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SOME RESULTS ON THE OPEN LOCATING-TOTAL DOMINATION NUMBER IN GRAPHS

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Abstract. In this paper, we generalize the concept of an open locating-dominating set in a graph. We introduce a concept as an open locating-total dominating set in graphs that is equivalent to the open neighborhood locating-dominating set. A vertex set $S \subseteq V(G)$ is an open locating-total dominating if the set S is a total dominating set of G and for any pair of distinct vertices x and y in $V(G)$, $N(x) \cap S \neq N(y) \cap S$. The open locating-total domination number, denoted $\gamma_t^{OL}(G)$, of G is the minimum cardinality of an open locating-total dominating set.
In this paper, we determine the open locating-total dominating set of some families of graphs. Also, the open locating-total domination number is calculated for two families of trees. The present paper is an extended version of our paper, presented at the 52nd Annual Iranian Mathematics Conference, Shahid Bahonar University of Kerman, Iran, 2021.

Keywords: Open locating-dominating set, Total dominating set, Cartesian product of graphs.
2020 MSC: 05C69.

1. Introduction

Graph theory is used as a theoretical tool to consider actual networks. One of the studies based on graphs is finding the location of monitoring devices such as surveillance cameras or fire alarms to safeguard a system serves.

The problem of placing monitoring devices in a system in such a way that every site in the system is adjacent to a monitor site can are modeled by a total dominating set in a graph. In this way, one is able to detect a failure that is crucial for the network. On the other hand, the location of failures can be identified by the set of monitors which is modeled by a combination of total dominating sets and locating sets.

Locating sets for such studies were introduced by Salter in 1975 [18] and the concepts of locating and dominating were combined in [14, 15]. Also, the locating-total dominating set in the graph was introduced by Haynes et al. in 2006 [6]. Some more results on the locating-total domination number are obtained in [1, 10, 13, 19]. In [5] the locating edge domination number of comb

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product of graph is investigated. Wardani et al. determined the locating dominating set on edge corona product. They showed that there is a relation between the locating dominating set on the basic graph and its operation [20]. Locating domination in bipartite graphs and their complements are studied in [8].

To identify the situations of failure in the system, the set of devices that have the failure in its neighborhood or their own location can be determined by the open locating dominating set. The problem of "open locating dominating sets" was introduced by Honkala et al. [9] in the context of coding theory for binary hypercubes. Applications of Location detection problems that can be modeled by the open locating total dominating set include finding faults in multiprocessors, intruders in buildings and facilities, and for environmental monitoring using wireless sensor networks. Devices placed at the vertices of the open locating-total dominating set can uniquely detect and locate disturbances in a system. Motivated by the definition of the open locating set and total dominating set, we introduce the open locating-total dominating set of a graph G.

All graphs considered in this paper are finite, undirected and simple graphs. Let $G = (V, E)$ be a graph with vertex set V and edge set E. The open neighborhood of vertex v, denoted $N(v)$, is $\{u \in V | uv \in E\}$ while its closed neighborhood is given by $N[v] = N(v) \cup \{v\}$. For a vertex $v \in V(G)$, the degree of v, written by $deg(v)$ or $d(v)$, is the cardinality of $N(v)$. A leaf of G is the vertex of degree 1, while a support vertex of G is a vertex adjacent to the leaf. A clique is a subset C of vertices in a graph G such that the induced subgraph by C is a complete graph.

A set S of vertices of a graph G is a dominating set of G if every vertex in $V \setminus S$ is adjacent to a vertex of S, and S is a total dominating set if every vertex in V has a neighbor in S. A subset S of V is an open neighborhood locating dominating set (OLDS) of G if for each vertex $w \in V(G)$ there is at least one vertex v in $S \cap N(w)$ and for any pair of distinct vertices x and y in V we have $N(x) \cap S \neq N(y) \cap S$. The open neighborhood locating domination number $\gamma^{OL}(G)$ is the minimum cardinality of an open neighborhood locating dominating set for G [16]. Seo and Slater obtained the open neighborhood locating domination number of paths and gave lower and upper bound on the open neighborhood locating domination number of a tree [16]. In [17], it is characterized by the trees that achieve these extreme values. Chellali et al. in [2] characterized graphs with $\gamma^{OL}(G) = 2, 3$, or n and graphs that are C_4-free with $\gamma^{OL}(G) = n - 1$.

This definition is equivalent to $S \subseteq V(G)$ is a total dominating set of G and for any pair of distinct vertices x and y in V, $N(x) \cap S \neq N(y) \cap S$. For this reason, in this paper, we call such a set the open locating-total dominating set (OLTDS). So, the open locating-total domination number $\gamma_t^{OL}(G)$ is the minimum cardinality of an OLTDS for G. An open locating-total dominating set for G of cardinality $\gamma_t^{OL}(G)$ will be called a $\gamma_t^{OL}(G)$-set. It is clear that $\gamma_t^{OL}(G)=\gamma^{OL}(G)$.

