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THE FUZZY D'ALEMBERT SOLUTIONS OF THE FUZZY WAVE EQUATION UNDER GENERALIZED DIFFERENTIABILITY

M. Miri Karbasaki, M.R. Balooch shahryari ✉, and O. Sedaghatfar

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Abstract. In this paper, a one-dimensional homogeneous fuzzy wave equation is solved with an analytical procedure using the fuzzy D'Alembert method by considering the generalized differentiability. Then, some definitions related to fuzzy numbers, theorems, and used lemmas are given. Additionally, the physical interpretation and dependency domain of fuzzy wave solutions are investigated by providing examples, where the fuzzy wave solutions are in the form of fuzzy standing, traveling, and recursive waves. The abstract of this article was presented in the 9th Iranian Joint Congress on Fuzzy and Intelligent Systems (2022, March 2-4), Bam, Kerman, Iran.

Keywords: Generalized Hukuhara differentiability; Fuzzy partial differential equation; Fuzzy wave equation; Fuzzy D'Alembert method.
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1. Introduction

Partial differential equations provide an important tool for modeling numerous issues in various fields of engineering, physics, and systems involving variables of space and time. A class of differential equations with hyperbolic partial derivatives, which describe vibrations within objects and how waves propagate inside them, is called telegraph equations. Telegraph equations appear in the study of the propagation of electrical signals in a transmission line wire and wave phenomena. Transactions between heat transfer and diffusion or interaction reaction of diffusion introduces several non-linear phenomena in physics, chemistry, and biological processes [1–3, 9]. Wave equations and a particular class of telegraph equations are much more useful for modeling reaction propagation in the branches of the mentioned sciences than conventional diffusion equations. For example, biologists encounter such equations when studying the pulsatile blood flow in the arteries and the one-dimensional random movement of insects along with a rivet [24]. Porous and parallel flows

✉ baloochshahryari.1980@gmail.com, ORCID: 0000-0002-9599-8090
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of Maxwell viscous liquids [11] are only a few small examples that can be modeled with telegraph equations [10, 18, 19]. Fuzzy sets theory is a natural way of detecting dynamic systems under uncertainty and processing ambiguous information in mathematical models that are used for extensive real-time problems in science. In 1965, professor Lotfizade first introduced this theory to model ambiguous concepts in the real world [12]. To introduce fuzzy partial differential equations, we need to define the fuzzy derivative. The fuzzy derivative was initially introduced by Change and Zadeh [25], followed by Dubois and Prade [8], Puri and Ralescu [16], and Goetschel and Voxman [21] used this definition. For the first time, the concept of differential equations was formulated in a fuzzy environment by Kaleva [17]. In recent years, several authors have presented a wide range of results in both theoretical and applied fields. The pioneers in this theory typically followed the methods of Hukuhara derivation. But this method had a fundamental disadvantage, the length of support of the final solution increased indefinitely and this objection made this method fail to reflect the behavior of the system. This problem was corrected by the generalized derivative definition, based on the generalized difference, defined by Bede and Gal [5], then a complete theory of fuzzy differential equations was presented. After that, Stefanini described the generalized Hukuhara definitions in 2010 [13], and the generalized derivation, based on the generalized Hukuhara derivative, is redefined in 2013 [6]. Fuzzy partial differential equations often arise from the formulation of the fundamental laws of nature or mathematical analysis of uncertainty in applied mathematics and engineering sciences. Most laws of nature and physics, Newtonian motion laws and equations such as telegraph, heat, wave, etc. are in the form of fuzzy partial differential equations in uncertainty. These laws express the phenomena of physics by connecting fuzzy space and fuzzy derivatives with time. Professor Allahviranloo et al. presented an analytical method for solving the fuzzy heat equation under generalized Hukuhara derivation (gH-derivation) [26]. In this regard, we have solved the one-dimensional fuzzy wave equation in electromagnetic and telegraph equations using an analytical technique under generalized derivatives (gH-derivatives). Besides, the physical interpretations of the ambiguous wave responses are presented by giving examples, in which the solutions are shown as the fuzzy standing wave, fuzzy traveling wave, and fuzzy backward wave. Here, we present some basic concepts of fuzzy theory, including fuzzy sets, fuzzy numbers, fuzzy new definitions, gH-derivatives, and related concepts used in this study. This paper aims to attain a solution for a fuzzy wave equation under generalized partial Hukuhara differentiability by the fuzzy D'Alembert method. To find the solution to some properties for generalized partial Hukuhara differentiability are provided. In Section 2, we introduce the basic concept of generalized Hukuhara derivative. Also, multi-variables calculus for fuzzy function and some significant properties of this concept is discussed. In Section 3, the physical interpretation of fuzzy D'Alembert solutions for the fuzzy wave under generalized Hukuhara derivative is defined. In Section 4, the fuzzy D'Alembert solutions for the fuzzy wave on an infinite string under the generalized derivative is studied. In Section 5, the solution method and domain of dependence are explored. In Section 6, the conclusion is given, and the results are shown in some examples. The abstract of this article was presented in [15].

