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A NOTE ON MULTIVARIATE MAJORIZATION

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Abstract. A matrix A is said to be multivariate majorized by a matrix B, written $A \prec B$, if there exists a doubly stochastic matrix D such that $A = BD$. In the present paper, we obtain a totally ordered subset of M_{nm} which contains a given matrix A. Also, we show that the totality of all extreme points of the collection of all matrices which are multivariately majorized by a matrix A is the set of all matrices obtained by permuting the columns of A.

Keywords: Doubly stochastic, Extreme point, Multivariate majorization.
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1. Introduction and preliminaries

Majorization is a concept of interest in various areas of mathematics and statistics. Some kinds of majorization such as multivariate or matrix majorization were motivated by the concepts of vector majorization and were introduced in [9].

A matrix R with nonnegative entries is called *row stochastic* if the sum of every row of R is 1. A nonnegative real matrix D is called a *doubly stochastic matrix* if each of its row sums and column sums is equal to one.

A matrix A is said to be multivariate majorized by a matrix B, written $A \prec B$, if there exists a doubly stochastic matrix D such that $A = BD$.

For more information on both vector and matrix majorization and their applications see [1, 13–15].

In [3, 6, 7] the authors discuss a set of doubly stochastic matrices associated with a given majorization, see also [10] for a related study.

In recent decades, characterizing the structure of majorization preserving linear maps on certain spaces of matrices has been intensively studied (see [11, 12, 16]).

Here are some notations that will be used throughout this paper.

Let M_{nm} be the set of all $n \times m$ real matrices, $M_n = M_{nn}$ be the set of all $n \times n$ real matrices, $\mathcal{RS}(n)$ be the set of all $n \times n$ row stochastic matrices, $\mathcal{DS}(n)$ be the set of all $n \times n$ doubly stochastic matrices, $\mathcal{S}(n)$ be the set

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of all $n \times n$ matrices which in every row one entry is 1 and all other entries are zero, $\mathcal{P}(n)$ be the set of all $n \times n$ permutation matrices, $\mathbb{R}^n$ be the set of all $n \times 1$ (column) vectors, and $\mathbb{R}_n$ be the set of all $1 \times n$ (row) vectors. The notation $[a_1|a_2|\cdots|a_m]$ is used for the $n \times m$ matrix whose columns are $a_1, a_2, \ldots, a_m \in \mathbb{R}^n$ and the identity matrix is denoted by I_n, or simply I, if the size n of the matrix I is understood from the context. The letter J stands for the (rank-1) square matrix all of whose entries are 1. The symbol $\mathbb{N}_k$ is used for the set $\{1, 2, \ldots, k\}$.

Let $A, B \in M_{nm}$. We write $A \prec_r B$, if $A = BR$, for some $R \in \mathcal{RS}(m)$. Also, we write $A \prec B$, if $A = BD$ for some $D \in \mathcal{DS}(m)$.

Let S be a subset of a vector space on $\mathbb{R}$. The set $co(S)$ of S is the convex hull of S, i.e., $X \in co(S)$ if X can be represented as

$$X = \sum_{j=1}^{k} \lambda_j X_j, \text{ with } \sum_{j=1}^{k} \lambda_j = 1, \ \lambda_j \geq 0 \ (j = 1, \ldots, k)$$

for some positive integer k and members $X_1, \ldots, X_k \in S$. For any convex subset C of a vector space on $\mathbb{R}$, a member $X \in C$ is an extreme point of C, if X cannot be expressed as a convex combination of two members of C both different from X. The set of all extreme points of C is denoted by $ext(C)$.

