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POSITIVE IMPLICATIVE TRUE-FALSE IDEALS IN BCK-ALGEBRAS

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Dedicated to sincere professor Mashaallah Mashinchi

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Abstract. In BCK-algebra, the concept of a positive implicative $T\&F$-ideal is introduced, and further several properties are investigated. The relationship between $T\&F$-ideals and positive implicative $T\&F$-ideals is established, and an example is given to reveal that a $T\&F$-ideal is not a positive implicative $T\&F$-ideal. Various conditions under which a $T\&F$-ideal can be a positive implicative $T\&F$-ideal are explored and various characterizations of a positive implicative $T\&F$-ideal are studied. The extended property of a positive implicative $T\&F$-ideal is constructed.

Keywords: True-False structure, (limited) $T\&F$-ideal, (limited) positive implicative $T\&F$-ideal.
2020 MSC: 03G25, 06F35, 03E72, 08A72.

1. Introduction

As an extension of the classical concept of the set, the fuzzy set was introduced by Zade in 1965 (see [12]). In mathematics, fuzzy sets are somewhat similar to sets in which elements have a degree of membership. As an extension of fuzzy sets, the interval valued fuzzy sets [4] have emerged and are being applied to several sides. Cubic sets, one of the hybrid structures by using both a fuzzy set and an interval valued fuzzy set at the same time, have been introduced by Jun et al. [7], and are currently being applied to various fields. Mohseni et al. [10] constructed True-False structures based on a fuzzy set and an interval valued fuzzy set, and then they studied the basic properties. They also applied it to groups and BCK/BCI-algebras at the same time, and used this structure to study ideal theory in BCK/BCI-algebras. For more study about BCK/BCI-algebras, see [5, 6, 8, 11] and for different extensions of fuzzy sets and ideals of BCK/BCI-algebra, see [1–3, 9].

In this manuscript, we introduce the concept of a positive implicative $T\&F$-ideal in BCK-algebra, and investigate several properties. We establish the relationship between a $T\&F$-ideal and a positive implicative $T\&F$-ideal. We

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show that a $T\&F$-ideal is not a positive implicative $T\&F$-ideal by giving an example. We explore various conditions under which a $T\&F$-ideal can be a positive implicative $T\&F$-ideal, and study various characterizations of a positive implicative $T\&F$-ideal. We construct an extended property of a positive implicative $T\&F$-ideal.

2. Preliminaries

2.1. Basic concepts about BCK/BCI-algebras. A *BCI-algebra* is defined to be an algebra $(X;*,0)$ that satisfies the following conditions:

(I) $(\forall \mathring{x},\mathring{y},\mathring{z}\in X)\ (((\mathring{x}*\mathring{y})*(\mathring{x}*\mathring{z}))*(\mathring{z}*\mathring{y})=0)$,
(II) $(\forall \mathring{x},\mathring{y}\in X)\ ((\mathring{x}*(\mathring{x}*\mathring{y}))*\mathring{y}=0)$,
(III) $(\forall \mathring{x}\in X)\ (\mathring{x}*\mathring{x}=0)$,
(IV) $(\forall \mathring{x},\mathring{y}\in X)\ (\mathring{x}*\mathring{y}=0,\ \mathring{y}*\mathring{x}=0\ \Rightarrow\ \mathring{x}=\mathring{y})$.

If a BCI-algebra X satisfies the following condition:

(V) $(\forall \mathring{x}\in X)\ (0*\mathring{x}=0)$,

then X is called a *BCK-algebra*. We define an order relation "$\leq$" on a BCK/BCI-algebra X as follows:

$$(\forall \mathring{x},\mathring{y}\in X)(\mathring{x}\leq\mathring{y}\ \Leftrightarrow\ \mathring{x}*\mathring{y}=0). \tag{1}$$

Every BCK/BCI-algebra X satisfies:

$$(\forall \mathring{x}\in X)(\mathring{x}*0=\mathring{x}), \tag{2}$$

$$(\forall \mathring{x},\mathring{y},\mathring{z}\in X)(\mathring{x}\leq\mathring{y}\ \Rightarrow\ \mathring{x}*\mathring{z}\leq\mathring{y}*\mathring{z},\ \mathring{z}*\mathring{y}\leq\mathring{z}*\mathring{x}), \tag{3}$$

$$(\forall \mathring{x},\mathring{y},\mathring{z}\in X)((\mathring{x}*\mathring{y})*\mathring{z}=(\mathring{x}*\mathring{z})*\mathring{y}). \tag{4}$$

A subset L of a BCK/BCI-algebra X is called an *ideal* of X (see [6, 8]) if it satisfies:

$$0\in L, \tag{5}$$

$$(\forall \mathring{x},\mathring{y}\in X)(\mathring{x}*\mathring{y}\in L,\ \mathring{y}\in L\ \Rightarrow\ \mathring{x}\in L). \tag{6}$$

A subset L of a BCK-algebra X is called a *positive implicative ideal* of X (see [6, 8]) if it satisfies (5) and

$$(\forall \mathring{x},\mathring{y},\mathring{z}\in X)((\mathring{x}*\mathring{y})*\mathring{z}\in L,\ \mathring{y}*\mathring{z}\in L\ \Rightarrow\ \mathring{x}*\mathring{z}\in L). \tag{7}$$

For more information on BCI-algebra and BCK-algebra, please refer to the books [5] and [8].

2.2. Basic concepts about True-False structures. Let U be a universal set. A *True-False structure* (briefly, *$T\&F$-structure*) over U (see [10]) is defined to be a pair $(U,\mathcal{A})$ with $\mathcal{A}:=(\varphi_A,\widetilde{\varphi}_A,\partial_A,\widetilde{\partial}_A)$ given by the following function:

$$\begin{aligned}\mathcal{A}:U&\to[0,1]\times int\,([0,1])\times[0,1]\times int\,([0,1])\,,\\ \mathring{x}&\mapsto(\varphi_A(\mathring{x}),\widetilde{\varphi}_A(\mathring{x}),\partial_A(\mathring{x}),\widetilde{\partial}_A(\mathring{x})),\end{aligned} \tag{8}$$

where $int\,([0,1])$ is the set of all sub-intervals of $[0,1]$.

A $T\&F$-structure $(U, \mathcal{A})$ over U with $\mathcal{A} := (\varphi_A, \widetilde{\varphi}_A, \partial_A, \widetilde{\partial}_A)$ is said to be *limited* (see [10]) if $\varphi_A(\mathring{x}) + \partial_A(\mathring{x}) \leq 1$ and $\sup \widetilde{\varphi}_A(\mathring{x}) + \sup \widetilde{\partial}_A(\mathring{x}) \leq 1$ for all $\mathring{x} \in U$.

By an *interval number*, we mean a closed subinterval $\tilde{a} = [a^-, a^+]$ of $I = [0, 1]$, where $0 \leq a^- \leq a^+ \leq 1$. The interval number $\tilde{a} = [a^-, a^+]$ with $a^- = a^+$ is denoted by $\mathbf{a}$. Denote by $[I]$ the set of all interval numbers. Let us define what is known as *refined minimum* (briefly, rmin) of two elements in $[I]$. We also define the symbols "$\succeq$", "$\preceq$", "$=$" in case of two elements in $[I]$. Consider two interval numbers $\tilde{a}_1 = [a_1^-, a_1^+]$ and $\tilde{a}_2 = [a_2^-, a_2^+]$. Then

$$\begin{aligned} &\operatorname{rmin}\{\tilde{a}_1, \tilde{a}_2\} = \left[\min\left\{a_1^-, a_2^-\right\}, \min\left\{a_1^+, a_2^+\right\}\right], \\ &\operatorname{rmax}\{\tilde{a}_1, \tilde{a}_2\} = \left[\max\left\{a_1^-, a_2^-\right\}, \max\left\{a_1^+, a_2^+\right\}\right], \\ &a_1 \succeq a_2 \ \text{ if and only if } \ a_1^- \geq a_2^- \ \text{ and } \ a_1^+ \geq a_2^+, \end{aligned}$$

and similarly we may have $\tilde{a}_1 \preceq \tilde{a}_2$ and $\tilde{a}_1 = \tilde{a}_2$. To say $\tilde{a}_1 \succ \tilde{a}_2$ (resp. $\tilde{a}_1 \prec \tilde{a}_2$) we mean $\tilde{a}_1 \succeq \tilde{a}_2$ and $\tilde{a}_1 \neq \tilde{a}_2$ (resp. $\tilde{a}_1 \preceq \tilde{a}_2$ and $\tilde{a}_1 \neq \tilde{a}_2$) (see [10]).

