{H}_0$ has lebesgue content 0.
3. $\mu(\{\lambda = k\})$ for all $k > 0$, in which μ denotes the lebesgue measure on $\mathbb{R}^2$.

Theorem 4.2. *Suppose that λ satisfies in the above assumptions and*

$$\sup_{\mathbf{x}\in\mathbb{R}^2} \mid \lambda_n(\mathbf{x}) - \lambda(\mathbf{x}) \mid \longrightarrow^{p.s} 0.$$

Then, for almost all $k \in (0,1)$,

$$k_{\alpha,n} \longrightarrow^p k_\alpha \quad as \quad n \longrightarrow \infty.$$

Proof. First of all, we introduce the following notations;

$$D^l(k) = \{\mathbf{x} \in \mathbb{R}^2 \ : \ \lambda(\mathbf{x}) \le k\}, \qquad D^l_n(k) = \{\mathbf{x} \in \mathbb{R}^2 \ : \ \lambda_n(\mathbf{x}) \le k\}. \tag{23}$$

Also, $F_{\lambda(\mathbf{x})}$ and $F_{\lambda_n(\mathbf{x})}$ are the cdf of $\lambda(\mathbf{x})$ and $\lambda_n(\mathbf{x})$, respectively, as

$$F_{\lambda(\mathbf{x})} = \mu\big(D^l(k)\big), \qquad F_{\lambda_n(\mathbf{x})} = \mu\big(D^l_n(k)\big). \tag{24}$$

According to the proof of theorem 2 of the paper by [10], the upper bound for levy metric $F_{\lambda(\mathbf{x})}$ and $F_{\lambda_n(\mathbf{x})}$ is

$$L(F_{\lambda(\mathbf{x})}, F_{\lambda_n(\mathbf{x})}) \le \max(\ || \ \lambda_n - \lambda \ ||_\infty \ , \ \ V_n), \tag{25}$$

where

$$V_n = \sup_{k \ge 0} \mid \mu\big(D^l_n(k)\big) - \mu\big(D^l(k)\big) \mid . \tag{26}$$

Based on "Assumptions", $F_{\lambda(\mathbf{x})}$ is a bijection form $\big(0, \sup_{\mathbf{x} \in \mathbb{R}^2} \lambda(\mathbf{x})\big)$ to $(0,1)$. Suppose G is its inverse function. Therefore, based on Lemma 3.1 of the paper by Cadre et al. [3], G is almost every where differntiable.

Consider $\alpha \in (0,1)$ such that G is differentiable at α. Then $G(\alpha) = k_\alpha$. Let G_n is pseudo-inverse of $F_{\lambda_n(\mathbf{x})}$, that is,

$$G_n(t) = \inf\{s \ge 0 \ : \ F_{\lambda_n(\mathbf{x})}(s) \ge t\}. \tag{27}$$

Then $G_n(\alpha) = k_{\alpha,n}$.

On the other hand, since $0 \le F_{\lambda(\mathbf{x})} \le 1$ and $0 \le F_{\lambda_n(\mathbf{x})} \le 1$, we have $L(F_{\lambda(\mathbf{x})}, F_{\lambda_n(\mathbf{x})}) \le 1$. Also, based on the property of Levy metric, the following relation is obtained:

$$L(F_{\lambda(\mathbf{x})}, F_{\lambda_n(\mathbf{x})}) = L(G, G_n). \tag{28}$$

Therefore, by using Lemma 3.2 of the paper by Cadre et al. [3] and inequality (25), it is concluded that:

$$\mid k_{\alpha,n} - k_\alpha \mid = \mid G_n(\alpha) - G(\alpha) \mid \le cL(F_{\lambda(\mathbf{x})}, F_{\lambda_n(\mathbf{x})}) \le c \max(\ || \ \lambda_n - \lambda \ ||_\infty, \ \ V_n), \tag{29}$$

where c is a positive constant.

Furthermore, by using the scheme of Lemma 3 of the paper Verdier [10],

$$V_n \longrightarrow^p 0 \qquad as \qquad n \longrightarrow \infty. \tag{30}$$

Therefore,

$$k_{\alpha,n} \longrightarrow^p k_\alpha \qquad as \qquad n \longrightarrow \infty. \tag{31}$$

□

5. Average run length

Commonly, control charts are evaluated by the average run length (ARL), which is the average number of sampling subgroups for a chart to signal, that is, the average number of points on a chart until a point indicates an out-of-control condition. The shorter the charts' ARL for an out-of-control condition, the better the performance in detecting the shift from the in-control condition.

To make a comparison, the ARLs value of the shift detecting in the stable state should be compared, i.e., the longer the better, and then these values in the unstable state should be compared, i.e., the shorter the better [5]. In the following subsection, some simulation results are presented regarding to the numerical performance of the new charts .

Simulation study. In this subsection, the performance of the two proposed control charts is compared with each other, based on the ARL. The computations are made using software Mathematica.

First, 10000 samples of size n from bivariate exponential distribution by FGM copula dependence structure are simulated and their likelihood ratio statistic $\lambda(\mathbf{x}_1, \mathbf{x}_2)$ are obtained. Then based on 0.0027-quantile of those statistics as Eq. (17), the threshold value $k_{0.0027,n}$ is gained.