2. Preliminaries

The basic definitions and theorems that we need in this article are considered. In addition, some new concepts are proved here. We denote $\mathbb{R}_{\mathcal{F}}$, the set of fuzzy numbers, that is, normal, fuzzy convex, upper semi-continuous, and compactly supported fuzzy sets that are defined over the real line. Let $\tilde{u} \in \mathbb{R}_{\mathcal{F}}$ be a fuzzy number; for $0 < r \leq 1$, the r-level set (or r-cut) of $\tilde{u}$ is defined by $[\tilde{u}]_r = \{x \in \mathbb{R}^n | \tilde{u}(x) \geq r\}$, and for $r = 0$ is defined by the closure of support $[\tilde{u}]_0 = cl\{x \in \mathbb{R}^n | \tilde{u}(x) > 0\}$. We denote $[\tilde{u}]_r = [u_r^-, u_r^+]$, so the r-level set $[\tilde{u}]_r$ is a closed interval for all $r \in [0, 1]$. If $\tilde{u}, \tilde{v} \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, the addition and scalar multiplication are defined as having the r-levels set of $[\tilde{u}+\tilde{v}]_r = [\tilde{u}]_r + [\tilde{v}]_r$ and $[\lambda\tilde{u}]_r = \lambda[\tilde{u}]_r$, respectively. The triangular fuzzy number $\tilde{u} \in \mathbb{R}_{\mathcal{F}}$ is defined as an ordered triple $A = (a_1, a_2, a_3)$ with $a_1 \leq a_2 \leq a_3$.

Definition 2.1. [30] We denote by $\mathcal{K}_C$ the family of all bounded closed intervals in $\mathbb{R}$, i.e.,

$$\mathcal{K}_C = \Big\{A = [\underline{a}, \overline{a}] / \underline{a}, \overline{a} \in \mathbb{R} \textit{ and } \underline{a} \leq \overline{a}\Big\}.$$

Definition 2.2. [30] Suppose that $A = [\underline{a}, \overline{a}] \in \mathcal{K}_C$. Then interval length of A is showed by $len(A)$ and is defined as $len(A) = \overline{a} - \underline{a}$.

In the bellow statement, the properties of interval length are expressed.

Proposition 2.3. *[30] Let $a \in \mathbb{R}$, and $A, B \in \mathcal{K}_C$. Then below relations are established for interval length:*

(1) $len(A) \geq 0$,
(2) $len(A + B) = len(A) + len(B)$,
(3) $len(\alpha A) = |a|\ len(A)$,
(4) *if* $A \ominus_H B$ *exists, then* $len(A \ominus_H B) = |len(A) - len(B)|$.

Definition 2.4. [22] The generalized Hukuhara difference of two fuzzy numbers $A, B \in \mathbb{R}_{\mathcal{F}}$ is the fuzzy number C, (if it exists), such that

$$A \ominus_{gH} B = C \Longleftrightarrow \begin{cases} (i)\ A = B + C, \\ \text{or } (ii)\ B = A + (-1)C, \end{cases}$$

then

$$A \ominus_{gH} B = C \Longleftrightarrow$$
$$\begin{cases} (i)\ C = (a_1 - b_1, a_2 - b_2, a_3 - b_3) \\ \text{or } (ii)\ C = (a_3 - b_3, a_2 - b_2, a_1 - b_1) \end{cases}$$

Provided that C is a triangular fuzzy number [5, 7]. The results obtained in [7] show that if $A, B \in \mathbb{R}_{\mathcal{F}}$, then $A \ominus_{gH} B$ always exists in $\mathbb{R}_{\mathcal{F}}$.

Definition 2.5. [6] The generalized difference (or g-difference for short) of two fuzzy numbers $\tilde{u}, \tilde{v} \in \mathbb{R}_{\mathcal{F}}$ is defined by its level-sets as

$$[\tilde{u} \ominus_g \tilde{v}]_r = cl\Big(conv \bigcup_{\beta \geq r} [\tilde{u}]_\beta \ominus_{gH} [\tilde{v}]_\beta\Big) \text{ for all } r \in [0,1], \tag{1}$$

where the gH-difference $\ominus_{gH}$ is with interval operands $[\tilde{u}]_\beta$ and $[\tilde{v}]_\beta$.