Motivated by the above discussions, in this paper we investigate the open locating-total dominating set for some of the families of the graph. The rest of the paper is organized as follows. We investigate the open locating-total dominating set for the join, the corona of graphs, complementary prisms of a graph and the cartesian product of graphs in Section 2. In Section 3, the open locating-total domination number is obtained for two families of trees that are introduced in [3]. In Section 4, we give some conclusions and future work.
The present paper is an extended version of our paper, presented at the 52nd Annual Iranian Mathematics Conference [12].

2. Open locating-total dominating set in some families of graph

In this section, we study the open locating-total dominating set for some known families of graphs. We first investigate the open locating-total dominating set for the corona of graphs. Let G and H be graphs of order m and n, respectively. The corona of two graphs G and H is the graph $G \circ H$ obtained by taking one copy of G and m copies of H and then joining the ith vertex of G to every vertex of the ith copy of H [4].

Theorem 2.1. *Let G and H be two connected graphs, $m = |V(G)| \geq 4$ and $n = |V(H)| \geq 4$. Then $S \subseteq V(G \circ H)$ is an OLTD-set in $G \circ H$ if and only if $S = \bigcup_{x \in V(G)} S_x$, where S_x is OLTD-set in H^x and H^x is the copy of H whose vertices are attached one by one to the vertex x.*

Proof. Let $S \subseteq V(G \circ H)$ be an OLTD-set in $G \circ H$ and $S_x = V(H^x) \cap S$. In the graph $G \circ H$, every vertex of H^x is adjacent to x in G. Since S is an OLTD-set in $G \circ H$, for any $u, v \in V(H^x) \subseteq V(G \circ H)$, we have

$$\begin{aligned}
N_{H^x}(u) \cap S_x &= \big(N_{G\circ H}(u) \cap V(H^x)\big) \cap S_x \\
&= \big(N_{G\circ H}(u) \cap V(H^x)\big) \cap \big(V(H^x) \cap S\big) \\
&= N_{G\circ H}(u) \cap V(H^x) \cap S \\
&= \big(N_{G\circ H}(u) \cap S\big) \cap V(H^x) \\
&\neq \big(N_{G\circ H}(v) \cap S\big) \cap V(H^x) \\
&= \big(N_{G\circ H}(v) \cap V(H^x)\big) \cap \big(V(H^x) \cap S\big) \\
&= \big(N_{G\circ H}(v) \cap V(H^x)\big) \cap S_x \\
&= N_{H^x}(v) \cap S_x.
\end{aligned}$$

Therefore, S_x is an open locating set in H^x.
If $x \notin S$, then S_x is an OLTD-set in H^x. Because S is an OLTD-set in $G \circ H$ and S_x is open locating set.
If $x \in S$, since S_x is an open locating set we obtain, $N_G(u) \cap S_x \neq \emptyset$ for each

vertex $u \in V(H^x)$. Therefore, S_x is the dominating set. If $u \in S_x$ is not adjacent to any vertex in S_x then it is contrary to S_x is an open locating set. Thus, $S = \bigcup_{x \in V(G)} S_x$ is an OLTD-set in $G \circ H$.

For the converse, suppose that $S = \bigcup_{x \in V(G)} S_x$ in which S_x is an OLTD-set in H^x. Since S_x is the OLTD-set in H^x for each vertex $x \in V(G)$, S is a total dominating set. Thus, it is sufficient to show that S is an open locating set. For any two distinct vertices $u, v \in V(G \circ H)$, since $S_u \neq S_v$, we have

$$\begin{aligned} N_{G \circ H}(u) \cap S &= \big(N_G(u) \cup V(H^u)\big) \cap S \\ &= \big(N_G(u) \cap S\big) \cup \big(V(H^u) \cap S\big) \\ &= (N_G(u) \cap S) \cup S_u \\ &= \big(N_G(u) \cup S_u\big) \cap \big(S \cup S_u\big) \\ &= \big(N_G(u) \cup S_u\big) \cap S \\ &\neq \big(N_G(v) \cup S_v\big) \cap S \\ &\subseteq \big(N_G(v) \cup V(H^v)\big) \cap S \\ &= N_{G \circ H}(v) \cap S. \end{aligned}$$

Hence, S is an $\gamma_t^{OL}(G \circ H)$-set. □

Example 2.2. We suppose that the graphs G and H are cycles of order 4 with $V(G) = \{v_1, v_2, v_3, v_4\}$ and $V(H) = \{u_1, u_2, u_3, u_4\}$. We consider the vertices of ith copy of H as $\{u_1^i, u_2^i, u_3^i, u_4^i\}$. By applying Theorem 2.1, the OLTDS is $S = \bigcup_{i=1}^{4} \{u_1^i, u_2^i, u_3^i\}$. Therefore, the open locating-total domination number of $G \circ H$ is equal to 12. The open locating-total dominating set of $G \circ H$ is shown in Figure 1 with vertices of black color.

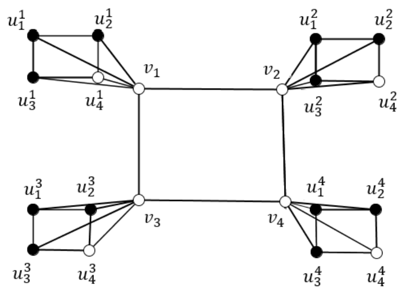

FIGURE 1. Example of open locating-total domination number of $C_4 \circ C_4$ and dominator with black color.