Set the shift δ_1 for λ_1 and the shift δ_2 for λ_2, i.e., $\lambda_1' = \lambda_1^0 + \delta_1$ and $\lambda_2' = \lambda_2^0 + \delta_2$. Then count the number of samples whose the likelihood ratio test statistic $\lambda(\mathbf{x}_1, \mathbf{x}_2)$ is less than the threshold value $k_{0.0027,n}$ and also, count the number of ones which the statistic S as Eq. (18) is greater than 11.829, and then obtain ARLs for the charts CCLR and CCALR.

Here, samples are simulated for $\lambda_1^0 = 7$, $\lambda_2^0 = 5$ and various δ_1 and δ_2 values with a step of 0.2 between 0 and 2, and various values of sample size and dependence parameter θ. Results are shown in Tables 4, 5 and 6. In these tables, CC stands for control chart.

It is noted that in this paper, only the downward shifts are considered to show the ideas of new approach but it is not difficult to do so for upward shifts.

Since for CCLR, the simulated lower control limit is applied and for CCALR, the asymptotic distribution is used, it is expected that the two charts may not attain the same in-control ARL.

All tables show that ARL of CCALR is less than ARL of CCLR, i.e., CCALR shows the shifts faster than CCLR. Therefore, it is concluded that CCALR has the better performane to detect the shifts in parameters.

To help the reader to gain a better perspective of the ARLs' comparison, figures 1 and 2 represent the ARLs curve of CCLR and CCALR respect to the shift δ_1 for some values of the shift δ_2, and similarly, respect to δ_2 for some values of δ_1. Intuitively, These figures show better performance for CCALR than CCLR.

TABLE 4. ARL values of CCLR and CCALR for $n = 5$, $\theta = 0.3$, $\lambda_1^0 = 7$, $\lambda_2^0 = 5$ and various values of δ_1 and δ_2.

δ_2	CC	δ_1										
		0.0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
0.0	CCLR	434.783	400.000	256.410	384.615	277.778	303.030	270.270	285.714	204.082	163.934	142.857
	CCALR	400.000	333.333	212.766	322.581	232.558	250.000	232.558	212.766	161.290	144.928	129.870
0.2	CCLR	285.714	357.143	333.333	294.118	357.143	232.558	263.158	212.766	175.439	151.515	192.308
	CCLAR	243.902	294.118	270.270	238.095	322.581	212.766	232.558	181.818	158.730	129.870	158.730
0.4	CCLR	294.118	312.500	270.270	322.581	270.270	303.030	250.000	175.439	200.000	161.290	175.439
	CCLAR	227.273	256.410	243.902	277.778	256.410	217.391	196.078	147.059	172.414	128.205	144.928
0.6	CCLR	277.778	303.030	344.828	303.030	277.778	212.766	250.000	185.185	181.818	217.391	163.934
	CCLAR	217.391	263.158	270.270	227.273	250.000	172.414	192.308	156.250	156.250	181.818	138.889
0.8	CCLR	277.778	256.410	232.558	208.333	263.158	217.391	250.000	175.439	178.571	166.667	147.059
	CCLAR	238.095	243.902	185.185	178.571	222.222	188.679	217.391	149.254	153.846	142.857	131.579
1.0	CCLR	232.558	277.778	263.158	250.000	232.558	222.222	153.846	188.679	151.515	121.951	107.527
	CCLAR	204.082	256.410	212.766	212.766	204.082	188.679	133.333	169.492	135.135	111.111	91.743
1.2	CCLR	192.308	238.095	243.902	222.222	161.290	156.250	108.696	156.250	133.333	151.515	93.458
	CCLAR	178.571	192.308	192.308	204.082	144.928	133.333	98.039	133.333	106.383	119.048	86.956
1.4	CCLR	142.587	175.439	181.818	156.250	163.934	121.951	175.439	136.986	117.647	108.696	95.238
	CCLAR	128.205	140.845	144.928	126.582	125.000	106.383	149.254	113.636	100.000	96.154	79.365
1.6	CCLR	108.696	140.845	106.383	172.414	128.205	131.579	125.000	128.205	112.360	95.238	77.519
	CCLAR	100.000	119.048	93.458	144.928	111.111	116.279	117.647	108.696	97.087	88.496	68.966
1.8	CCLR	106.383	108.696	108.890	135.135	116.279	114.943	116.279	96.154	95.238	82.645	74.074
	CCLAR	88.496	93.458	93.458	105.263	104.167	98.039	100.000	85.470	80.645	72.464	68.493
2.0	CCLR	92.593	113.636	99.010	79.366	83.333	86.956	83.333	74.627	70.922	68.027	70.422
	CCLAR	77.519	96.154	80.645	72.993	74.627	75.188	68.966	67.114	62.500	59.880	59.172

6. Illustrative example

To demonstrate the performance and effectiveness of the proposed control charts, a simulated process with two quality characteristics destributed bivariate exponential with dependence structure based on FGM copula model by dependence parameter $\theta = 0.3$ is considered. The process works in-control when $\lambda_1 = 7$ and $\lambda_2 = 5$. For this process 20 samples of $n = 50$ observations are generated. The computed values of chart statistics $\lambda(\mathbf{X}_1, \mathbf{X}_2)$ and S, as Eqs. (11) and (18), are given in Table 7.