Let $\tilde{a}_i \in [I]$ where $i \in \Lambda$. We define

$$\operatorname*{rinf}_{i\in\Lambda} \tilde{a}_i = \left[\inf_{i\in\Lambda} a_i^-, \inf_{i\in\Lambda} a_i^+\right] \quad \text{and} \quad \operatorname*{rsup}_{i\in\Lambda} \tilde{a}_i = \left[\sup_{i\in\Lambda} a_i^-, \sup_{i\in\Lambda} a_i^+\right].$$

For any $\tilde{a} \in [I]$, its *complement*, denoted by $\tilde{a}^c$, is defined to be the interval number (see [10])

$$\tilde{a}^c = [1 - a^+, 1 - a^-].$$

Given a (limited) $T\&F$-structure $(U, \mathcal{A})$ over U with $\mathcal{A} := (\varphi_A, \widetilde{\varphi}_A, \partial_A, \widetilde{\partial}_A)$, consider the sets which are called *$T\&F$-level sets* of $\mathcal{A}$ over U:

$$\begin{aligned} &U(\varphi_A; \alpha) := \{\mathring{x} \in U \mid \varphi_A(\mathring{x}) \geq \alpha\}, \\ &U(\widetilde{\varphi}_A; \tilde{t}) := \{\mathring{x} \in U \mid \widetilde{\varphi}_A(\mathring{x}) \succcurlyeq \tilde{t}\}, \\ &L(\partial_A; \beta) := \{\mathring{x} \in U \mid \partial_A(\mathring{x}) \leq \beta\}, \\ &L(\widetilde{\partial}_A; \tilde{s}) := \{\mathring{x} \in U \mid \widetilde{\partial}_A(\mathring{x}) \preccurlyeq \tilde{s}\}, \\ &\mathcal{L}_{\mathcal{A}}(\alpha, \tilde{t}, \beta, \tilde{s}) := U(\varphi_A; \alpha) \cap U(\widetilde{\varphi}_A; \tilde{t}) \cap L(\partial_A; \beta) \cap L(\widetilde{\partial}_A; \tilde{s}), \end{aligned}$$

where $\alpha, \beta \in [0, 1]$ and $\tilde{t} = [t^-, t^+], \tilde{s} = [s^-, s^+] \in int([0, 1])$.

Note. We have to notice that the symbol $\frac{\mathring{x}}{\{\mathring{y}, \mathring{z}\}} \in \frac{X}{X \times X}$ means that $\mathring{x} \in X$ and $\{\mathring{y}, \mathring{z}\} \in X \times X$ such that the value of the true membership function (false) of the face expression is greater (less) than the minimum (maximum) of the denominator expression.

Consider a $T\&F$-structure $(X, \mathcal{A})$ over a set X with $\mathcal{A} := (\varphi_A, \widetilde{\varphi}_A, \partial_A, \widetilde{\partial}_A)$. We let the following sets (see [10]):

$$\Omega_T^{\varphi} := \{(\mathring{x}, \mathring{y}) \in X \times X : \varphi_A(\mathring{x}) \geq \varphi_A(\mathring{y}), \, \widetilde{\varphi}_A(\mathring{x}) \succcurlyeq \widetilde{\varphi}_A(\mathring{y})\}, \tag{9}$$

$$\Omega_F^{\partial} := \{(\mathring{x}, \mathring{y}) \in X \times X : \partial_A(\mathring{x}) \leq \partial_A(\mathring{y}), \, \widetilde{\partial}_A(\mathring{x}) \preccurlyeq \widetilde{\partial}_A(\mathring{y})\}, \tag{10}$$

$$\Omega^{\varphi}_{T(\min,\mathrm{rmin})} := \left\{ \tfrac{\mathring{x}}{\{\mathring{y},\mathring{z}\}} \in \tfrac{X}{X\times X} \;\middle|\; \begin{array}{l} \varphi_A(\mathring{x}) \geq \min\{\varphi_A(\mathring{y}), \varphi_A(\mathring{z})\} \\ \widetilde{\varphi}_A(\mathring{x}) \succcurlyeq \mathrm{rmin}\{\widetilde{\varphi}_A(\mathring{y}), \widetilde{\varphi}_A(\mathring{z})\}, \end{array} \right\} \tag{11}$$

and

$$\Omega^{\partial}_{F(\max,\mathrm{rmax})} := \left\{ \tfrac{\mathring{x}}{\{\mathring{y},\mathring{z}\}} \in \tfrac{X}{X\times X} \;\middle|\; \begin{array}{l} \partial_A(\mathring{x}) \leq \max\{\partial_A(\mathring{y}), \partial_A(\mathring{z})\} \\ \widetilde{\partial}_A(\mathring{x}) \preccurlyeq \mathrm{rmax}\{\widetilde{\partial}_A(\mathring{y}), \widetilde{\partial}_A(\mathring{z})\} \end{array} \right\}. \tag{12}$$

It is clear that

$$(\forall \mathring{x}, \mathring{y} \in X)\left((\mathring{x}, \mathring{y}) \in \Omega^{\varphi}_{T} \;\Leftrightarrow\; \tfrac{\mathring{x}}{\{\mathring{y},\mathring{y}\}} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \right), \tag{13}$$

$$(\forall \mathring{x}, \mathring{y}, \mathring{z} \in X)\left((\mathring{x}, \mathring{y}) \in \Omega^{\varphi}_{T},\, (\mathring{y}, \mathring{z}) \in \Omega^{\varphi}_{T} \;\Rightarrow\; (\mathring{x}, \mathring{z}) \in \Omega^{\varphi}_{T} \right), \tag{14}$$

$$(\forall \mathring{x}, \mathring{y}, \mathring{z} \in X)\left((\mathring{x}, \mathring{y}) \in \Omega^{\partial}_{F},\, (\mathring{y}, \mathring{z}) \in \Omega^{\partial}_{F} \;\Rightarrow\; (\mathring{x}, \mathring{z}) \in \Omega^{\partial}_{F} \right), \tag{15}$$

$$(\forall \mathring{x}, \mathring{y} \in X)\left((\mathring{x}, \mathring{y}) \in \Omega^{\partial}_{F} \;\Leftrightarrow\; \tfrac{\mathring{x}}{\{\mathring{y},\mathring{y}\}} \in \Omega^{\partial}_{F(\max,\mathrm{rmax})} \right), \tag{16}$$

$$(\forall \mathring{x}, \mathring{y}, \mathring{z} \in X)\left(\tfrac{\mathring{x}}{\{\mathring{y},\mathring{z}\}} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \;\Leftrightarrow\; \tfrac{\mathring{x}}{\{\mathring{z},\mathring{y}\}} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \right), \tag{17}$$

$$(\forall \mathring{x}, \mathring{y}, \mathring{z} \in X)\left(\tfrac{\mathring{x}}{\{\mathring{y},\mathring{z}\}} \in \Omega^{\partial}_{F(\max,\mathrm{rmax})} \;\Leftrightarrow\; \tfrac{\mathring{x}}{\{\mathring{z},\mathring{y}\}} \in \Omega^{\partial}_{F(\max,\mathrm{rmax})} \right). \tag{18}$$

Proposition 2.1 ([10]). *Let $(X, \mathcal{A})$ be a $T\&F$-structure over a set X with $\mathcal{A} := (\varphi_A,\, \widetilde{\varphi}_A,\, \partial_A,\, \widetilde{\partial}_A)$. For any $a, \mathring{x}, \mathring{y}, \mathring{z} \in X$, we have*

$$\begin{array}{l} \tfrac{a}{\{\mathring{x},\mathring{y}\}} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})},\, (\mathring{y}, \mathring{z}) \in \Omega^{\varphi}_{T} \;\Rightarrow\; \tfrac{a}{\{\mathring{x},\mathring{z}\}} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})}, \\ \tfrac{a}{\{\mathring{x},\mathring{y}\}} \in \Omega^{\partial}_{F(\max,\mathrm{rmax})},\, (\mathring{y}, \mathring{z}) \in \Omega^{\partial}_{F} \;\Rightarrow\; \tfrac{a}{\{\mathring{x},\mathring{z}\}} \in \Omega^{\partial}_{F(\max,\mathrm{rmax})}. \end{array} \tag{19}$$

Definition 2.2 ([10]). A $T\&F$-structure $(X, \mathcal{A})$ over a BCK/BCI-algebra X with $\mathcal{A} := (\varphi_A,\, \widetilde{\varphi}_A,\, \partial_A,\, \widetilde{\partial}_A)$ is called a *$T\&F$-ideal* of X if the following assertions are valid.

$$(\forall \mathring{x} \in X)\left((0, \mathring{x}) \in \Omega^{\varphi}_{T} \cap \Omega^{\partial}_{F} \right). \tag{20}$$

$$(\forall \mathring{x}, \mathring{y} \in X)\left(\tfrac{\mathring{x}}{\{\mathring{x}*\mathring{y},\, \mathring{y}\}} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial}_{F(\max,\mathrm{rmax})} \right). \tag{21}$$

If a $T\&F$-ideal is limited, then we say that it is a *limited $T\&F$-ideal*.