In the following theorem, we study the open locating-total dominating set in the join of two graphs. The join of two graphs G and H is the graph $G + H$

with the vertex set $V(G+H) = V(G) \cup V(H)$ and the edge set

$$E(G+H) = E(G) \cup E(H) \cup \{xy : x \in V(G), y \in V(H)\}.$$

Theorem 2.3. *Let G and H be non-trivial connected graphs. If $S \subseteq V(G+H)$ is an OLTD-set in $G+H$, then $S_1 = S \cap V(G)$ and $S_2 = S \cap V(H)$ are open locating sets of G and H, respectively.*

Proof. Let $S \subseteq V(G+H)$, $S_1 = \emptyset$ and $S = S_2 = S \cap V(H)$. Since every vertex of G is adjacent to each vertex in $V(H)$, for any distinct vertices u and v in $V(G)$ we have

$$N_{G+H}(u) \cap S = N_{G+H}(v) \cap S.$$

Thus, it is contrary to the assumption of S. Thus, $S_1 \neq \emptyset$. Similarly, $S_2 \neq \emptyset$. Suppose that one of S_1 and S_2, say S_1, is not an open locating set in G. Then, there exist distinct vertices u and v of G such that $N_G(u) \cap S_1 = N_G(v) \cap S_1$. Since $S_2 \subseteq N_{G+H}(u)$ and $S_2 \subseteq N_{G+H}(v)$ it follows that

$$\begin{aligned} N_{G+H}(u) \cap S &= (N_G(u) \cup V(H)) \cap S \\ &= (N_G(u) \cap S) \cup (V(H) \cap S) \\ &= (N_G(u) \cap S_1) \cup S_2 \\ &= (N_G(v) \cap S_1) \cup S_2 \\ &= N_{G+H}(v) \cap S. \end{aligned}$$

Thus, S is not a locating set in $G+H$, contrary to our assumption. Therefore, S_1 and S_2 are locating sets in G and H, respectively. □

Theorem 2.4. *Let G be a connected nontrivial graph. Then $S \subseteq V(G+K_1)$ is an open locating-total dominating set in $G+K_1$ if and only if for $v \in V(K_1)$ either $S = S_1 \cup \{v\}$, where S_1 is an open locating set in G, or $v \notin S$ and $S = S_1$ is an OLTD-set in G.*

Proof. Let $H = K_1 = \{v\}$ and $S \subseteq V(G+K_1)$ be an OLTD-set in $G+K_1$. If $v \in S$, then $S = S_1 \cup \{v\}$ in which $S_1 = S \cap V(G)$. For any two distinct vertices $x, y \in V(G)$,

$$\begin{aligned} (N_G(x) \cap S_1) \cup \{v\} &= N_{G+K_1}(x) \cap S \\ &\neq N_{G+K_1}(y) \cap S \\ &= (N_G(x) \cap S_1) \cup \{v\}. \end{aligned}$$

Therefore,

$$N_G(x) \cap S_1 \neq N_G(y) \cap S_1.$$

Hence, S_1 is an open locating set in G. If $v \notin S$, then $OLTD(G+K_1) = OLTD(G)$.

For the converse, assume first that $S = S_1 \cup \{v\}$ and $S \subseteq V(G+K_1)$ where S_1 is an open locating set in G. Clearly, S is a total dominating set in $G+H$.

For any two distinct vertices $x, y \in V(G)$,

$$\begin{aligned} N_{G+K_1}(x) \cap S &= (N_G(x) \cup \{v\}) \cap (S_1 \cup \{v\}) \\ &= (N_G(x) \cap S_1) \cup \{v\} \\ &\neq (N_G(y) \cap S_1) \cup \{v\} \\ &= N_{G+K_1}(y) \cap S. \end{aligned}$$

This shows that S is an open locating-total dominating set in $G + H$.

If $v \notin S$, then $S = S_1$ where S_1 is an open locating set in G and S is an open locating set in $G + K_1$. So, S is an open locating-total dominating set in $G + K_1$. □

Now, we study the open locating-total dominating sets in complementary prisms. Complementary prisms were first introduced by Haynes, Henning, Slater, and van der Merwe in [7]. For a graph G, its complementary prism, denoted $G\overline{G}$, is formed from a copy of G and a copy of $\overline{G}$ by adding a perfect matching between corresponding vertices. For each $v \in V(G)$, let $\overline{v}$ correspond to the vertex v in $\overline{G}$. Formally, $G\overline{G}$ is formed from $G \cup \overline{G}$ by adding the edge $v\overline{v}$ for every $v \in V(G)$. In [11], it is studied Locating-Domination in complementary prisms.