To investigate the performance of new charts in detecting parameter shift, our immediate impresion is that the process is now operating in a new quality level. Consider six cases of parameters shifts in which in two cases, the first parameter is shifted. In another two cases, the second parameter is shifted and in the other two cases, both of the parameters are shifted.

Suppose after 20^{th} sample the process becoms out-of-control. For this purpose, 20 samples are generated of $n = 50$ from bivariate exponential with dependence structure based on FGM copula model by dependence parameter $\theta = 0.3$, λ_1' and λ_2', displaied in Table 8. Figures 3-8 show the CCLR and CCALR of each process. It is seen that in most cases, CCALR detects the parameters shifs faster than CCLR, as it is expected.

Table 5. ARL values of CCLR and CCALR for $n = 10$, $\theta = 0.3$, $\lambda_1^0 = 7$, $\lambda_2^0 = 5$ and various values of δ_1 and δ_2.

δ_2	CC	δ_1										
		0.0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
0.0	CCLR	700.135	625.000	555.556	500.000	322.581	250.000	333.333	250.000	217.391	142.857	140.845
	CCALR	383.212	370.370	312.500	333.333	222.222	181.818	178.571	166.667	125.000	99.010	90.090
0.2	CCLR	625.000	434.783	526.316	333.333	416.667	294.118	263.158	200.000	212.766	175.439	105.263
	CCLAR	344.828	270.270	357.143	263.158	294.118	212.766	181.818	140.845	128.205	106.383	72.464
0.4	CCLR	500.000	434.783	400.000	400.000	312.500	294.118	270.270	232.558	175.439	192.308	108.696
	CCLAR	294.118	250.000	270.270	277.778	196.078	192.308	188.679	135.135	113.636	114.943	76.336
0.6	CCLR	476.190	454.545	333.333	400.000	303.03	232.558	208.333	227.273	138.889	144.928	114.943
	CCLAR	243.902	250.000	222.222	227.273	217.391	149.254	144.928	147.059	101.010	85.470	79.365
0.8	CCLR	344.828	243.902	232.558	204.082	200.000	208.333	217.391	172.414	119.048	131.579	105.263
	CCLAR	204.082	153.846	126.582	151.515	131.579	135.135	138.889	99.010	85.470	86.207	71.942
1.0	CCLR	196.078	357.143	217.391	172.414	256.410	227.273	172.414	147.059	123.457	94.340	75.188
	CCLAR	138.889	232.558	138.889	114.943	144.928	138.889	111.111	92.593	81.301	64.103	50.251
1.2	CCLR	161.290	136.986	188.679	133.333	138.889	111.111	120.482	116.279	90.909	83.333	65.790
	CCLAR	112.360	103.093	147.059	88.496	93.458	79.365	79.365	86.207	61.350	56.180	47.393
1.4	CCLR	138.889	163.934	144.928	125.000	109.890	135.135	95.238	87.719	78.125	60.606	59.172
	CCLAR	83.333	97.087	86.956	86.956	79.365	81.967	62.893	57.143	54.945	43.478	40.816
1.6	CCLR	104.167	89.286	91.743	82.645	84.746	74.627	83.333	64.935	57.143	51.814	42.194
	CCLAR	63.943	58.480	57.804	54.054	58.480	51.282	56.180	46.512	40.486	35.587	30.960
1.8	CCLR	71.429	69.444	66.667	66.225	65.360	59.172	52.083	48.309	47.170	39.526	34.247
	CCLAR	48.544	44.843	44.053	42.918	42.373	37.313	37.313	33.113	32.362	27.027	25.000
2.0	CCLR	53.476	45.662	44.444	47.170	39.370	41.322	37.879	41.494	34.843	32.154	28.329
	CCLAR	37.313	32.680	31.446	30.581	26.738	29.851	27.027	26.882	24.096	23.148	20.243

7. **Concluding remarks**

In this paper, two control charts CCLR and CCALR were proposed for use with data that are assumed to follow bivariate exponential distribution by FGM copula model dependence structure. The performance of the proposed charts was studied in simulation scheme. Different values of dependence parameter and distribution parameters and also, various values of sample sizes are studied. Overall, the CCALR has more effective performance monitoring shifts of the parameters than CCLR.

It is noted that the proposed control charts are without memory, which are not rapid to show the small changes. Then to overcome this problem and have a chart to show the small changes very fast, control charts with memory are needed. One way is to combine these charts with the EWMA or CUSUM schemes, which will be one of the future research topics.

8. **Aknowledgement**

We would like to thank the reviewers for their thoughtful comments and efforts towards improving our manuscript.

TABLE 6. ARL values of CCLR and CCALR for $n = 50$, $\theta = 0.3$, $\lambda_1^0 = 7$, $\lambda_2^0 = 5$ and various values of δ_1 and δ_2.