Proposition 2.3 ([10]). *Every $T\&F$-ideal $(X, \mathcal{A})$ of a BCK/BCI-algebra X satisfies:*

$$(\forall \mathring{x}, \mathring{y} \in X)\left(\mathring{x} \leq \mathring{y} \;\Rightarrow\; (\mathring{x}, \mathring{y}) \in \Omega^{\varphi}_{T} \cap \Omega^{\partial}_{F} \right). \tag{22}$$

$$(\forall \mathring{x}, \mathring{y}, \mathring{z} \in X)\left(\mathring{x} * \mathring{y} \leq \mathring{z} \;\Rightarrow\; \tfrac{\mathring{x}}{\{\mathring{y},\mathring{z}\}} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial}_{F(\max,\mathrm{rmax})} \right). \tag{23}$$

3. Positive implicative $T\&F$-ideals

Definition 3.1. A $T\&F$-structure $(X, \mathcal{A})$ over a BCK-algebra X with $\mathcal{A} := (\varphi_A,\, \widetilde{\varphi}_A,\, \partial_A,\, \widetilde{\partial}_A)$ is called a *positive implicative $T\&F$-ideal* of X if it satisfies (20) and

$$(\forall x, y, z \in X)\left(\tfrac{x*z}{\{(x*y)*z,\, y*z\}} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial}_{F(\max,\mathrm{rmax})} \right). \tag{24}$$

Example 3.2. *Consider a BCK-algebra* $X = \{0, 1, 2, 3, 4\}$ *with the binary operation "$*$" given as follows:*

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	1	3	0	3
4	4	4	4	4	0

Let $(X, \mathcal{A})$ *be a T&F-structure over* X *with* $\mathcal{A} := (\varphi_A,\ \widetilde{\varphi}_A,\ \partial_A,\ \widetilde{\partial}_A)$ *given as follows:*

X	$\varphi_A(x)$	$\widetilde{\varphi}_A(x)$	$\partial_A(x)$	$\widetilde{\partial}_A(x)$
0	0.66	$[0.38, 0.53]$	0.47	$[0.28, 0.37]$
1	0.66	$[0.38, 0.53]$	0.47	$[0.28, 0.37]$
2	0.55	$[0.27, 0.49]$	0.56	$[0.31, 0.49]$
3	0.66	$[0.38, 0.53]$	0.47	$[0.28, 0.37]$
4	0.44	$[0.25, 0.43]$	0.67	$[0.44, 0.75]$

It is routine to verify that

$$0.66 = \varphi_A(3) \geq \min\{\varphi_A(3), \varphi_A(4)\} = \min\{0.66, 0.44\} = 0.44,$$
$$[0.38, 0.53] = \widetilde{\varphi}_A(3) \succcurlyeq \mathrm{rmin}\{\widetilde{\varphi}_A(3), \widetilde{\varphi}_A(4)\} = \mathrm{rmin}\{[0.38, 0.53], [0.25, 0.43]\} = [0.25, 0.43],$$

and

$$0.47 = \partial_A(3) \leq \max\{\partial_A(3), \partial_A(4)\} \leq \max\{0.47, 0.67\} = 0.67$$
$$[0.28, 0.37] = \widetilde{\partial}_A(3) \preccurlyeq \mathrm{rmax}\{\widetilde{\partial}_A(3), \widetilde{\partial}_A(4)\} = \mathrm{rmax}\{[0.28, 0.37], [0.44, 0.75]\} = [0.44, 0.75].$$

Similarly we can calculate all $\Omega^{\varphi}_{T(\min, r\min)}, \Omega^{\partial}_{F(\max, r\max)}$, *for any* $x \in X$. *Clearly,* $(X, \mathcal{A})$ *is a positive implicative T&F-ideal of* X *which is not limited.*

Example 3.3. *Given a positive implicative ideal* L *of a BCK-algebra* X, *define a T&F-structure* $(X, \mathcal{A})$ *over* X *as follows:*

$$\mathcal{A} : X \to [0,1] \times int([0,1]) \times [0,1] \times int([0,1]),$$
$$x \mapsto \begin{cases} (\alpha, \tilde{t}, \beta, \tilde{s}) & \text{if } x \in L, \\ (1, \tilde{1}, 0, \tilde{0}) & \text{otherwise} \end{cases}$$

where $(\alpha, \beta) \in (0, 1] \times [0, 1)$ *and* $(\tilde{t}, \tilde{s}) \in int((0, 1]) \times int([0, 1))$. *According to definition of* $\mathcal{A}$, *we have,*
If $x, y, z \in L$, *then*

$$\alpha = \varphi_A(x) \geq \min\{\varphi_A(y), \varphi_A(z)\} = \min\{\alpha, \alpha\} = \alpha,$$
$$\tilde{t} = \widetilde{\varphi}_A(x) \succcurlyeq \mathrm{rmin}\{\widetilde{\varphi}_A(y), \widetilde{\varphi}_A(z)\} = \mathrm{rmin}\{\tilde{t}, \tilde{t}\} = \tilde{t},$$

and

$$\beta = \partial_A(x) \leq \max\{\partial_A(y), \partial_A(z)\} \leq \max\{\beta, \beta\} = \beta,$$
$$\tilde{s} = \widetilde{\partial}_A(x) \preccurlyeq \text{rmax}\{\widetilde{\partial}_A(y), \widetilde{\partial}_A(z)\} = \text{rmax}\{\tilde{s}, \tilde{s}\} = \tilde{s}.$$

Otherwise, if $x \notin L$ and $y, z \in L$, we have

$$1 = \varphi_A(x) \geq \min\{\varphi_A(y), \varphi_A(z)\} = \min\{\alpha, \alpha\} = \alpha,$$
$$\tilde{1} = \widetilde{\varphi}_A(x) \succcurlyeq \text{rmin}\{\widetilde{\varphi}_A(y), \widetilde{\varphi}_A(z)\} = \text{rmin}\{\tilde{t}, \tilde{t}\} = \tilde{t},$$

and

$$0 = \partial_A(x) \leq \max\{\partial_A(y), \partial_A(z)\} \leq \max\{\beta, \beta\} = \beta,$$
$$\tilde{0} = \widetilde{\partial}_A(x) \preccurlyeq \text{rmax}\{\widetilde{\partial}_A(y), \widetilde{\partial}_A(z)\} = \text{rmax}\{\tilde{s}, \tilde{s}\} = \tilde{s}.$$

If $x, y, z \notin L$, then we have

$$1 = \varphi_A(x) \geq \min\{\varphi_A(y), \varphi_A(z)\} = \min\{0, 0\} = 0,$$
$$\tilde{1} = \widetilde{\varphi}_A(x) \succcurlyeq \text{rmin}\{\widetilde{\varphi}_A(y), \widetilde{\varphi}_A(z)\} = \text{rmin}\{\tilde{0}, \tilde{0}\} = \tilde{0},$$

and

$$0 = \partial_A(x) \leq \max\{\partial_A(y), \partial_A(z)\} \leq \max\{1, 1\} = 1,$$
$$\tilde{0} = \widetilde{\partial}_A(x) \preccurlyeq \text{rmax}\{\widetilde{\partial}_A(y), \widetilde{\partial}_A(z)\} = \text{rmax}\{\tilde{1}, \tilde{1}\} = \tilde{1}.$$

Similarly we can calculate other cases. Obviously, $(X, \mathcal{A})$ is a positive implicative $T\&F$-ideal of X.

We establish a relationship between a $T\&F$-ideal and a positive implicative $T\&F$-ideal in BCK-algebras.