Proposition 1.1. *[15, Theorem 3.22] Let $A, B \in M_{nm}$. Then the following are equivalent*

(i) *$A \prec B$ and $B \prec A$.*
(ii) *There exists a permutation matrix $P \in \mathcal{P}(m)$ such that $AP = B$.*

Proposition 1.2. *[13, pp. 548] The set $\mathcal{DS}(n)$ of all $n \times n$ doubly stochastic matrices is compact. Then by Heine-Borel theorem, $\mathcal{DS}(n)$ is closed and bounded.*

Lemma 1.3. *[4, Lemma 2.2] Let R be a nonsingular row stochastic matrix. If R^{-1} is nonnegative, then R is a permutation matrix.*

In Section 2, we obtain a totally ordered subset of $(M_{nm}, \prec)$ that contains a given matrix A and then we generalize the Birkhoff's theorem. For more information on multivariate majorization, we refer the reader to [2, 5, 8].

2. Maximal totally ordered subsets of $(M_{nm}, \prec)$

In this section we prove the main results. First, we characterize the maximal totally ordered subsets of $(M_{nm}, \prec)$.

Proposition 2.1. *Let $A \in M_{nm}$. Then $B \in M_{nm}$ is minimal with respect to $B \prec A$ if and only if $B = m^{-1}AJ$.*

Note. Every entry of a row of $m^{-1}AJ$ is the arithmetic mean of the corresponding row of A.

Proof. Let $J_m = m^{-1}J$ and observe that $J_m \in \mathcal{DS}(m)$. Then $AJ_m \prec A$. Let $X \prec AJ_m$. Then $X = AJ_mD = AJ_m$ and hence AJ_m is minimal.

Conversely, assume $B \in M_{nm}$ is minimal with respect to $B \prec A$. Hence, $BJ_m \prec B \prec A$ and thus $BJ_m = B = AD$ for some doubly stochastic matrix D. Then $B = BJ_m^2 = ADJ_m = AJ_m$. □

Theorem 2.2. *Let $A \in M_{nm}$. There exists a maximal totally ordered subset $\mathcal{L}$ of $(M_{nm}, \prec)$ containing A.*

Proof. Let $A_0 = AJ_m$. We define $\mathcal{L} = \{\lambda A + (1-\lambda)A_0 : \lambda \geq 0\}$. If $A_0 = A$, we can slightly modify A to make A and A_0 distinct. We claim $\mathcal{L}$ is the desired set.
Clearly, $\prec$ is reflexive and transitive relation on M_{nm} and therefore on $\mathcal{L}$. We will show that it corresponds to the totally ordered set $[0, \infty)$. Let $0 \leq t \leq 1$ and define $A(\lambda) = \lambda A + (1-\lambda)A_0$ for $\lambda \geq 0$. Then $tI + (1-t)J_m$ is doubly stochastic and for $\lambda > 0$,

$$A(\lambda)[tI + (1-t)J_m] = A(t\lambda),$$

which shows that $A(t\lambda) \prec A(\lambda)$. Assume $A(\lambda) \prec A(t\lambda)$. Then by Proposition 1.1, there exists a permutation matrix P such that

$$A(\lambda) = A(t\lambda)P = A(\lambda)[tI + (1-t)J_m]P.$$

Hence, $A - A_0 = t(AP - A_0)$ and thus $\| A - A_0 \| = t \| AP - A_0 \|$. It follows that $t = 1$. Thus the relation $\lambda \mapsto A(\lambda)$ is a bijection between $[0, \infty)$ and $\mathcal{L}$. Hence, $\mathcal{L}$ is a totally ordered set that contains A.