Theorem 2.5. *If G is the nontrivial complete graph K_n, then $\gamma_t^{OL}(G\overline{G}) = n$ and $\gamma_t^{OL}(G\overline{G})$-set $= V(G)$.*

Proof. Since the corona $K_n \circ K_1$ is the complementary prism of $K_n\overline{K_n}$, the obtained graph has n leaves and therefore n support vertices. These support vertices are in the OLTD-set of graph $G\overline{G}$. Thus,

$$\gamma_t^{OL}(G\overline{G}) \geq n.$$

The set $V(G)$ forms the OLTD-set for $G\overline{G}$ so, $\gamma_t^{OL}(G\overline{G}) \leq n$. The proof is complete. □

Note that the converse of Theorem 2.5 is not true in general. Because with considering cycle C_n of order n and its complementary prism $C_n\overline{C_n}$, one can easy check that $\gamma_t^{OL}(G\overline{G}) = V(C_n)$ and consequently, $\gamma_t^{OL}(C_n\overline{C_n}) = n$. For example, in the graph $C_4\overline{C_4}$ shown in Figure 2, $\gamma_t^{OL}(C_4\overline{C_4}) = 4$.

Theorem 2.6. *Let G be the complete bipartite graph $K_{r,s}$, where $r + s = n$ and $1 \leq r \leq s$. Then*

$$\gamma_t^{OL}(G\overline{G}) = \begin{cases} n+1 & \textit{if } r = 1, 2 \\ n & \textit{if } 3 \leq r \leq s. \end{cases}$$

Proof. Let $G = K_{r,s}$, $1 \leq r \leq s$, where $R = \{x_1, x_2, \ldots, x_r\}$ and $S = \{y_1, y_2, \ldots, y_s\}$ are the bipartite sets of G with cardinality r and s, respectively. We consider $\overline{v}$ in $\overline{G}$ correspond each vertex $v \in G$. Therefore, the vertex set $V(\overline{G}) = \overline{R} \cup \overline{S}$ such that $\overline{R} = \{\overline{x_1}, \ldots, \overline{x_r}\}$ and $S = \{\overline{y_1}, \ldots, \overline{y_s}\}$. Let D be a $\gamma_t^{OL}(G\overline{G})$-set.

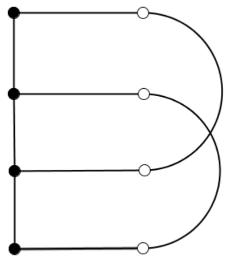

FIGURE 2. Example of open locating-total domination number of $C_4\overline{C_4}$ and dominator with black color.

(i) Assume that $r = 1$. The graph $G = K_{1,s}$ is the star graph. Thus, $\overline{G}$ has an isolated vertex. Note that $\overline{x_1}$ is a leaf in $G\overline{G}$.
According to the structure graph $G\overline{G}$, $x_1 \in D$. Also, since D is an open locating set and $N(y_i) \cap D \neq N(\overline{x_1}) \cap D$ for all $1 \leq i \leq s$, we deduce that $\overline{y_i} \in D$ for $1 \leq i \leq s$. On the other hand, since D is a total dominating set in $G\overline{G}$ and x_1 is not adjacent to any vertex $\overline{y_i}$ for $1 \leq i \leq s$, thus there is a vertex $v \in D$ such that x_1 is adjacent to v. Therefore, $\gamma_t^{OL}(G\overline{G}) = |D| \geq s + 1 + 1 = n + 1$. On the other hand, $\overline{S} \cup \{x_1, \overline{x_1}\}$ is an OLTD-set for $G\overline{G}$.
Thus, $\gamma_t^{OL}(G\overline{G}) \leq n + 1$ and consequently, $\gamma_t^{OL}(G\overline{G}) = n + 1$.

Let $r = 2$ and $G = K_{2,s}$, where $s + 2 = n$. Set $\overline{S} \cup \{x_1, x_2, \overline{x}_1\}$ is the OLTD-set for $G\overline{G}$. So, $\gamma_t^{OL}(G\overline{G}) \leq s + 3 = n + 1$. Let D be an open locating-total dominating set of graph $G\overline{G}$ and $|D| \leq n = s + 2$. Since the set $\overline{S}$ is a clique in the graph $G\overline{G}$, $|D \cap (S \cup \overline{S})| \geq s$. Therefore, $|D \cap (R \cup \overline{R})| = 2$. Thus, we have the following cases.
Case 1: If $\{x_1, x_2\} \subseteq D$, then $N_{G\overline{G}}(x_1) \cap D = N_{G\overline{G}}(x_2) \cap D$. So, it is a contradiction.
Case 2: Assume that $\{\overline{x_1}, \overline{x_2}\} \subseteq D$. Since $\overline{S}$ is a clique in the graph $G\overline{G}$ and $|D \cap (S \cup \overline{S})| \geq s$, $\overline{S} \subseteq D$. Otherwise, suppose $y \in \overline{S}$ and $\overline{y} \notin D$. Since y is only adjacent to $\overline{y}$ among the set of vertices of $\overline{S}$ in the graph $G\overline{G}$, it is a contradiction because D is the total dominating set. Therefore, $\overline{S} \subseteq D$. So, $N_{G\overline{G}}(x_1) \cap D = \{\overline{x_1}\} = N_{G\overline{G}}(\overline{x_2}) \cap D$ which is a contradiction.
Case 3: Suppose that $\{x_1, \overline{x_1}\} \subseteq D$ or $\{x_2, \overline{x_2}\} \subseteq D$. Without loss of generality, let $\{x_1, \overline{x_1}\} \subseteq D$. According to the above discussion, D cannot dominate the vertices x_2 and $\overline{x_2}$. So, this case is a contradiction.
Case 4: Let $\{x_1, \overline{x_2}\} \subseteq D$ or $\{\overline{x_1}, x_2\} \subseteq D$. Without loss of generality, let $\{x_1, \overline{x_2}\} \subseteq D$. Since x_1 and $\overline{x_2}$ are isolated vertices in the set D

then, it is a contradiction.
Therefore, $|D| \geq n+1$ and the result holds.