Theorem 3.4. *In a BCK-algebra, every positive implicative $T\&F$-ideal is a $T\&F$-ideal.*

Proof. Let $(X, \mathcal{A})$ be a positive implicative $T\&F$-ideal of a BCK-algebra X. Using (2) and (24), we have

$$\tfrac{x}{\{x*y,\, y\}} = \tfrac{x*0}{\{(x*y)*0,\, y*0\}} \in \Omega^{\varphi}_{T(\min,\text{rmin})} \cap \Omega^{\partial}_{F(\max,\text{rmax})},$$

for all $x, y \in X$. Therefore $(X, \mathcal{A})$ is a $T\&F$-ideal of X. □

The following example shows that any $T\&F$-ideal may not be a positive implicative $T\&F$-ideal.

Example 3.5. *Consider a BCK-algebra $X = \{0, 1, 2, 3\}$ with the binary operation "$*$" given as follows:*

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

Let $(X, \mathcal{A})$ be a T&F-structure over X with $\mathcal{A} := (\varphi_A, \widetilde{\varphi}_A, \partial_A, \widetilde{\partial}_A)$ given as follows:

X	$\varphi_A(x)$	$\widetilde{\varphi}_A(x)$	$\partial_A(x)$	$\widetilde{\partial}_A(x)$
0	0.76	$[0.47, 0.63]$	0.22	$[0.23, 0.39]$
1	0.56	$[0.38, 0.53]$	0.37	$[0.34, 0.47]$
2	0.56	$[0.38, 0.53]$	0.37	$[0.34, 0.47]$
3	0.46	$[0.26, 0.47]$	0.58	$[0.39, 0.66]$

It is routine to verify that $(X, \mathcal{A})$ is a T&F-ideal of X. But it is not a positive implicative T&F-ideal of X since $\frac{2*1}{\{(2*1)*1,\, 1*1\}} \notin \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial}_{F(\max,\mathrm{rmax})}$.

We provide conditions for a T&F-ideal to be a positive implicative T&F-ideal.

Theorem 3.6. *Let $(X, \mathcal{A})$ be a T&F-ideal of a BCK-algebra X. Then the following are equivalent.*

(i) *$(X, \mathcal{A})$ is a positive implicative T&F-ideal of X.*

(ii) *$(X, \mathcal{A})$ satisfies:*

$$(\forall x, y \in X)\left((x * y,\, (x * y) * y) \in \Omega^{\varphi}_{T} \cap \Omega^{\partial}_{F}\right). \tag{25}$$

(iii) *$(X, \mathcal{A})$ satisfies:*

$$(\forall x, y, z \in X)\left(((x * z) * (y * z),\, (x * y) * z) \in \Omega^{\varphi}_{T} \cap \Omega^{\partial}_{F}\right). \tag{26}$$

Proof. (i) $\Rightarrow$ (ii). Assume that $(X, \mathcal{A})$ is a positive implicative T&F-ideal of X. If we replace z with y in (24) and use (III), then

$$\frac{x*y}{\{(x*y)*y,\, 0\}} = \frac{x*y}{\{(x*y)*y,\, y*y\}} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial}_{F(\max,\mathrm{rmax})}.$$

Since $(0, (x * y) * y) \in \Omega^{\varphi}_{T} \cap \Omega^{\partial}_{F}$ by (20), it follows from Proposition 2.1 and (16) that $(x * y,\, (x * y) * y) \in \Omega^{\varphi}_{T} \cap \Omega^{\partial}_{F}$. Hence (ii) is valid.

(ii) $\Rightarrow$ (iii). Let $(X, \mathcal{A})$ be a T&F-ideal of X that satisfies the condition (25). The combination of (I), (3) and (4) induces

$$((x * (y * z)) * z) * z = ((x * z) * (y * z)) * z \leq (x * y) * z,$$

for all $x, y, z \in X$. It follows from (22) that

$$(((x * (y * z)) * z) * z,\, (x * y) * z) \in \Omega^{\varphi}_{T} \cap \Omega^{\partial}_{F}.$$

Since $((x * (y * z)) * z,\, ((x * (y * z)) * z) * z) \in \Omega^{\varphi}_{T} \cap \Omega^{\partial}_{F}$ by (25), we have

$$((x * z) * (y * z),\, (x * y) * z) = ((x * (y * z)) * z,\, (x * y) * z) \in \Omega^{\varphi}_{T} \cap \Omega^{\partial}_{F},$$

by (4), (14) and (15).

(iii) $\Rightarrow$ (i). Let $(X, \mathcal{A})$ be a T&F-ideal of X that satisfies the condition (26). Then

$$\frac{x*z}{\{(x*z)*(y*z),\, y*z\}} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial}_{F(\max,\mathrm{rmax})},$$

for all $x, y, z \in X$ by (21). It follows from (17), (18), Proposition 2.1 and (26) that

$$\frac{x*z}{\{(x*y)*z,\, y*z\}} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial}_{F(\max,\mathrm{rmax})}.$$

Therefore $(X, \mathcal{A})$ is a positive implicative $T\&F$-ideal of X. □

Theorem 3.7. *A $T\&F$-structure $(X, \mathcal{A})$ over a BCK-algebra X is a positive implicative $T\&F$-ideal of X if and only if it satisfies the condition* (20) *and*

$$(27) \qquad (\forall x, y, z \in X) \left(\frac{x*y}{\{((x*y)*y)*z,\, z\}} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial}_{F(\max,\mathrm{rmax})} \right).$$

Proof. If $(X, \mathcal{A})$ is a positive implicative $T\&F$-ideal of X, then $(X, \mathcal{A})$ is a $T\&F$-ideal of X (see Theorem 3.4) and the condition (20) is obviously established. Using (4) and (21) leads to

$$(28) \qquad \frac{x*y}{\{(x*z)*y,\, z\}} = \frac{x*y}{\{(x*y)*z,\, z\}} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial}_{F(\max,\mathrm{rmax})},$$

for all $x, y, z \in X$. Since $((x*z)*y,\, ((x*z)*y)*y) \in \Omega^{\varphi}_{T} \cap \Omega^{\partial}_{F}$ by (25), it follows from (4) and Proposition 2.1 that

$$\frac{x*y}{\{((x*y)*y)*z,\, z\}} = \frac{x*y}{\{((x*z)*y)*y,\, z\}} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial}_{F(\max,\mathrm{rmax})}.$$

Conversely, suppose that a $T\&F$-structure $(X, \mathcal{A})$ over X satisfies the conditions (20) and (27). Then

$$\frac{x}{\{x*z,\, z\}} = \frac{x*0}{\{((x*0)*0)*z,\, z\}} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial}_{F(\max,\mathrm{rmax})},$$

for all $x, z \in X$. Thus $(X, \mathcal{A})$ is a $T\&F$-ideal of X. After we replace z with 0 in (27), we use (2) to obtain

$$\frac{x*y}{\{(x*y)*y,\, 0\}} = \frac{x*y}{\{((x*y)*y)*0,\, 0\}} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial}_{F(\max,\mathrm{rmax})},$$

for all $x, y \in X$. If you combine this with (20) and use Proposition 2.1, then we have

$$\frac{x*y}{\{(x*y)*y,\, (x*y)*y\}} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial}_{F(\max,\mathrm{rmax})},$$

that is, $(x*y,\, (x*y)*y) \in \Omega^{\varphi}_{T} \cap \Omega^{\partial}_{F}$. Therefore $(X, \mathcal{A})$ is a positive implicative $T\&F$-ideal of X by Theorem 3.6. □

Lemma 3.8 ([11]). *If a $T\&F$-structure $(X, \mathcal{A})$ over a BCK-algebra X satisfies the condition* (23)*, then $(X, \mathcal{A})$ is a $T\&F$-ideal of X.*

Theorem 3.9. *A $T\&F$-structure $(X, \mathcal{A})$ over a BCK-algebra X is a positive implicative $T\&F$-ideal of X if and only if it satisfies:*

$$(29) \qquad ((x*y)*y)*a \le b \;\Rightarrow\; \frac{x*y}{\{a, b\}} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial}_{F(\max,\mathrm{rmax})},$$

for all $x, y, a, b \in X$.