It remains to show that $\mathcal{L}$ is maximal. If not, there exists an element $B \in M_{nm} \backslash \mathcal{L}$ such that $\mathcal{L} \cup \{B\}$ is totally ordered. Due to minimality of A_0, $A(0) \prec B$. Also, if $A(\lambda) \prec B$ for all $\lambda \in [0, \infty)$, it follows that for every $\lambda \in [0, \infty)$ there exists a doubly stochastic D_λ such that $A(\lambda) = BD_\lambda$. By Proposition 1.2, the class of all doubly stochastic matrices is bounded, it follows that $\{A(\lambda) : \lambda \in [0, \infty)\}$ is bounded; a contradiction. Thus B gives rise to a pair of nonempty subsets $C_- = \{\lambda : A(\lambda) \prec B\}$ and $C_+ = \{\lambda : B \prec A(\lambda)\}$ of $[0, \infty)$ defining a number $\lambda_0 = \sup C_- = \inf C_+ \in [0, \infty)$. Choose a monotone sequence $\{\lambda_n\}$ in C_- (resp., C_+) that converges to λ_0 and find a sequence $\{D_n\}$ of doubly stochastic matrices such that $A(\lambda_n) = BD_n$, (resp., $B = A(\lambda_n)D_n$). Due to the compactness (resp., in $\mathbb{R}^{nm}$-topology) of the class of doubly stochastic matrices by Proposition 1.2, assume without loss of generality that $\{D_n\}$ converges to some doubly stochastic D_- (resp., D_+). Then $A(\lambda_0) = BD_-$ (resp., $B = A(\lambda_0)D_+$). Since $\mathcal{L} \cup \{B\}$ is totally ordered, it follows that $B = A(\lambda_0)$; a contradiction. □

Now, we prove a result similar to Theorem 2.2 for $\prec_r$, i.e., we show that for $A \in M_{nm}$, there exists a maximal totally ordered subset of $(M_{nm}, \prec_r)$ containing A. To prove the fact we need to state the following lemma.

Lemma 2.3. *Let $A, B \in M_n$ and A be an invertible matrix such that $A \prec_r B$ and $B \prec_r A$. Then $A = BP$ for some $P \in \mathcal{P}(n)$.*

Proof. Suppose $A \prec_r B$, then $A = BR$ for some $R \in \mathcal{RS}(n)$. Since $B \prec_r A$, there exists $S \in \mathcal{RS}(n)$ such that $B = AS$. Hence $A = BR = ASR$, therefore $SR = I$. Now, by Lemma 1.3, $R \in \mathcal{P}(n)$. □

Theorem 2.4. *Let $A \in M_{nm}$. There exists a maximal totally ordered subset $\mathcal{L}$ of $(M_{nm}, \prec_r)$ containing A.*

Proof. For every $X \in M_{nm}$, define

$$X(\lambda) = \lambda X + (1-\lambda)XJ_m, \qquad \lambda \in \mathbb{R}.$$

Consider the following two cases.

Case 1. Assume that A is invertible. We show that

$$\mathcal{L} = \{A(\lambda) : \lambda \geq 0\}$$

is a maximal totally ordered set with respect to $\prec_r$. Since $I(\lambda)I(\frac{1}{\lambda}) = I$ for every $\lambda > 0$, it follows that $I(\lambda)$ is invertible for every $\lambda > 0$. Since $A(\lambda) = AI(\lambda)$, it follows that $A(\lambda)$ is invertible for each $\lambda > 0$.

It is clear that $\prec_r$ is a reflexive and transitive relation on $\mathcal{L}$. In the proof of Theorem 2.2 we have shown that $A(t\lambda) \prec A(\lambda)$ for every $0 \leq t \leq 1$. Thus $A(t\lambda) \prec_r A(\lambda)$. Now, if $A(\lambda) \prec_r A(t\lambda)$, then by Lemma 2.3, we have $A(\lambda) = A(t\lambda)P$ for some $P \in \mathcal{P}(n)$. It follows that $t = 1$, hence $\mathcal{L}$ is totally ordered set which contains A.

It remains to show that $\mathcal{L}$ is maximal with respect to $\prec_r$. The set $\mathcal{L} \cup \{B\}$ for every $B \in M_{nm} \backslash \mathcal{L}$ with relation $\prec$ is not totally ordered. So, there exists $\lambda_0 \geq 0$ such that $A(\lambda_0) \prec B$ and $B \prec A(\lambda_0)$. Hence, $A(\lambda_0) \prec_r B$ and $B \prec_r A(\lambda_0)$. This shows that $\mathcal{L} \cup \{B\}$ with relation $\prec_r$ is not totally ordered.