(ii) For $3 \leq r \leq s$, we show that $\overline{R} \cup \overline{S}$ is an OLTD-set of $G\overline{G}$. The sets $\overline{R}$ and $\overline{S}$ are cliques in the graph $G\overline{G}$. So, these sets dominate all vertices in $G\overline{G}$ and are total dominating sets. On the other hand, for any two distinct vertices $u, v \in (\overline{R} \cup \overline{S})$,

$$N(u) \cap (\overline{R} \cup \overline{S}) \neq N(v) \cap (\overline{R} \cup \overline{S}).$$

Therefore,

$$\gamma_t^{OL}(G\overline{G}) = r + s = n.$$

□

Theorem 2.7. *For any graph G,*

$$max\{\gamma_t^{OL}(G), \gamma_t^{OL}(\overline{G})\} \leq \gamma_t^{OL}(G\overline{G}) \leq \gamma_t^{OL}(G) + \gamma_t^{OL}(\overline{G}).$$

Proof. If $G = K_n$, then $\gamma_t^{OL}(G) = n-1$ and $\gamma_t^{OL}(\overline{G}) = n$. Hence,

$$max\{\gamma_t^{OL}(G), \gamma_t^{OL}(\overline{G})\} = n = \gamma_t^{OL}(G\overline{G}) \leq \gamma_t^{OL}(G) + \gamma_t^{OL}(\overline{G}) = 2n-1.$$

Thus, we may assume G is not complete and D is an OLTD-set in $G\overline{G}$. Suppose that $D_1 = D \cap V(G)$ and $D_2 = D \cap V(\overline{G})$. If D_1 is an open locating-total dominating set in G, then we are finished. So, assume there exists a set $T \subseteq V(G)$ such that T is not open locating-total dominated by D_1. Thus, T will get these features by D_2.
Since each vertex in D_2 is adjacent to at most one vertex in T, $|T| \leq |D_2|$. But, set $T \cup D_1$ is an open locating-total dominating set in G. Assume, without loss of generality, that $\gamma_t^{OL}(G) \geq \gamma_t^{OL}(\overline{G})$. Thus,

$$\gamma_t^{OL}(G) \leq |T \cup D_1| \leq |T| + |D_1| \leq |D_2| + |D_1| = |D| = \gamma_t^{OL}(G\overline{G}).$$

Therefore,

$$max\{\gamma_t^{OL}(G), \gamma_t^{OL}(\overline{G})\} \leq \gamma_t^{OL}(G\overline{G}).$$

For the upper bound, let S_1 be an OLTD-set in G and S_2 be an OLTD-set for $\overline{G}$. Let $S = S_1 \cup S_2$. Since every vertex of $G(\overline{G})$ is dominated by $S_1(S_2)$, S is a total dominating set. It is sufficient to show that S is an open locating set in $G\overline{G}$ for any two vertices $u \in V(G)$ and $\overline{v} \in V(\overline{G})$. According to the structure of graph $G\overline{G}$, we have

$$N_{G\overline{G}}(u) = N_G(u) \cup \{\overline{u}\},$$

and

$$N_{G\overline{G}}(\overline{v}) = N_{\overline{G}}(\overline{v}) \cup \{v\}.$$

Since $N_{G\overline{G}}(u) \cap S_1 \neq N_{G\overline{G}}(\overline{v}) \cap S_2$, $N_{G\overline{G}}(u) \cap S \neq N_{G\overline{G}}(\overline{v}) \cap S$. Therefore, S is an open locating set. Consequently,

$$\gamma_t^{OL}(G\overline{G}) \leq |S_1 \cup S_2| = |S_1| + |S_2| = \gamma_t^{OL}(G) + \gamma_t^{OL}(\overline{G}).$$

□

Seo and Salter obtained the exact value of open neighborhood locating domination number of paths [16]. In the same way, we can have the following results for wheel graphs and cycles. Let W_n denote a wheel graph of order n.