Proof. Assume that $(X, \mathcal{A})$ is a positive implicative $T\&F$-ideal of X. Then $(X, \mathcal{A})$ is a $T\&F$-ideal of X by Theorem 3.4. Let $x, y, a, b \in X$ be such that $((x * y) * y) * a \leq b$. Then

$$\tfrac{(x*y)*y}{\{a,\, b\}} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial}_{F(\max,\mathrm{rmax})},$$

by (23). If we combine this with (25), then

$$\tfrac{x*y}{\{a,\, b\}} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial}_{F(\max,\mathrm{rmax})}.$$

Conversely, suppose that $(X, \mathcal{A})$ satisfies the condition (29). Let $x, a, b \in X$ be such that $x * a \leq b$. Then $(x * a) * b = 0$, and so $(((x * 0) * 0) * a) * b = 0$, that is, $((x * 0) * 0) * a \leq b$. It follows from (2) and (29) that

$$\tfrac{x}{\{a,\, b\}} = \tfrac{x*0}{\{a,\, b\}} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial}_{F(\max,\mathrm{rmax})}.$$

Hence $(X, \mathcal{A})$ is a $T\&F$-ideal of X by Lemma 3.8. Since

$$(((x * y) * y) * ((x * y) * y)) * 0 = 0$$

for all $x, y \in X$, we have $\tfrac{x*y}{\{(x*y)*y,\, 0\}} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial}_{F(\max,\mathrm{rmax})}$ by (29). Since $(0,\, (x * y) * y) \in \Omega^{\varphi}_{T} \cap \Omega^{\partial}_{F}$ by (20), it follows from Proposition 2.1 that

$$\tfrac{x*y}{\{(x*y)*y,\, (x*y)*y\}} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial}_{F(\max,\mathrm{rmax})},$$

that is, $(x * y,\, (x * y) * y) \in \Omega^{\varphi}_{T} \cap \Omega^{\partial}_{F}$. Therefore $(X, \mathcal{A})$ is a positive implicative $T\&F$-ideal of X by Theorem 3.6. □

Theorem 3.10. *A $T\&F$-structure $(X, \mathcal{A})$ over a BCK-algebra X is a positive implicative $T\&F$-ideal of X if and only if it satisfies:*

(30) $$((x * y) * z) * a \leq b \;\Rightarrow\; \tfrac{(x*z)*(y*z)}{\{a,\, b\}} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial}_{F(\max,\mathrm{rmax})},$$

for all $x, y, z, a, b \in X$.

Proof. Let $(X, \mathcal{A})$ be a positive implicative $T\&F$-ideal of X. Then $(X, \mathcal{A})$ is a $T\&F$-ideal of X by Theorem 3.4. Let $x, y, z, a, b \in X$ be such that $((x * y) * z) * a \leq b$. Then

$$\tfrac{(x*y)*z}{\{a,\, b\}} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial}_{F(\max,\mathrm{rmax})},$$

by (23). If we combine this with (26), then

$$\tfrac{(x*z)*(y*z)}{\{a,\, b\}} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial}_{F(\max,\mathrm{rmax})}.$$

Conversely, suppose that $(X, \mathcal{A})$ satisfies the condition (30) and let $x, y, a, b \in X$ be such that $((x * y) * y) * a \leq b$. Then

$$\tfrac{x*y}{\{a,\, b\}} = \tfrac{(x*y)*(y*y)}{\{a,\, b\}} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial}_{F(\max,\mathrm{rmax})}.$$

Therefore, $(X, \mathcal{A})$ is a positive implicative $T\&F$-ideal of X by Theorem 3.9. □

Summarizing the above results, we obtain a characterization of positive implicative $T\&F$-ideals.

Theorem 3.11. *Let $(X, \mathcal{A})$ be a $T\&F$-structure over a BCK-algebra X. Then the following arguments are equivalent:*

(i) *$(X, \mathcal{A})$ is a positive implicative $T\&F$-ideal of X.*
(ii) *$(X, \mathcal{A})$ is a $T\&F$-ideal of X satisfying the condition* (25).
(iii) *$(X, \mathcal{A})$ is a $T\&F$-ideal of X satisfying the condition* (26).
(iv) *$(X, \mathcal{A})$ satisfies the conditions* (20) *and* (27).
(v) *$(X, \mathcal{A})$ satisfies the condition* (29).
(vi) *$(X, \mathcal{A})$ satisfies the condition* (30).

Given a $T\&F$-structure $(X, \mathcal{A})$ over a set X with $\mathcal{A} := (\varphi_A, \widetilde{\varphi}_A, \partial_A, \widetilde{\partial}_A)$ and a natural number k, we consider the following sets:

$$\Omega^{\varphi(k)}_{T(\min,\mathrm{rmin})} := \left\{ \tfrac{\mathring{x}}{\{\mathring{y}_1,\mathring{y}_2,\cdots,\mathring{y}_k\}} \in \tfrac{X}{X^k} \;\middle|\; \begin{array}{l} \varphi_A(\mathring{x}) \geq \min\{\varphi_A(\mathring{y}_1), \varphi_A(\mathring{y}_2), \cdots, \varphi_A(\mathring{y}_k)\} \\ \widetilde{\varphi}_A(\mathring{x}) \succcurlyeq \mathrm{rmin}\{\widetilde{\varphi}_A(\mathring{y}_1), \widetilde{\varphi}_A(\mathring{y}_2), \cdots, \widetilde{\varphi}_A(\mathring{y}_k)\} \end{array} \right\},$$

and

$$\Omega^{\partial(k)}_{F(\max,\mathrm{rmax})} := \left\{ \tfrac{\mathring{x}}{\{\mathring{y}_1,\mathring{y}_2,\cdots,\mathring{y}_k\}} \in \tfrac{X}{X^k} \;\middle|\; \begin{array}{l} \partial_A(\mathring{x}) \leq \max\{\partial_A(\mathring{y}_1), \partial_A(\mathring{y}_2), \cdots, \partial_A(\mathring{y}_k)\} \\ \widetilde{\partial}_A(\mathring{x}) \preccurlyeq \mathrm{rmax}\{\widetilde{\partial}_A(\mathring{y}_1), \widetilde{\partial}_A(\mathring{y}_2), \cdots, \widetilde{\partial}_A(\mathring{y}_k)\} \end{array} \right\}.$$

Lemma 3.12. *A $T\&F$-structure $(X, \mathcal{A})$ over a BCK-algebra X is a $T\&F$-ideal of X if and only if it satisfies:*

$$x * \prod_{i=1}^{k} a_i = 0 \;\Rightarrow\; \tfrac{x}{\{a_1,a_2,\cdots,a_k\}} \in \Omega^{\varphi(k)}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial(k)}_{F(\max,\mathrm{rmax})}, \tag{31}$$

for all $x, a_1, a_2, \cdots, a_k \in X$, where $x * \prod_{i=1}^{k} a_i = (\cdots (x * a_1) * \cdots) * a_k$.

Proof. Since a $T\&F$-structure $(X, \mathcal{A})$ is a $T\&F$-ideal of X if and only if it satisfies:

$$(\forall x, a, b \in X)\left((x * a) * b = 0 \;\Rightarrow\; \tfrac{x}{\{a,\, b\}} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial}_{F(\max,\mathrm{rmax})}\right),$$

this lemma is demonstrated by inductive methods. □

In the theorems below, we investigate that Theorems 3.9 and 3.10 are expressed in a more general form.

Theorem 3.13. *A $T\&F$-structure $(X, \mathcal{A})$ over a BCK-algebra X is a positive implicative $T\&F$-ideal of X if and only if it satisfies:*

$$\tfrac{x*y}{\{a_1,a_2,\cdots,a_k\}} \in \Omega^{\varphi(k)}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial(k)}_{F(\max,\mathrm{rmax})}, \tag{32}$$

for all $x, y, a_1, a_2, \cdots, a_k \in X$ with $((x * y) * y) * \prod_{i=1}^{k} a_i = 0$.

Proof. Let $(X, \mathcal{A})$ be a positive implicative $T\&F$-ideal of X. Then $(X, \mathcal{A})$ is a $T\&F$-ideal of X by Theorem 3.4. Suppose that $((x*y)*y)*\prod_{i=1}^{k} a_i = 0$ for all $x, y, a_1, a_2, \cdots, a_k \in X$. Using Lemma 3.12, we have

$$\tfrac{(x*y)*y}{\{a_1,a_2,\cdots,a_k\}} \in \Omega^{\varphi(k)}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial(k)}_{F(\max,\mathrm{rmax})}.$$

If we combine this with (25), then

$$\tfrac{x*y}{\{a_1,a_2,\cdots,a_k\}} \in \Omega^{\varphi(k)}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial(k)}_{F(\max,\mathrm{rmax})},$$

which proves (32).