δ_2	CC	δ_1										
		0.0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
0.0	CCLR	500.000	344.828	250.000	116.279	57.143	26.178	14.184	8.217	4.715	2.873	1.904
	CCALR	384.615	303.030	204.082	100.000	50.505	22.727	12.920	7.457	4.321	2.691	1.815
0.2	CCLR	357.143	270.270	172.414	126.582	49.505	25.840	15.480	8.850	4.606	2.893	1.977
	CCLAR	303.030	238.095	158.730	103.093	44.643	22.883	13.514	7.862	4.272	2.725	1.885
0.4	CCLR	256.410	277.778	200.000	98.039	49.020	27.397	14.451	8.258	4.726	2.888	1.990
	CCLAR	232.558	222.222	178.571	81.301	43.290	24.510	12.937	7.429	4.348	2.707	1.890
0.6	CCLR	188.679	163.934	147.059	74.074	47.393	23.364	13.850	7.530	4.550	2.789	1.928
	CCLAR	161.290	149.254	128.205	64.516	40.486	20.161	12.300	6.780	4.226	2.617	1.830
0.8	CCLR	133.333	117.647	80.645	60.976	38.023	20.243	11.723	6.954	4.102	2.799	1.867
	CCLAR	113.636	100.000	66.225	53.192	33.670	18.518	10.571	6.365	3.824	2.613	1.773
1.0	CCLR	83.333	69.930	66.667	40.984	26.596	17.182	9.881	6.142	3.873	2.484	1.822
	CCLAR	70.922	59.880	55.556	35.089	22.936	15.480	8.961	5.602	3.591	2.341	1.742
1.2	CCLR	43.860	44.248	38.911	29.154	19.380	13.793	8.562	5.328	3.466	2.346	1.698
	CCLAR	37.736	37.736	33.113	25.575	17.036	12.300	7.868	4.880	3.236	2.227	1.635
1.4	CCLR	29.674	29.940	24.155	18.349	14.225	10.020	6.863	4.688	3.213	2.167	1.633
	CCLAR	26.110	25.974	21.978	16.447	12.771	8.842	6.266	4.279	2.993	2.059	1.568
1.6	CCLR	19.763	18.553	15.723	13.532	10.526	7.825	5.438	3.837	2.650	2.029	1.555
	CCLAR	17.391	16.474	14.144	12.063	9.634	7.122	4.973	3.578	2.519	1.934	1.502
1.8	CCLR	12.019	11.521	10.373	9.251	7.570	5.754	4.154	3.148	2.347	1.785	1.442
	CCLAR	10.822	10.363	9.452	8.389	6.901	5.244	3.882	2.970	2.229	1.724	1.401
2.0	CCLR	7.524	7.819	7.262	6.196	5.246	4.316	3.378	2.666	2.030	1.638	1.347
	CCLAR	6.826	7.148	6.557	5.740	4.892	3.960	3.152	2.493	1.929	1.582	1.309

TABLE 7. The values of chart statistics for generated subsamples data.

case	$\lambda(x_1, x_2)$	s	case	$\lambda(x_1, x_2)$	s
1	0.99742	0.00516	11	0.89851	0.21404
2	0.80627	0.43068	12	0.92981	0.14555
3	0.79194	0.46654	13	0.98319	0.03390
4	0.99553	0.00896	14	0.80589	0.43163
5	0.91146	0.18542	15	0.87839	0.25932
6	0.80026	0.44563	16	0.94933	0.10401
7	0.73641	0.61193	17	0.93394	0.13670
8	0.80127	0.44312	18	0.90851	0.19190
9	0.86225	0.29642	19	0.89132	0.23010
10	0.99587	0.00827	20	0.93181	0.14126

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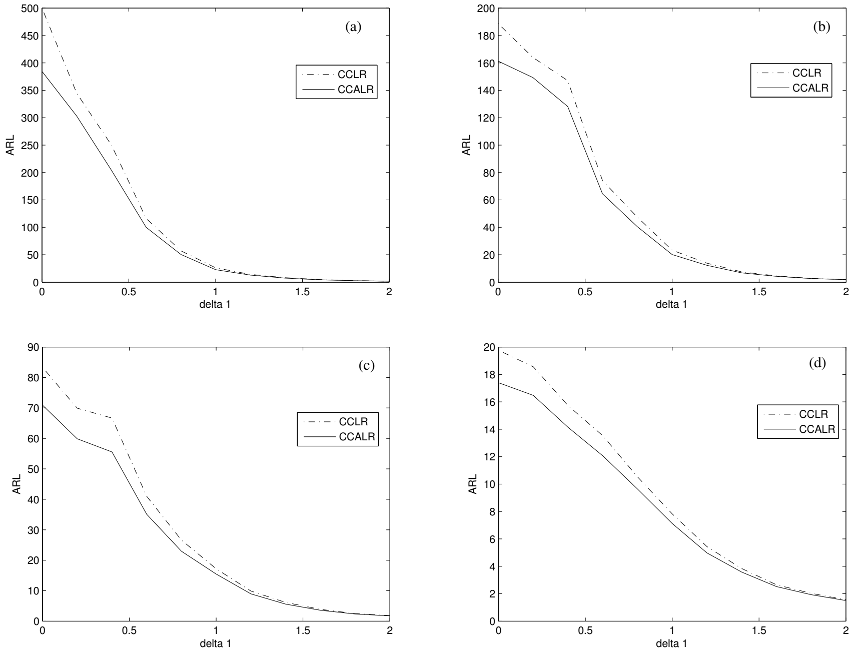

Figure 1. ARL curves of CCLR and CCALR in cases where $n = 50$, $\alpha = 0.3$, $\lambda_1 = 7$, $\lambda_2 = 5$ and (a) $\delta_2 = 0.0$, (b)$\delta_2 = 0.6$, (c)$\delta_2 = 1.0$, (d)$\delta_2 = 1.6$.