Proposition 2.8. *Let D be an OLTD-set for W_n for $n \geq 6$. If $D \cap \{v_{i-1}, v_i, v_{i+1}\} = \{v_{i-1}, v_{i+1}\}$, then $3 \leq i \leq n-2$ and*

$$\{v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}\} \subseteq D.$$

Proposition 2.9. *If D is an OLTD-set for W_n for $n \geq 6$, then if $D \cap \{v_i, v_{i+1}\} = \emptyset$, then $3 \leq i \leq n-3$ and*

$$\{v_{i-4}, v_{i-3}, v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}\} \subseteq D.$$

Proposition 2.10. *Let D be an OLTD-set for W_n for $n \geq 6$. If $D \cap \{v_i, v_{i+1}\} = \emptyset$, then $3 \leq i \leq n-3$ and*

$$\{v_{i-4}, v_{i-3}, v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}\} \subseteq D.$$

Proposition 2.11. *If D is an OLTD-set for W_n for $n \geq 6$, then*

$$|\{v_i, v_{i+1}, v_{i+2}, v_{i+3}\} \cap D| \geq 2.$$

Proposition 2.12. *$\gamma_t^{OL}(W_n)$ for $n \geq 6$ is as following*

(i) *if $n = 3k$ or $n = 3k+1$, then $\gamma_t^{OL}(W_n) = 2k$,*
(ii) *if $n = 3k+2$, then $\gamma_t^{OL}(W_n) = 2(k+1)$.*

Lemma 2.13. *$\gamma_t^{OL}(C_n)$ for $n \geq 6$ is as following*

(i) *if $n = 3k$ or $n = 3k+1$, then $\gamma_t^{OL}(C_n) = 2k$,*
(ii) *if $n = 3k+2$, then $\gamma_t^{OL}(C_n) = 2(k+1)$.*

Finally, we investigate the open locating-total domination number in the Cartesian product of graphs. For graphs G_1 and G_2, the Cartesian product $G_1 \times G_2$ is the graph with vertex set $V(G_1) \times V(G_2)$, where two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if either $u_1 = u_2$ and $v_1v_2 \in E(G_2)$ or $v_1 = v_2$ and $u_1u_2 \in E(G_1)$.

Let $\{v_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ be the vertex set of $C_m \times P_n$. We observe that the subgraph included by $A_i = \{v_{i1}, v_{i2}, \dots, v_{in}\}$ is isomorphic to the path P_n for each $1 \leq i \leq m$ and the subgraph induced by $B_j = \{v_{1j}, v_{2j}, \dots, v_{mj}\}$ is isomorphic to the cycle C_m for each $1 \leq j \leq n$.

Theorem 2.14. *For any positive integers m, n such that $m = 3t$ or $m = 3t+1$ where $t \geq 1$ and $n \geq 2$,*

$$\gamma_t^{OL}(C_m \times P_n) \leq \frac{2}{3}mn.$$

Proof. Let G be the Cartesian product $C_m \times P_n$, where $m = 3t$ for a positive integer t. For every $1 \leq j \leq n$, we define

$$D_j := B_j - \cup_{l=0}^{t-1}(v_{(3l+1)j})$$

where $B_j = \{v_{1j}, v_{2j}, \ldots, v_{mj}\}$ and $1 \leq j \leq n$. We show that $S = \cup_{j=1}^{n} D_j$ is an open locating-total dominating set in $C_m \times P_n$. We complete the proof by induction on n.
If $n = 2$, then $S = \cup_{j=1}^{2} D_j$ is an open locating-total dominating set for $C_m \times P_2$. Therefore, $\gamma_t^{OL}(C_m \times P_2) \leq |D_1 \cup D_2|$. By Lemma 2.13, we can obtain

$$\gamma_t^{OL}(C_m \times P_2) \leq 4t = \frac{2}{3}mn.$$

If $n = 3$, then $S = \cup_{j=1}^{3} D_j$ is an OLTD-set for $C_m \times P_3$. Thus

$$|S| = 6t = \frac{2}{3}mn = \gamma_t^{OL}(C_m \times P_3).$$

Assume that for $n-1$, $\gamma_t^{OL}(C_m \times P_{n-1}) \leq \frac{2}{3}m(n-1)$. We add a leaf to the last vertex from P_{n-1} to obtain path P_n. So, one cycle C_m is added to graph $C_m \times P_{n-1}$ where every vertex in new cycle C_m is adjacent to m the last vertex from P_{n-1}'s. So, the new graph is a $C_m \times P_n$. Using Lemma 2.13, we have

$$\begin{aligned}
\gamma_t^{OL}(C_m \times P_n) &\leq 2t + \gamma_t^{OL}(C_m \times P_{n-1}) \\
&\leq 2t + \frac{2}{3}m(n-1) \\
&= 2t + \frac{2}{3}mn - \frac{2}{3}(3t) \\
&= \frac{2}{3}mn.
\end{aligned}$$

For $m = 3t+1$, by a similar proof as above and using Lemma 2.13 the result is true. □

Theorem 2.15. *For any positive integers m, n such that $m = 3t+2$, where $t \geq 1$ and $n \geq 2$,*

$$\gamma_t^{OL}(C_m \times P_n) \leq \frac{2}{3}(m+1)n.$$

Proof. We proceed by induction on n. Suppose that $D_j = B_j - \cup_{l=1}^{t}(v_{(3l)j})$ in which $B_j = \{v_{1j}, v_{2j}, \ldots, v_{mj}\}$ and $1 \leq j \leq n$. We show that $S = \cup_{j=1}^{n} D_j$ is an open locating-total dominating set in $C_m \times P_n$.
If $n = 2$, then the set $S = \cup_{j=1}^{2} D_j = D_1 \cup D_2$ is an open locating-total dominating set in $C_m \times P_2$. Therefore

$$|S| = 4t + 4 = \frac{2}{3}(m+1)n.$$

Assume that for $n-1$, $\gamma_t^{OL}(C_m \times P_n) \leq \frac{2}{3}(m+1)(n-1)$. According to the method of proof Theorem 2.14, by adding the vertices C_m to each of the last vertices of P_n.