Conversely, suppose that $(X, \mathcal{A})$ satisfies the condition (32). Let $x, y, a, b \in X$ be such that $((x*y)*y)*a \leq b$. Then $(((x*y)*y)*a)*b = 0$ and so

$$\tfrac{x*y}{\{a_1,\, a_2\}} \in \Omega^{\varphi(2)}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial(2)}_{F(\max,\mathrm{rmax})} = \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial}_{F(\max,\mathrm{rmax})}.$$

It follows from Theorem 3.9 that $(X, \mathcal{A})$ is a positive implicative $T\&F$-ideal of X. □

Theorem 3.14. *A $T\&F$-structure $(X, \mathcal{A})$ over a BCK-algebra X is a positive implicative $T\&F$-ideal of X if and only if it satisfies:*

$$\tfrac{(x*z)*(y*z)}{\{a_1,a_2,\cdots,a_k\}} \in \Omega^{\varphi(k)}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial(k)}_{F(\max,\mathrm{rmax})}, \tag{33}$$

for all $x, y, z, a_1, a_2, \cdots, a_k \in X$ with $((x*y)*z)*\prod_{i=1}^{k} a_i = 0$.

Using the notion of $T\&F$-level sets, we provide a characteization of a positive implicative $T\&F$-ideal.

Theorem 3.15. *Let $(X, \mathcal{A})$ be a $T\&F$-structure over a BCK-algebra X with $\mathcal{A} := (\varphi_A, \widetilde{\varphi}_A, \partial_A, \widetilde{\partial}_A)$. Then $(X, \mathcal{A})$ is a positive implicative $T\&F$-ideal of X if and only if the nonempty $T\&F$-level sets $U(\varphi_A; \alpha)$, $U(\widetilde{\varphi}_A; \tilde{t})$, $L(\partial_A; \beta)$ and $L(\widetilde{\partial}_A; \tilde{s})$ of $\mathcal{A}$ over X are positive implicative ideals of X for all $\alpha, \beta \in [0, 1]$ and $\tilde{t}, \tilde{s} \in int([0, 1])$.*

Proof. Suppose $(X, \mathcal{A})$ is a positive implicative $T\&F$-ideal of X, and choose $\alpha, \beta \in [0, 1]$ and $\tilde{t}, \tilde{s} \in int([0, 1])$ where $U(\varphi_A; \alpha)$, $U(\widetilde{\varphi}_A; \tilde{t})$, $L(\partial_A; \beta)$ and $L(\widetilde{\partial}_A; \tilde{s})$ are nonempty sets. Then there exist $x, y, a, b \in X$ such that $x \in U(\varphi_A; \alpha)$, $y \in U(\widetilde{\varphi}_A; \tilde{t})$, $a \in L(\partial_A; \beta)$ and $b \in L(\widetilde{\partial}_A; \tilde{s})$. It follows from (20) that $\varphi_A(0) \geq \varphi_A(x) \geq \alpha$, $\widetilde{\varphi}_A(0) \succcurlyeq \widetilde{\varphi}_A(y) \succcurlyeq \tilde{t}$, $\partial_A(0) \leq \partial_A(a) \leq \beta$ and $\widetilde{\partial}_A(0) \preccurlyeq \widetilde{\partial}_A(b) \preccurlyeq \tilde{s}$, that is, $0 \in U(\varphi_A; \alpha) \cap U(\widetilde{\varphi}_A; \tilde{t}) \cap L(\partial_A; \beta) \cap L(\widetilde{\partial}_A; \tilde{s})$. Let $x, y, z \in X$ be such that

$$(x*y)*z \in U(\varphi_A; \alpha) \cap U(\widetilde{\varphi}_A; \tilde{t}) \cap L(\partial_A; \beta) \cap L(\widetilde{\partial}_A; \tilde{s}),$$

and

$$y*z \in U(\varphi_A; \alpha) \cap U(\widetilde{\varphi}_A; \tilde{t}) \cap L(\partial_A; \beta) \cap L(\widetilde{\partial}_A; \tilde{s}).$$

Then $\varphi_A((x*y)*z) \geq \alpha$, $\varphi_A(y*z) \geq \alpha$, $\widetilde{\varphi}_A((x*y)*z) \succcurlyeq \tilde{t}$, $\widetilde{\varphi}_A(y*z) \succcurlyeq \tilde{t}$, $\partial_A((x*y)*z) \leq \beta$, $\partial_A(y*z) \leq \beta$, $\widetilde{\partial}_A((x*y)*z) \preccurlyeq \tilde{s}$, and $\widetilde{\partial}_A(y*z) \preccurlyeq \tilde{s}$. Using (21), we have

$$\begin{aligned}
&\varphi_A(x*z) \geq \min\{\varphi_A((x*y)*z), \varphi_A(y*z)\} \geq \alpha, \\
&\widetilde{\varphi}_A(x*z) \succcurlyeq \mathrm{rmin}\{\widetilde{\varphi}_A((x*y)*z), \widetilde{\varphi}_A(y*z)\} \succcurlyeq \tilde{t}, \\
&\partial_A(x*z) \leq \max\{\partial_A((x*y)*z), \partial_A(y*z)\} \leq \beta, \\
&\widetilde{\partial}_A(x*z) \preccurlyeq \mathrm{rmax}\{\widetilde{\partial}_A((x*y)*z), \widetilde{\partial}_A(y*z)\} \preccurlyeq \tilde{s},
\end{aligned}$$

and so $x*z \in U(\varphi_A;\alpha) \cap U(\widetilde{\varphi}_A;\tilde{t}) \cap L(\partial_A;\beta) \cap L(\widetilde{\partial}_A;\tilde{s})$. Therefore $U(\varphi_A;\alpha)$, $U(\widetilde{\varphi}_A;\tilde{t})$, $L(\partial_A;\beta)$ and $L(\widetilde{\partial}_A;\tilde{s})$ are positive implicative ideals of X for all $\alpha,\beta \in [0,1]$ and $\tilde{t},\tilde{s} \in int([0,1])$.

Conversely, suppose that the nonempty $T\&F$-level sets $U(\varphi_A;\alpha)$, $U(\widetilde{\varphi}_A;\tilde{t})$, $L(\partial_A;\beta)$ and $L(\widetilde{\partial}_A;\tilde{s})$ of $\mathcal{A}$ over X are positive implicative ideals of X for all $\alpha,\beta \in [0,1]$ and $\tilde{t},\tilde{s} \in int([0,1])$. If there exist $a,b,c,d \in X$ such that $\varphi_A(0) < \varphi_A(a)$, $\widetilde{\varphi}_A(0) \prec \widetilde{\varphi}_A(b)$, $\partial_A(0) > \varphi_A(c)$, and $\widetilde{\partial}_A(0) \succ \widetilde{\partial}_A(d)$. Then $\varphi_A(0) < \alpha \leq \varphi_A(a)$, $\widetilde{\varphi}_A(0) < \tilde{t} \preccurlyeq \widetilde{\varphi}_A(a)$, $\partial_A(0) > \beta \geq \varphi_A(c)$ and $\widetilde{\partial}_A(0) > \tilde{s} \succcurlyeq \widetilde{\varphi}_A(c)$ where $\alpha := \frac{1}{2}(\varphi_A(0)+\varphi_A(a))$, $\tilde{t} := \frac{1}{2}(\widetilde{\varphi}_A(0)+\widetilde{\varphi}_A(b))$, $\beta := \frac{1}{2}(\partial_A(0)+\partial_A(c))$ and $\tilde{s} := \frac{1}{2}(\widetilde{\partial}_A(0)+\widetilde{\partial}_A(d))$. Hence $0 \notin U(\varphi_A;\alpha) \cap U(\widetilde{\varphi}_A;\tilde{t}) \cap L(\partial_A;\beta) \cap L(\widetilde{\partial}_A;\tilde{s})$ which is a contradiction. Thus $(0,x) \in \Omega_T^{\varphi} \cap \Omega_F^{\partial}$ for all $x \in X$.