Case 2. Assume A is not invertible. Since I is invertible, by case 1 $K = \{\lambda I + (1-\lambda)J_m : \lambda \geq 0\}$ is a maximal totally ordered subset of $(M_{nm}, \prec_r)$. Thus $K \cup \{A\}$ is not a totally ordered set. It follows that there exists $\lambda_0 \geq 0$ such that $A \prec_r I(\lambda_0)$ and $I(\lambda_0) \prec_r A$. Therefore,

$$\mathcal{L} = \{\lambda I + (1-\lambda)J_m : \lambda \geq 0, \lambda \neq \lambda_0\} \cup \{A\}$$

is maximal totally ordered subset of $(M_{nm}, \prec_r)$ which contains A. □

Theorem 2.5. *(Birkhoff's theorem). The totality of all extreme points of the collection of all doubly stochastic matrices is the set of all permutation matrices.*

For $A \in M_{nm}$, let

$$C(A) = \{X \in M_{nm} : X \prec A\}.$$

So, by Birkhoff's theorem $ext\ C(I) = \mathcal{P}(n)$. In the following theorem we generalize this concept.

Theorem 2.6. *Let $A \in M_{nm}$. Then*

$$ext\ C(A) = \{AP : P \in \mathcal{P}(m)\}.$$

Proof. Let $A = [a_1|\dots|a_m] \in M_{nm}$ be arbitrary.

By Birkhoff's theorem, $ext\ C(A) \subseteq \{AP : P \in \mathcal{P}(m)\}$. Thus, it is enough to show that for every permutation matrix P, AP is an extreme point.

Let $P, P_1, P_2, \dots, P_r \in \mathcal{P}(m)$ be such that $AP = \sum_{i=1}^{r} \lambda_i AP_i$ for some positive numbers $\lambda_1, \dots, \lambda_r$ with $\sum_{i=1}^{r} \lambda_i = 1$.

We show that $AP_1 = AP_2 = \cdots = AP_r$. Replacing each P_k by $P_k P^{-1}$, one can assume without loss of generality that $P = I$.

We prove by induction on m (the number of columns). If $m = 1$ then there is nothing to prove. Assume that, it holds for all matrices with the number of columns less than m. Let $A = [a_1|\dots|a_m]$. If $a_1 = \cdots = a_m$, then the result follows. Without loss of generality assume that $a_1 = \cdots = a_k$ for some $1 \le k < m$ and a_1 is not a convex combination of $a_{k+1}, \dots, a_m$. Since a_1 is not a convex combination of $a_{k+1}, \dots, a_m$ and $A = \sum_{i=1}^{r} \lambda_i AP_i$, for every$1 \le i < r$, there exist permutation matrices $Q_i \in M_k$ and $Q_i' \in M_{m-k}$ such that $P_i = Q_i \oplus Q_i^{'}$. Let $B = [a_{k+1}|\dots|a_m]$. So, $B = \sum_{i=1}^{r} \lambda_i BQ_i^{'}$ and hence by induction assumption $B = BQ_i^{'}$. Therefore $AP_i = [a_1|\dots|a_1|B](Q_i \oplus Q_i^{'}) = [a_1|\dots|a_1|BQ_i^{'}] = A$. □

Corollary 2.7. *(Rado [14]). For $a \in \mathbb{R}_n$, the set $\{x \in \mathbb{R}_n : x \prec a\}$ is the convex hull of points obtained by permuting the components of a.*

Proof. Since $a \in M_{1n} = \mathbb{R}_n$, by Theorem 2.6 we have

$$ext\{x \in \mathbb{R}_n : x \prec a\} = \{aP : P \in \mathcal{P}(m)\}.$$

□

3. Right matrix majorization

In this paper, V stands for a finite dimensional vector space on $\mathbb{R}$.

Definition 3.1. The linearly dependent subset S of a finite dimensional vector space V on $\mathbb{R}$ is called minimal if there is a linearly dependent set T such that $T \subseteq S$, then $T = S$.