$$\begin{aligned}\gamma_t^{OL}(C_m \times P_n) &\le 2(t+1) + \gamma_t^{OL}(C_m \times P_{n-1})\\ &\le 2(t+1) + \frac{2}{3}(m+1)(n-1)\\ &= 2(t+1) + \frac{2}{3}(m+1)n - \frac{2}{3}(m+1)\\ &= \frac{2}{3}(m+1)n.\end{aligned}$$

□

3. To determine the open locating-total domination number for some families of trees

In this section, we obtain γ_t^{OL} for two families of trees. For any tree T, let $L(T)$ denote the set of leaves of the tree T. Also, let n and l denote the order of the tree and the number of leaves, respectively.

We consider the family Γ of labeled trees that is introduced in [3]. Let Γ be the family of labeled trees denoted by $sta(v)$. There are three statuses for v, A, B and C used to label the tree. Let $T = T_k$ and T_0 be a P_6 in which two leaves have status C, the two support vertices have status A, and the other vertices have status B (see Figure 3).

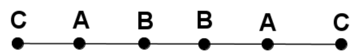

FIGURE 3. Labeled path P_6.

For $k \ge 1$, T_k can be obtained recursively from T_{k-1} by one of the following operations.

i) Operation τ_1. For any $y \in V(T_{k-1})$ if $sta(y) = C$ and $deg(y) = 1$ in T_{k-1}, then add a path x, w, v, z and edge xy. Let $sta(x) = sta(w) = B$, $sta(v) = A$ and $sta(z) = C$ (see Figure 4).

ii) Operation τ_2. For any $y \in V(T_{k-1})$ if $sta(y) = B$, then add a path x, w, v and edge xy. Let $sta(x) = B$, $sta(w) = A$ and $sta(v) = C$ (see Figure 4).

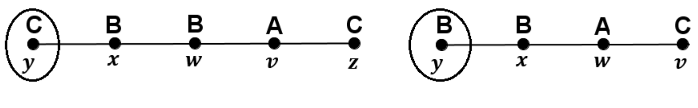

FIGURE 4. Operation τ_1 and operation τ_2.

So, we consider

$$A(T) = \{v \in V(T)|sta(v) = A\},$$
$$B(T) = \{v \in V(T)|sta(v) = B\},$$
$$C(T) = \{v \in V(T)|sta(v) = C\}.$$

In [3], it is obtained a locating-total domination number for $T \in \Gamma$ (see Lemma 3.1). We determine an open locating-total domination number for such trees.

Lemma 3.1. *If $T \in \Gamma$, then $\gamma^{OL}(T) = 2|A(T)| = \frac{n+l}{2}$.*

Theorem 3.2. *If $T \in \Gamma$, then $\gamma_t^{OL}(T) = \frac{3n-l}{4}$ and S is an open locating-total dominating set for T as following*

$$S = B(T) \cup \{L(T) \cap A(T)\} \cup \{N(B(T)) \cap C(T)\},$$

where $L(T)$ is set of leaves of T.

Proof. Let $T = T_k$ for $k \geq 0$. We proceed by induction on the order k. Let D be an OLTD-set in T_k. If $k = 0$, it is easy to prove that the result holds for $T_0 = P_6$. For $k = 1$, $T = T_1$ is obtained from T_0 by two operations τ_1 and τ_2. It is clear that S is an OLTD-set in T_1 as following

$$S = B(T_1) \cup \{L(T_1) \cap A(T_1)\} \cup \{N(B(T_1)) \cap C(T_1)\}.$$

Thus,

$$\begin{aligned} |S| = \gamma_t^{OL}(T_1) &= |B(T_1)| + |L(T_1) \cap A(T_1)| + |N(B(T_1)) \cap C(T_1)| \\ &= |B(T_1)| + l + |C(T_1)| - l \\ &= |B(T_1)| + |C(T_1)|. \end{aligned}$$

There is a support vertex for each of the leaves in the tree. Therefore, in $T \in \Gamma$, $|B(T)| = |C(T)|$. For tree T in the family of Γ, we have

$$\begin{aligned} |B(T)| &= |A(T)| + |N(C(T)) \cap B(T)| \\ &= |A(T)| + |C(T)| - l \\ &= 2|A(T)| - l. \end{aligned}$$