If (24) is false, then there exist $a,b,c \in X$ such that

$$\tfrac{a*c}{\{(a*b)*c,\, b*c\}} \notin \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial}_{F(\max,\mathrm{rmax})}.$$

Then $\frac{a*c}{\{(a*b)*c,\, b*c\}} \in \frac{X}{X\times X} \setminus \left(\Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cup \Omega^{\partial}_{F(\max,\mathrm{rmax})}\right)$,

$$\tfrac{a*c}{\{(a*b)*c,\, b*c\}} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \setminus \Omega^{\partial}_{F(\max,\mathrm{rmax})},$$

or

$$\tfrac{a*c}{\{(a*b)*c,\, b*c\}} \in \Omega^{\partial}_{F(\max,\mathrm{rmax})} \setminus \Omega^{\varphi}_{T(\min,\mathrm{rmin})}.$$

If $\frac{a*c}{\{(a*b)*c,\, b*c\}} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \setminus \Omega^{\partial}_{F(\max,\mathrm{rmax})}$, then $\frac{a*c}{\{(a*b)*c,\, b*c\}} \notin \Omega^{\partial}_{F(\max,\mathrm{rmax})}$ and so $\partial_A(a*c) > \max\{\partial_A((a*b)*c), \partial_A(b*c)\}$ or

$$\widetilde{\partial}_A(a*c) \succ \mathrm{rmax}\{\widetilde{\partial}_A((a*b)*c), \widetilde{\partial}_A(b*c)\}.$$

It follows that $(a*b)*c \in L(\partial_A;\beta)$ and $b*c \in L(\partial_A;\beta)$, but $a*c \notin L(\partial_A;\beta)$ for $\beta := \max\{\partial_A((a*b)*c), \partial_A(a*c)\}$; or $(a*b)*c \in L(\widetilde{\partial}_A;\tilde{s})$ and $b*c \in L(\widetilde{\partial}_A;\tilde{s})$, but $a*c \notin L(\widetilde{\partial}_A;\tilde{s})$ for $\tilde{s} := \mathrm{rmax}\{\widetilde{\partial}_A((a*b)*c), \widetilde{\partial}_A(b*c)\}$. This is a contradiction. Now, if $\frac{a*c}{\{(a*b)*c,\, b*c\}} \in \Omega^{\partial}_{F(\max,\mathrm{rmax})} \setminus \Omega^{\varphi}_{T(\min,\mathrm{rmin})}$, then $\frac{a*c}{\{(a*b)*c,\, b*c\}} \notin \Omega^{\varphi}_{T(\min,\mathrm{rmin})}$. Hence $\varphi_A(a*c) < \min\{\varphi_A((a*b)*c), \varphi_A(b*c)\}$ or $\widetilde{\varphi}_A(a*c) \prec \mathrm{rmin}\{\widetilde{\varphi}_A((a*b)*c), \widetilde{\varphi}_A(b*c)\}$, It follows that $(a*b)*c \in U(\varphi_A;\alpha)$, $b*c \in U(\varphi_A;\alpha)$, and $a*c \notin U(\varphi_A;\alpha)$ for $\alpha := \min\{\varphi_A((a*b)*c), \varphi_A(b*c)\}$; or $(a*b)*c \in U(\widetilde{\varphi}_A;\tilde{t})$ and $b*c \in U(\widetilde{\varphi}_A;\tilde{t})$, but $a*c \notin U(\widetilde{\varphi}_A;\tilde{t})$ for $\tilde{t} := \mathrm{rmin}\{\widetilde{\varphi}_A((a*b)*c), \widetilde{\varphi}_A(b*c)\}$. This is a contradiction.

Based on the above calculations, it is clear that the case of $\frac{a*c}{\{(a*b)*c,\, b*c\}} \in \frac{X}{X\times X} \setminus \left(\Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cup \Omega^{\partial}_{F(\max,\mathrm{rmax})}\right)$ also leads to a contradiction. Therefore, (24) is valid, and consequently $(X, \mathcal{A})$ is a positive implicative $T\&F$-ideal of X. □

Example 3.16. *According to Example 3.2, let $\alpha = 0.66$, $\beta = 0.55$ and $\tilde{t} = [0.27, 0.49]$, $\tilde{s} = [0.3, 0.5]$. Then $U(\varphi_A;\alpha) = \{0,1,3\}$, $U(\widetilde{\varphi}_A;\tilde{t}) = \{0,1,2,3\}$, $L(\partial_A;\beta) = \{2,4\}$ and $L(\widetilde{\partial}_A;\tilde{s}) = \{2,4\}$ of $\mathcal{A}$ over X that are positive implicative ideals of X.*

Corollary 3.17. *Let $(X, \mathcal{A})$ be a $T\&F$-structure over a BCK-algebra X with $\mathcal{A} := (\varphi_A, \widetilde{\varphi}_A, \partial_A, \widetilde{\partial}_A)$. Then $(X, \mathcal{A})$ is a positive implicative $T\&F$-ideal of X if and only if the sets $0_{\varphi_A} := \{x \in X \mid \varphi_A(x) = \varphi_A(0)\}$, $0_{\widetilde{\varphi}_A} := \{x \in X \mid \widetilde{\varphi}_A(x) = \widetilde{\varphi}_A(0)\}$, $0_{\partial_A} := \{x \in X \mid \partial_A(x) = \partial_A(0)\}$, and $0_{\widetilde{\partial}_A} := \{x \in X \mid \widetilde{\partial}_A(x) = \widetilde{\partial}_A(0)\}$ are positive implicative ideals of X.*

In Example 3.5, we see that a $T\&F$-ideal may not be a positive implicative $T\&F$-ideal. But we have the following extension property for positive implicative $T\&F$-ideals.

Theorem 3.18. *Let $(X, \mathcal{A})$ and $(X, \mathcal{B})$ be $T\&F$-ideals of X with $\mathcal{A} := (\varphi_A, \widetilde{\varphi}_A, \partial_A, \widetilde{\partial}_A)$ and $\mathcal{B} := (\varphi_B, \widetilde{\varphi}_B, \partial_B, \widetilde{\partial}_B)$, respectively, such that*

- (i) *$\varphi_A(0) = \varphi_B(0)$, $\widetilde{\varphi}_A(0) = \widetilde{\varphi}_B(0)$, $\partial_A(0) = \partial_B(0)$, $\widetilde{\partial}_A(0) = \widetilde{\partial}_B(0)$.*
- (ii) *$\varphi_A(x) \le \varphi_B(x)$, $\partial_A(x) \ge \partial_B(x)$, $\widetilde{\varphi}_A(x) \preccurlyeq \widetilde{\varphi}_B(x)$, $\widetilde{\partial}_A(x) \succcurlyeq \widetilde{\partial}_B(x)$ for all $x \in X$.*

If $(X, \mathcal{A})$ is a positive implicative $T\&F$-ideal of X, then so is $(X, \mathcal{B})$.

Proof. Assume that $(X, \mathcal{A})$ is a positive implicative $T\&F$-ideal of X and $x, y, z \in X$. Using the given conditions and (III), (4) and (26), we have

$$\begin{aligned}
\varphi_B(0) = \varphi_A(0) &= \varphi_A(((x*y)*z)*((x*y)*z)) \\
&= \varphi_A(((x*((x*y)*z))*y)*z) \\
&\le \varphi_A(((x*((x*y)*z))*z)*(y*z)) \\
&\le \varphi_B(((x*((x*y)*z))*z)*(y*z)) \\
&= \varphi_B(((x*z)*(y*z))*((x*y)*z)),
\end{aligned}$$

$$\begin{aligned}
\partial_B(0) = \partial_A(0) &= \partial_A(((x*y)*z)*((x*y)*z)) \\
&= \partial_A(((x*((x*y)*z))*y)*z) \\
&\ge \partial_A(((x*((x*y)*z))*z)*(y*z)) \\
&\ge \partial_B(((x*((x*y)*z))*z)*(y*z)) \\
&= \partial_B(((x*z)*(y*z))*((x*y)*z)),
\end{aligned}$$

$$\begin{aligned}\widetilde{\varphi}_B(0) &= \widetilde{\varphi}_A(0) = \widetilde{\varphi}_A(((x*y)*z)*((x*y)*z)) \\ &= \widetilde{\varphi}_A(((x*((x*y)*z))*y)*z) \\ &\preccurlyeq \widetilde{\varphi}_A(((x*((x*y)*z))*z)*(y*z)) \\ &\preccurlyeq \widetilde{\varphi}_B(((x*((x*y)*z))*z)*(y*z)) \\ &= \widetilde{\varphi}_B(((x*z)*(y*z))*((x*y)*z)),\end{aligned}$$

and

$$\begin{aligned}\widetilde{\partial}_B(0) &= \widetilde{\partial}_A(0) = \widetilde{\partial}_A(((x*y)*z)*((x*y)*z)) \\ &= \widetilde{\partial}_A(((x*((x*y)*z))*y)*z) \\ &\succcurlyeq \widetilde{\partial}_A(((x*((x*y)*z))*z)*(y*z)) \\ &\succcurlyeq \widetilde{\partial}_B(((x*((x*y)*z))*z)*(y*z)) \\ &= \widetilde{\partial}_B(((x*z)*(y*z))*((x*y)*z)).\end{aligned}$$