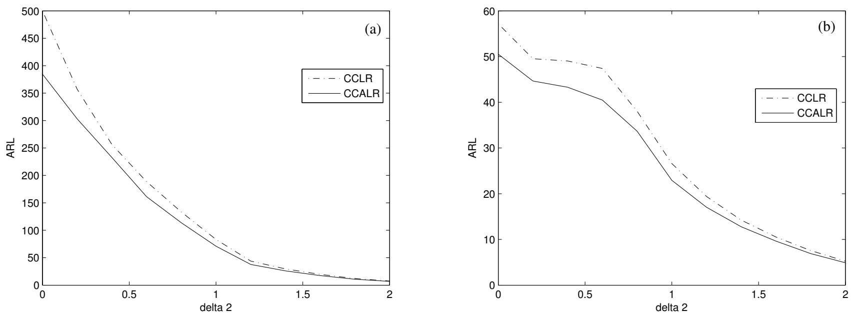

Figure 2. ARL curves of CCLR and CCALR in cases where $n = 50$, $\alpha = 0.3$, $\lambda_1 = 7$, $\lambda_2 = 5$ and (a) $\delta_1 = 0.0$, (b)$\delta_1 = 0.8$.

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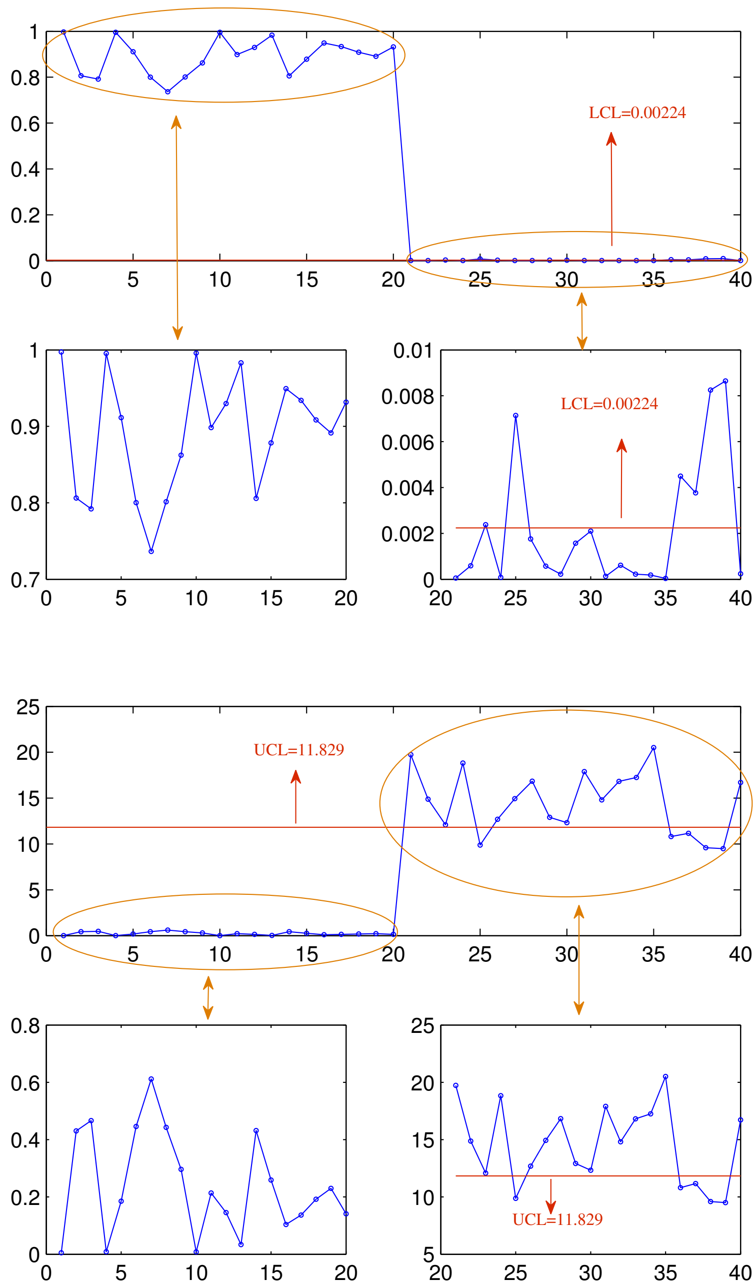

Figure 3. CCLR (the above one) and CCALR (the below one) for process 1.