Lemma 3.2. *Let V be a vector space. If $dim(V) = n$, then every minimal linear dependent subset of V has at most $n + 1$ elements.*

Lemma 3.3. *Let S be a linear dependent subset of V. Then S is minimal if and only if $S - \{x_0\}$ is linearly independent for all $x_0 \in S$.*

Proof. Suppose that S is minimal and $x_0 \in S$. If $S - \{x_0\}$ is a linear dependent subset of V, then $S - \{x_0\} \subset S$ leads to a contradiction. Thus $S - \{x_0\}$ is linearly independent for all $x_0 \in S$.

Conversely, assume that S is a linearly dependent subset of V, and $S - \{x_0\}$ is linearly independent for all $x_0 \in S$. We show that S is minimal.

Assume that S is not minimal, so there exists a linearly dependent subset T of V such that $T \subsetneqq S$. Then there exists $x_0 \in S$ such that $T \subseteq S - \{x_0\}$. Therefore $S - \{x_0\}$ is linearly dependent for all $x_0 \in S$. □

Lemma 3.4. *Let $S = \{v_1, \ldots, v_m\}$ be a linearly independent subset of V and $v = c_1v_1 + \ldots + c_mv_m$ where $c_i \neq 0$ for all $i = 1, \cdots, m$. Then $S^{'} = S \cup \{v\}$ is minimal linearly dependent subset of V.*

Proof. By Lemma 3.3, it is sufficent to prove that $S^{'} - \{x_0\}$ is linearly independent subset of V for all $x_0 \in S$. Now, we consider two cases.

Case 1. If $x_0 = v$, then $S^{'} - \{x_0\} = S$ is is linearly independent subset of V.

Case 2. If $x_0 \in S$. Without loss of generality assume that $x_0 = v_1$. We show that $S^{'} - \{x_0\}$ is linearly independent subset of V. Let $d_2v_2 + \cdots + d_mv_m + dv = 0$, where $d_2, \ldots, d_m, d \in \mathbb{R}$. Thus

$$d_2v_2 + \cdots + d_mv_m + d(c_1v_1 + \cdots + c_mv_m) = 0.$$

It follows that

$$\begin{cases} dc_1 = 0 & \Rightarrow d = 0 \\ d_2 + dc_2 = 0 & \Rightarrow d_2 = 0 \\ \vdots & \vdots \\ d_m + dc_m = 0 & \Rightarrow d_m = 0 \end{cases}.$$

Therefore, $S^{'} - \{x_0\}$ is is linearly independent subset of V and the proof is complete. □

Lemma 3.5. *Let $\{v_1, \ldots, v_m\}$ be a minimal linearly dependent subset of V. If $\sum_{i=1}^m c_iv_i = 0$ and $\sum_{i=1}^m d_iv_i = 0$ where $c_i, d_i \in \mathbb{R}$ for all $1 \leq i \leq m$, then $\overrightarrow{d} || \overrightarrow{c}$.*

Proof. It is straightforward verified. □

For $A \in M_{nm}$, let

$$C(A, \prec_r) = \{X \in M_{nm} : X \prec_r A\}.$$

So $\mathcal{RS}(n) = C(I, \prec_r)$ and by [12, Lemma 4.3] we have *ext* $C(I, \prec_r) = \mathcal{S}(n)$.

Also, for a minimal linearly dependent subset S of V, we define

$$\mathbb{N}_S^+ = \{i | c_i > 0\}, \qquad \mathbb{N}_S^- = \{i | c_i < 0\},$$

where $c_i \in \mathbb{R}$ for all $1 \leq i \leq m$, hence $\{\mathbb{N}_S^+, \mathbb{N}_S^-\}$ is a partition of $\mathbb{N}_n$. It is called a sign defined partition of S.

Dahl [9] showed that for $A = [a_1 | \ldots | a_n] \in M_n$

$$\begin{aligned} E = ext\ C(A, \prec_r) &\subseteq \{AR : R \in \mathcal{S}(n)\} \\ &= \left\{ \left[\sum_{i \in J_1} a_i, \ldots, \sum_{i \in J_n} a_i \right] P : P \in \mathcal{P}(n) \right\} = A_{\mathcal{S}(n)}, \end{aligned}$$

where $J_1, \ldots, J_n$ is a partition of $\{1, \ldots, n\}$.