This shows that

(34) $$(((x*z)*(y*z))*((x*y)*z),\, 0) \in \Omega_T^{\varphi} \cap \Omega_F^{\partial}$$

where

$$\Omega_T^{\varphi} := \{(x,y) \in X \times X \mid \varphi_B(x) \geq \varphi_B(y),\, \widetilde{\varphi}_B(x) \succcurlyeq \widetilde{\varphi}_B(y)\},$$

$$\Omega_F^{\partial} := \{(x,y) \in X \times X \mid \partial_B(x) \leq \partial_B(y),\, \widetilde{\partial}_B(x) \preccurlyeq \widetilde{\partial}_B(y)\}.$$

We will use the following sets:

$$\Omega^{\varphi}_{T(\min,\mathrm{rmin})} := \left\{ \tfrac{x}{\{y,z\}} \in \tfrac{X}{X\times X} \;\middle|\; \begin{array}{l} \varphi_B(x) \geq \min\{\varphi_B(y), \varphi_B(z)\} \\ \widetilde{\varphi}_B(x) \succcurlyeq \mathrm{rmin}\{\widetilde{\varphi}_B(y), \widetilde{\varphi}_B(z)\}, \end{array} \right\}$$

and

$$\Omega^{\partial}_{F(\max,\mathrm{rmax})} := \left\{ \tfrac{x}{\{y,z\}} \in \tfrac{X}{X\times X} \;\middle|\; \begin{array}{l} \partial_B(x) \leq \max\{\partial_B(y), \partial_B(z)\} \\ \widetilde{\partial}_B(x) \preccurlyeq \mathrm{rmax}\{\widetilde{\partial}_B(y), \widetilde{\partial}_B(z)\} \end{array} \right\}.$$

Since $(X, \mathcal{B})$ is a $T\&F$-ideal of X, we get

(35) $$\tfrac{(x*z)*(y*z)}{\{((x*z)*(y*z))*((x*y)*z),\, (x*y)*z\}} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial}_{F(\max,\mathrm{rmax})}.$$

The combination of (34) and (35) induces

$$\tfrac{(x*z)*(y*z)}{\{0,\, (x*y)*z\}} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial}_{F(\max,\mathrm{rmax})},$$

and so $((x*z)*(y*z),\, (x*y)*z) \in \Omega_T^{\varphi} \cap \Omega_F^{\partial}$. Therefore $(X, \mathcal{B})$ is a positive implicative $T\&F$-ideal of X by Theorem 3.6. □

Theorem 3.19. *Let L and X be two BCK-algebras and $f : L \to X$ be a BCK-homomorphism. If $(L, \mathcal{A})$ and $(X, \mathcal{B})$ are two T& F-structures on L and X, and $\mathcal{A} := (\varphi_A,\, \widetilde{\varphi}_A,\, \partial_A,\, \widetilde{\partial}_A)$ and $\mathcal{B} := (\varphi_B,\, \widetilde{\varphi}_B,\, \partial_B,\, \widetilde{\partial}_B)$, respectively, then*
(i) If $(I, \mathcal{B})$ is a positive implicative T& F-ideal of X, then $f^{-1}(I, \mathcal{B})$ is a

positive implicative T& F-ideal of L.
(ii) If f is onto and $(I, \mathcal{A})$ is a positive implicative T& F-ideal of L, then $f(I, \mathcal{A})$ is a positive implicative T& F-ideal of X, where $f(I, \mathcal{A})$ is defined as follows:

$$f(I,\mathcal{A}) = \begin{cases} f(\Omega_T) = \begin{cases} f(\varphi_A)(\mathring{x}) &= \sup\limits_{a\in f^{-1}(\mathring{x})} \varphi_A(a), \\ f(\widetilde{\varphi}_A)(\mathring{x}) &= \operatorname{rsup}\limits_{a\in f^{-1}(\mathring{x})} \widetilde{\varphi}_A(a), \end{cases} \\ f(\Omega_F) = \begin{cases} f(\partial_A)(\mathring{x}) &= \inf\limits_{a\in f^{-1}(\mathring{x})} \partial_A(a), \\ f(\widetilde{\partial}_A)(\mathring{x}) &= \operatorname{rinf}\limits_{a\in f^{-1}(\mathring{x})} \widetilde{\partial}_A(a), \end{cases} \end{cases}$$

and if $f^{-1}(\mathring{x}) = \emptyset$, then $f(\varphi_A)(\mathring{x}) = 0$, $f(\widetilde{\varphi}_A)(\mathring{x}) = \tilde{0}$, $f(\partial_A)(\mathring{x}) = 1$ and $f(\widetilde{\partial}_A)(\mathring{x}) = \tilde{1}$.

Proof. (i) Suppose $(\mathring{x}*\mathring{y})*\mathring{z} \in f^{-1}(I,\mathcal{B})$ and $\mathring{y}*\mathring{z} \in f^{-1}(I,\mathcal{B})$. Then $(\mathring{x}*\mathring{y})*\mathring{z} \in f^{-1}\left(\Omega_T^{\varphi} \cap \Omega_F^{\partial}\right)$ and $\mathring{y} * \mathring{z} \in f^{-1}\left(\Omega_T^{\varphi} \cap \Omega_F^{\partial}\right)$. Thus, $f((\mathring{x} * \mathring{y}) * \mathring{z}) \in \Omega_T^{\varphi} \cap \Omega_F^{\partial}$ and $f(\mathring{y} * \mathring{z}) \in \Omega_T^{\varphi} \cap \Omega_F^{\partial}$. Since f is a BCK-homomorphism and $(I, \mathcal{B})$ is a positive implicative T& F-ideal of X, we have $(f(\mathring{x}) * f(\mathring{y})) * f(\mathring{z}) \in \Omega_T^{\varphi} \cap \Omega_F^{\partial}$ and $f(\mathring{y}) * f(\mathring{z}) \in \Omega_T^{\varphi} \cap \Omega_F^{\partial}$, and so $f(\mathring{x} * \mathring{z}) = f(\mathring{x}) * f(\mathring{z}) \in \Omega_T^{\varphi} \cap \Omega_F^{\partial}$. Hence, $\mathring{x} * \mathring{z} \in f^{-1}\left(\Omega_T^{\varphi} \cap \Omega_F^{\partial}\right)$. Therefore, $f^{-1}(I, \mathcal{B})$ is a positive implicative T& F-ideal of L.
(ii) We prove for $f(\varphi_A)(\mathring{x})$. Suppose $(\mathring{x} * \mathring{y}) * \mathring{z} \in f(I, \mathcal{A})$ and $\mathring{y} * \mathring{z} \in f(I, \mathcal{A})$. Since f is onto, there are a, b, c such that $(a * b) * c \in f^{-1}((\mathring{x} * \mathring{y}) * \mathring{z})$ and $b * c \in f^{-1}(\mathring{y} * \mathring{z})$. So, we have

$$f(\varphi_A)((\mathring{x} * \mathring{y}) * \mathring{z}) = \sup_{(a*b)*c\in f^{-1}((\mathring{x}*\mathring{y})*\mathring{z})} \varphi_A((a * b) * c) \geq \varphi_A((a * b) * c),$$
$$f(\varphi_A)(\mathring{y} * \mathring{z}) = \sup_{b*c\in f^{-1}(\mathring{y}*\mathring{z})} \varphi_A(b * c) \geq \varphi_A(b * c).$$

Then

$$\begin{aligned} f(\varphi_A)(\mathring{x} * \mathring{z}) = \sup_{a*c\in f^{-1}(\mathring{x}*\mathring{z})} \varphi_A(a * c) &\geq \varphi_A(a * c) \\ &\geq \min\{\sup_{(a*b)*c\in f^{-1}((\mathring{x}*\mathring{y})*\mathring{z})} \varphi_A((a * b) * c), \sup_{b*c\in f^{-1}(\mathring{y}*\mathring{z})} \varphi_A(b * c)\} \\ &\geq \min\{\varphi_A(b * c), \varphi_A((a * b) * c)\}. \end{aligned}$$

The proof of other cases is similar. Therefore, an image of a positive implicative T& F ideal is a positive implicative T& F ideal. □

4. **Conclusion**

The notion of a positive implicative $T\&F$-ideal is introduced in BCK-algebras, and several properties are investigated. The relationship between $T\&F$-ideals and positive implicative $T\&F$-ideals is established, and by an example, we showed that a $T\&F$-ideal is not a positive implicative $T\&F$-ideal. Then various conditions under which a $T\&F$-ideal can be a positive implicative $T\&F$-ideal are explored and various characterizations of a positive implicative $T\&F$-ideal are studied. The extended property of a positive implicative $T\&F$-ideal is constructed.

5. **Aknowledgement**

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