Using Lemma 3.1, we get

$$\gamma_t^{OL}(T_1) = 3|A(T)| - l = 3(\frac{n+l}{4}) - l = \frac{3n-l}{4}.$$

Assume that every tree $T_{k'}$ where $0 \leq k' \leq k-1$ with l' leaves satisfies this theorem. Let T_{k-1} be a tree of order n' having l' leaves. By adding a path with four vertices (the operation τ_1) to each of the leaves in T_{k-1} and doing τ_2 for any vertex with status B, we obtain tree T_k of order n with l leaves where

$$n = n' + 4l' + 3|B(T_{k-1})| = \frac{5}{2}(n' + l'), \tag{1}$$

$$l = l' + |B(T_{k-1})| = \frac{n' + l'}{2}. \tag{2}$$

According to operation τ_1, there is a path y, x, w, v, z in T_k in which $y \in T_{k-1}$ with $sta(y) = C$. It is clear that $v \in D$ and $y \notin \gamma_t^{OL}(T_{k-1})$-set. Since D is a total dominating set, $\{w, v\} \subseteq D$. We have

$$N(x) \cap D = \{w\} = N(v) \cap D,$$

which is a contradiction. Consequently, $y \in D$. Since y is a leaf in T_{k-1}, its support vertex is in $\gamma_t^{OL}(T_{k-1})$. Thus, we can have

$$\begin{aligned} D = &\Big[(\gamma_t^{OL}(T_{k-1}) - set) \cap \big\{L(T_{k-1}) \cap A(T_{k-1})\big\} \cup L(T_{k-1})\Big] \\ &\cup \Big\{\{x, w, v\} \mid in\ operacion\ \tau_1,\ y\ \in T_{k-1},\ edge\ yx\ and\ sta(y) = C\Big\} \\ &\cup \Big\{\{x, w\} \mid in\ operacion\ \tau_2,\ y \in T_{k-1},\ edge\ yx\ and\ sta(y) = B\Big\}. \end{aligned}$$

Applying the inductive hypothesis, the relations (1) and (2) we have

$$|D| = \frac{3n' - l'}{4} + 3l' + 2(\frac{n' + l'}{2} - l') = \frac{3n - l}{4}.$$

□

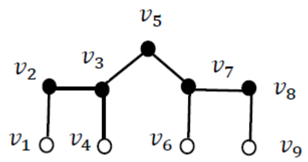

Figure 5. Open locating-total dominating set in $T \in \Lambda$ for $k = 2$.

In the end, we consider another family of trees and calculate the open locating-total domination number of them. Let Λ be the family of trees that can be obtained from k disjoint copies of P_4 by first adding $k-1$ edges in such a manner that they are incident only with support vertices and the resulting graph is connected, and then subdividing each new edge exactly once (see [3]).

Theorem 3.3. *If $T \in \Lambda$ and $n \geq 2$, then*

$$\gamma_t^{OL}(T) = \frac{1}{5}(3n - 2).$$

Proof. We proceed by induction on the order k. For $k = 2$, according to Figure 5 the set $\{v_2, v_3, v_5, v_7, v_8\}$ is an OLTS-set in T. Thus, $\gamma_t^{OL}(T) = 5 = \frac{1}{5}(3n-2)$. Assume that every tree T' for $k' \leq k - 1$ of order n' satisfies $\gamma_t^{OL}(T') = \frac{1}{5}(3n' - 2)$. According to Figure 6, $T = T' \cup \{v_{n-4}, v_{n-3}, v_{n-2}, v_{n-1}, v_n\}$ where

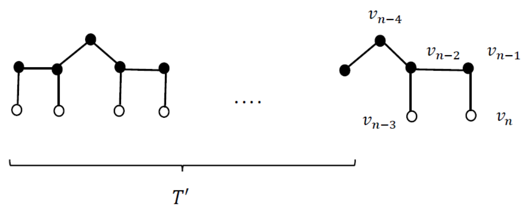

FIGURE 6. Open locating-total dominating set for $T \in \Lambda$.

$T' \in \Lambda$ for $k-1$. So, T' is a tree of order $n' = n - 5 = 5k - 6$ and applying the inductive hypothesis $\gamma_t^{OL}(T') = \frac{1}{5}(3n' - 2)$. Then clearly we see

$$\gamma_t^{OL}(T) - set = (\gamma_t^{OL}(T') - set) \cup \{v_{n-4}, v_{n-2}, v_{n-1}\}.$$

Therefore

$$\gamma_t^{OL}(T) = \gamma_t^{OL}(T') + 3 = \frac{1}{5}(3n' - 2) + 3 = \frac{1}{5}(3n - 2).$$

□

4. **Conclusion**

One of the effective methods in modeling the networks is the Locating Dominating sets. If any of these networks is considered as a graph, we will have the ability to detect a failure at any of the nodes by finding a Locating Dominating set on that graph. Due to using in modeling and applications, the theory of dominating sets and locating-dominating sets have been extensively studied.

The main purpose of this paper is to study the open locating-total dominating sets in graphs. We obtained an open locating-total dominating set for some graph families such as the join, the corona of graphs, complementary prisms of a graph, and the cartesian product of graphs. Some bounds on open locating-total domination numbers are obtained for these graphs. We also investigated the open locating-total domination number for two families of trees that is introduced in [3].

To conclude this paper, we suggest the following open problems.

Problem 3.1 Determine the bounds for the open locating-total domination number of trees.

Problem 3.2 Investigate open locating-total domination number in other families of graphs such as unicyclic graphs, Circulant Graphs, regular graphs, hypergraphs and etc.

Problem 3.3 Propose a heuristic algorithm for finding an open locating-total dominating set in a graph.

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