We are now ready to prove the main result of the section.

Theorem 3.6. *Let $A \in M_n$, and $E = ext\ C(A, \prec_r)$. If $F := A_{\mathcal{S}(n)} - E$, then*

$$F = \left\{ A \left[\sum_{i \in J_1} e_i, \ldots, \sum_{i \in J_n} e_i \right] P : P \in \mathcal{P}(n) \right\} = K,$$

where $J_1, \ldots, J_n$ is a partition of $\mathbb{N}_n$, and there exist $r, t \in \mathbb{N}_n, r \neq t$ and a minimal linearly dependent subset $S \subseteq \{a_1, \ldots, a_n\}$ such that $\mathbb{N}_S^+ \subseteq J_r$ and $\mathbb{N}_S^- \subseteq J_t$.

Proof. Let $AR_0 \in K$, where $R_0 \in \mathcal{S}(n)$. So, there exist $r, t \in \mathbb{N}_n, r \neq t$ and a minimal linearly dependent subset $S \subseteq \{a_1, \ldots, a_n\}$ such that $\mathbb{N}_S^+ \subseteq J_r$ and $\mathbb{N}_S^- \subseteq J_t$. We show that $AR_0 \in F$ or $AR_0 \notin E$.

Assume, $R_0 \in \mathcal{S}(n)$ and $a_i \neq 0$ for all $i = 1, \ldots, n$. We claim that, if there exists $R_0 \neq R \in \mathcal{RS}(n)$ such that $AR_0 = AR$, then $AR_0 \notin E$, i.e., $AR_0 \in F$.

Now, suppose that $R_0 \neq R \in \mathcal{RS}(n)$. So, at least one row of R must have at least two positive entries. Without loss of generality, assume that in the first row of R entries $r_{1j}, r_{1k} > 0$, where $j \neq k$. Put $\epsilon := min\left\{\frac{r_{1j}}{2}, \frac{r_{1k}}{2}\right\}$ and define $R_1 = [\lambda_{ij}]$, where $\lambda_{1j} := r_{1j} + \epsilon$, $\lambda_{1k} := r_{1k} - \epsilon$ and $\lambda_{ij} := r_{ij}$ otherwise. Also, put $R_2 = [\mu_{ij}]$, where

$$\mu_{ij} := \begin{cases} \mu_{1j} := r_{1j} - \epsilon & \\ \mu_{1k} := r_{1k} + \epsilon & \\ \quad r_{ij} & o.w. \end{cases}.$$

Therefore, $AR = \frac{1}{2}AR_1 + \frac{1}{2}AR_2$. Hence $AR_1 \neq AR_2$.

Existence: Let S be a minimal linearly dependent subset of $\{a_1, \ldots, a_n\}$. So, there exists $c_{i_1}, \ldots, c_{i_p} \in (0, 1)$ such that

$$\sum_{i_k \in \mathbb{N}_S^+} c_{i_k} a_{i_k} = \sum_{i_k \in \mathbb{N}_S^-} c_{i_k} a_{i_k}.$$

Put

$$r_{ij} := \begin{cases} 1 - c_{i_k} & j \in J_r \\ c_{i_k} & j \in J_t \\ c_{i_j} & o.w. \end{cases},$$

it follows that $R = [r_{ij}] \in \mathcal{RS}(n)$. □

4. Conclusion

In this paper, we characterized the maximal totally ordered subsets of $(M_{nm}, \prec)$. Next, we obtained a totally ordered subset of $(M_{nm}, \prec)$ that contains a given matrix A and then we generalized the Birkhoff's theorem. Also, we shown that the totality of all extreme points of the collection of all the matrices which are multivariately majorized by a matrix A is the set of all matrices obtained by permuting the columns of A.

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