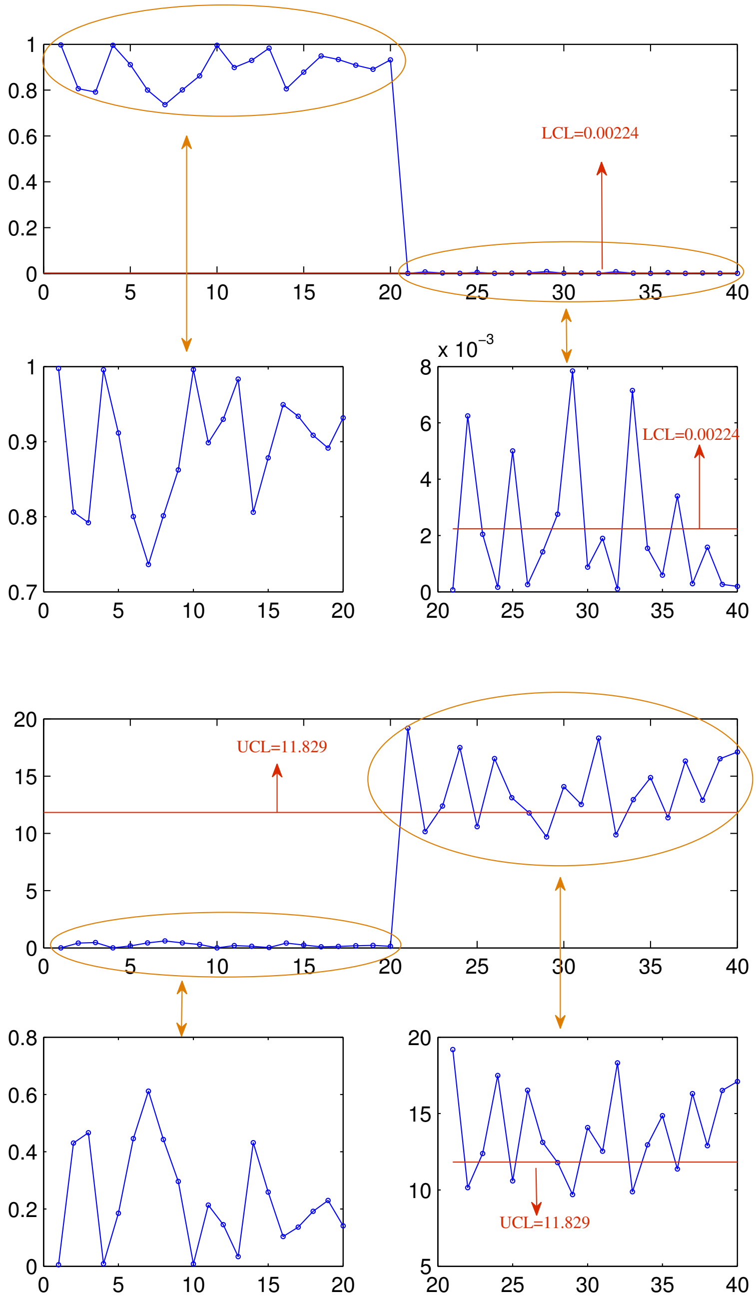

FIGURE 4. CCLR (the above one) and CCALR (the below one) for process 2.

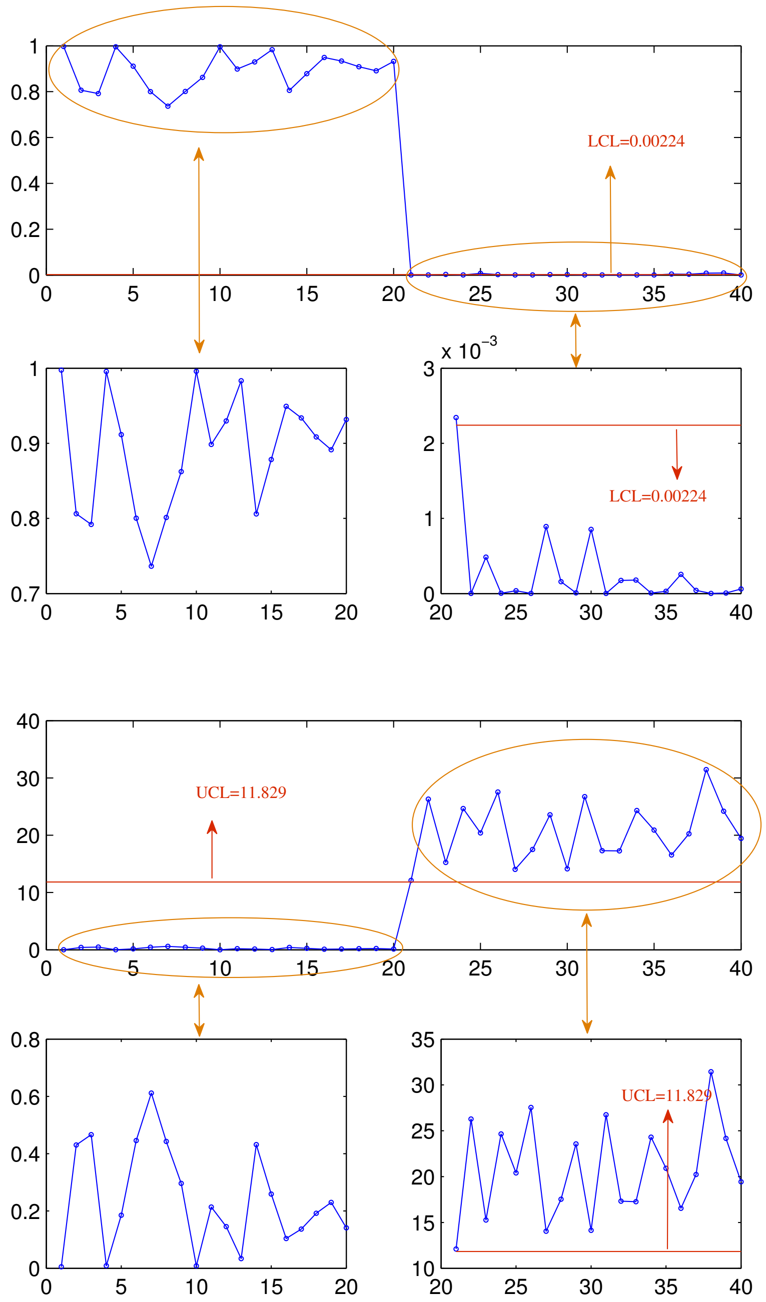

Figure 5. CCLR (the above one) and CCALR (the below one) for process 3.

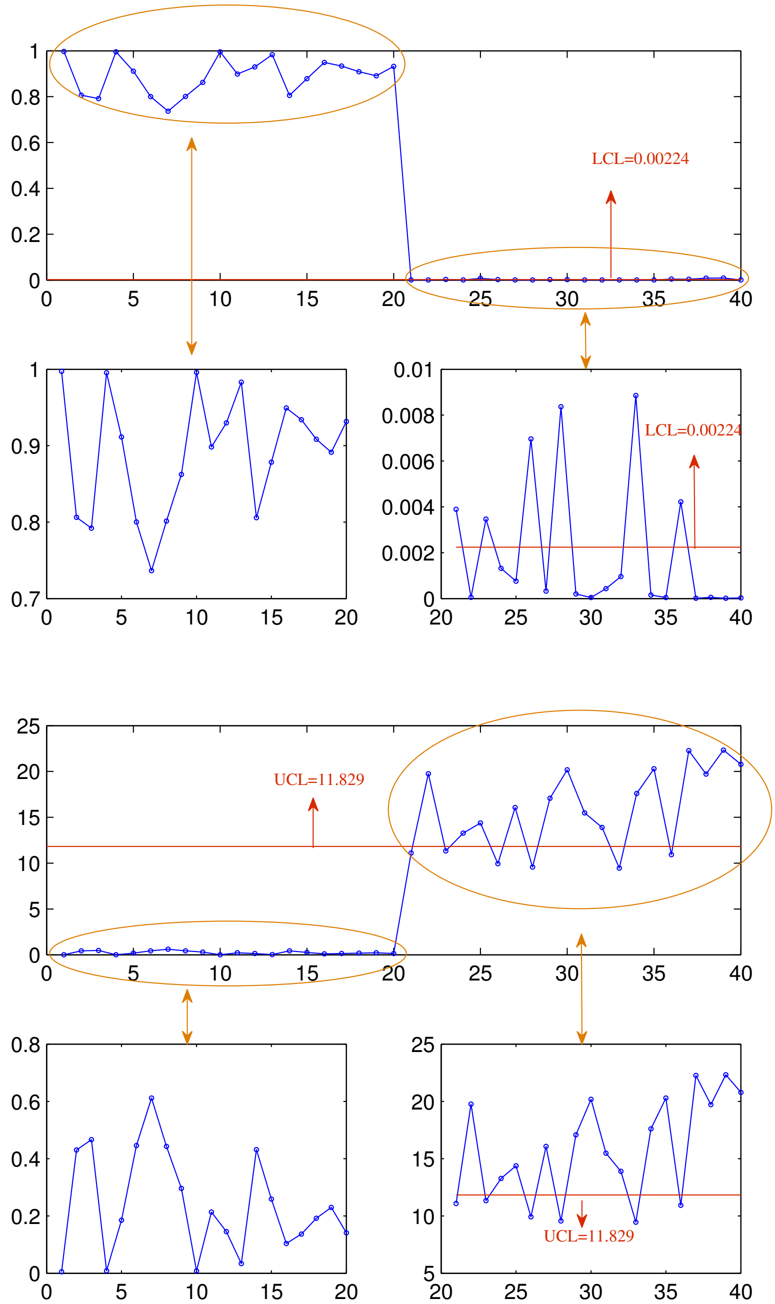

Figure 6. CCLR (the above one) and CCALR (the below one) for process 4.

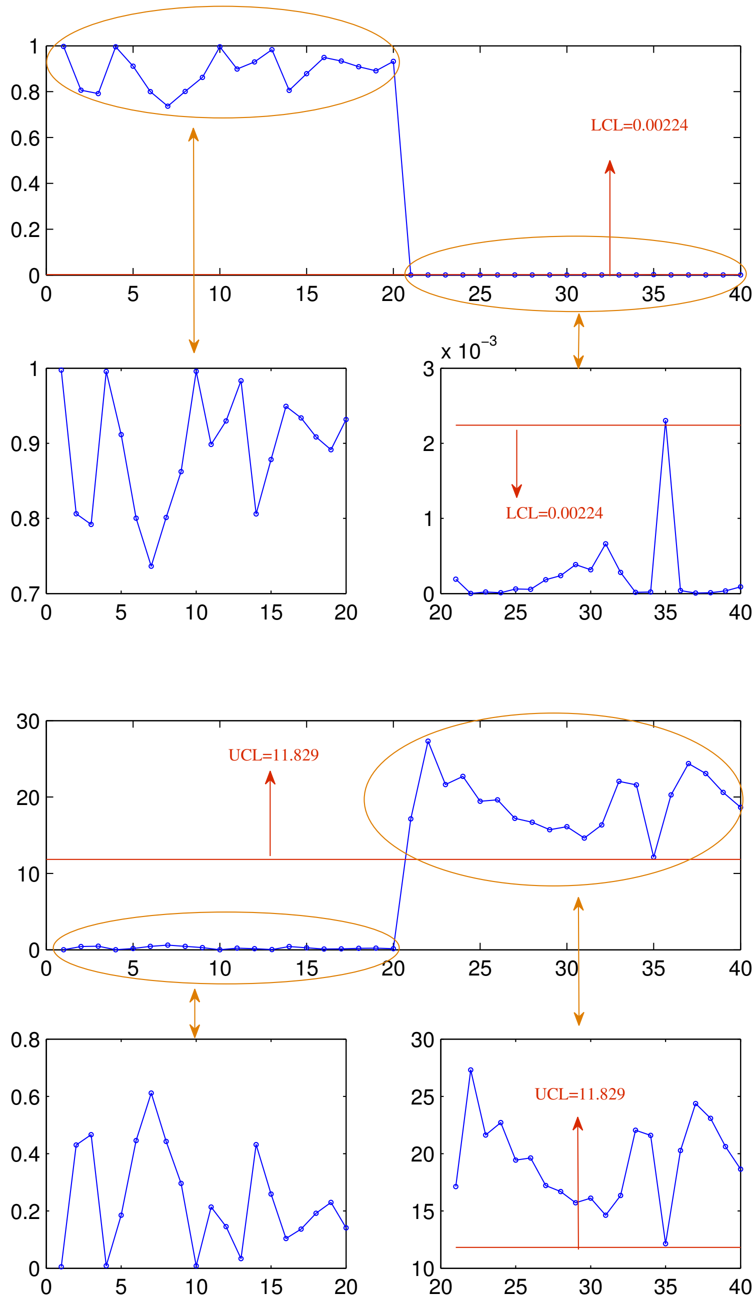

FIGURE 7. CCLR (the above one) and CCALR (the below one) for process 5.

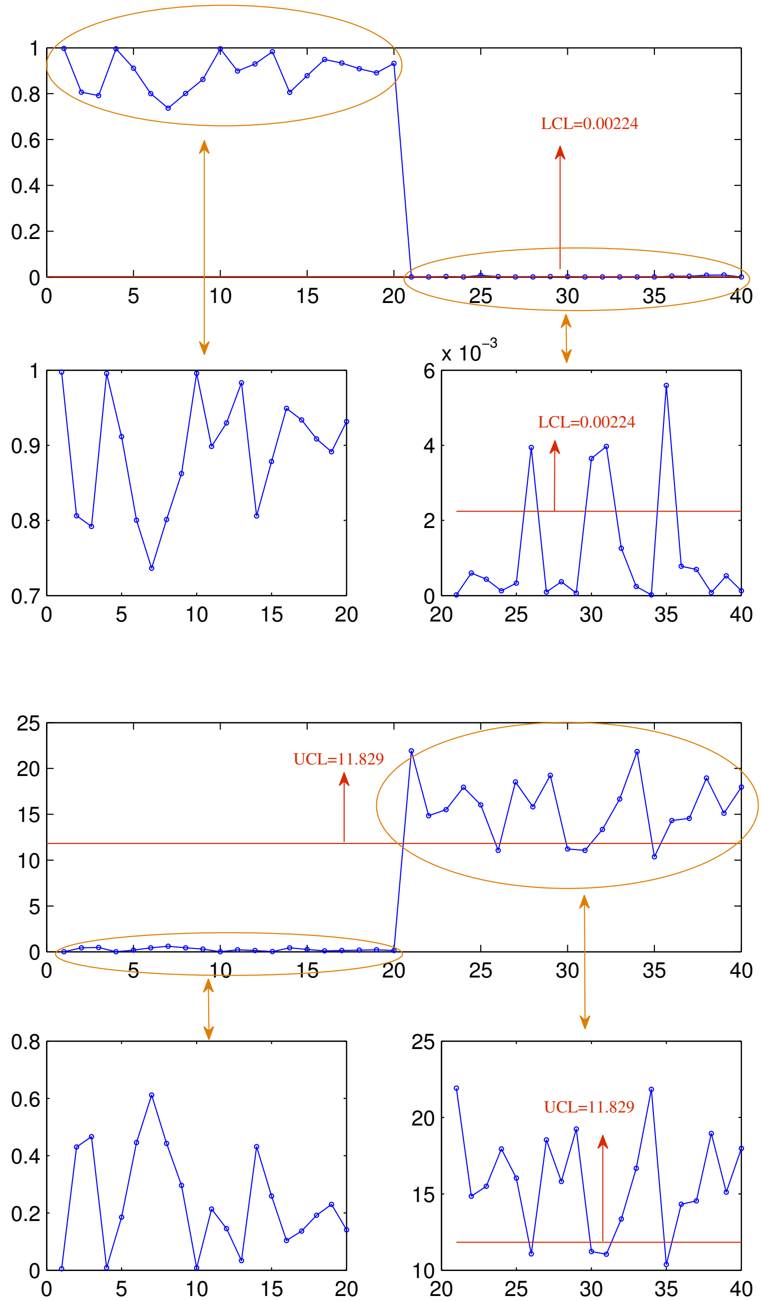

FIGURE 8. CCLR (the above one) and CCALR (the below one) for process 6.

TABLE 8. The values of six new quality levels for generated subsamples data.

process	λ'_1	λ'_2	process	λ'_1	λ'_2
1	3.9	5.0	4	7.0	2.8
2	4.0	5.0	5	4.5	3.0
3	7.0	2.5	6	5.0	3.0

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