Proposition 2.6. *[6] The g-difference* (1) *is given by the expression*

$$[\tilde{u} \ominus_g \tilde{v}]_r = \Big[\inf_{\beta \geq r} \min\{\tilde{u}_\beta^- - \tilde{v}_\beta^-, \tilde{u}_\beta^+ - \tilde{v}_\beta^+\}, \sup_{\beta \geq r} \max\{\tilde{u}_\beta^- - \tilde{v}_\beta^-, \tilde{u}_\beta^+ - \tilde{v}_\beta^+\}\Big].$$

Definition 2.7. [28] The Hausdorff distance between fuzzy numbers is given by $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \longrightarrow \mathbb{R}^+ \bigcup \{0\}$ as in [18]

$$D(\tilde{u}, \tilde{v}) = \sup_{r \in [0,1]} d\Big([\tilde{u}]^r, [\tilde{v}]^r\Big) = \sup_{r \in [0,1]} \max\{|u^-(r) - v^-(r)|, |u^+(r) - v^+(r)|\},$$

where d is the Hausdorff metric. The metric space $(\mathbb{R}_{\mathcal{F}}, D)$ is complete, separable and locally compact and the following properties from [18] for metric D are valid:

(1) $D(\tilde{u} \oplus \tilde{w}, \tilde{v} \oplus \tilde{w}) = D(\tilde{u}, \tilde{v}),\ \forall\ \tilde{u}, \tilde{v} \in \mathbb{R}_{\mathcal{F}}$;
(2) $D(\lambda\tilde{u}, \lambda\tilde{v}) = |\lambda| D(\tilde{u}, \tilde{v}),\ \forall \lambda \in \mathbb{R},\ \tilde{u}, \tilde{v} \in \mathbb{R}_{\mathcal{F}}$;
(3) $D(\tilde{u} \oplus \tilde{v}, \tilde{w} \oplus \tilde{z}) \leq D(\tilde{u}, \tilde{w}) + D(\tilde{v}, \tilde{z})$; $\forall\ \tilde{u}, \tilde{v}, \tilde{w}, \tilde{z} \in \mathbb{R}_{\mathcal{F}}$;
(4) $D(\tilde{u} \ominus \tilde{v}, \tilde{w} \ominus \tilde{z}) \leq D(\tilde{u}, \tilde{v}) + D(\tilde{v}, \tilde{z})$, as long as $\tilde{u} \ominus \tilde{v}$, $\tilde{w} \ominus \tilde{z}$ exist and $\tilde{u}, \tilde{v}, \tilde{w}, \tilde{z} \in \mathbb{R}_{\mathcal{F}}$,

where $\ominus_H$ is the Hukuhara difference (H-difference), it means that $\tilde{w} \ominus_H \tilde{v} = \tilde{u}$ if and only if $\tilde{u} \oplus \tilde{v} = \tilde{w}$.

Definition 2.8. A fuzzy number $\tilde{u}$ is called a singleton fuzzy number if the membership degree of $\tilde{u}$ is one and the membership degrees for the other members are zero. In other words

$$\tilde{u} = \begin{cases} 1, & x = u, \\ 0, & x \neq u, \end{cases}$$

especially

$$\tilde{0} = \chi_{\{0\}} = \begin{cases} 1, & x = 0, \\ 0, & x \neq 0. \end{cases}$$

Proposition 2.9. *[22] Let λ_1 and λ_2 are two real constants such that $\lambda_1, \lambda_2 \geq 0$ (or $\lambda_1, \lambda_2 \leq 0$). If $\tilde{f}(t)$ is a triangular fuzzy function, then*

$$\lambda_1 \tilde{f}(t) \ominus_{gH} \lambda_2 \tilde{f}(t) = (\lambda_1 - \lambda_2)\tilde{f}(t).$$

Definition 2.10. [4] The generalized Hukuhara derivative of a fuzzy-valued function $\tilde{f} : (a, b) \longrightarrow \mathbb{R}_{\mathcal{F}}$ at $x_0 \in (a, b)$ is defined as

$$\tilde{f}'_{gH}(x_0) = \lim_{h \to 0} \frac{\tilde{f}(x_0 + h) \ominus_{gH} \tilde{f}(x_0)}{h}, \tag{2}$$

if $\tilde{f}'_{gH}(x_0) \in \mathbb{R}_{\mathcal{F}}$ satisfying (2) exists, we say that $\tilde{f}$ is generalized Hukuhara differentiable (gH-differentiable for short) at x_0. In addition, we can say that $\tilde{f}(x)$

• $[(i) - gH]$-differentiable function if and only if for all $x \in (a, b)$

$$\tilde{f}'_{i.gH}(x) = \big(\tilde{f}'_1(x), \tilde{f}'_2(x), \tilde{f}'_3(x)\big), \tag{3}$$

defines a triangular fuzzy number.

• $[(ii) - gH]$-differentiable function if and only if for all $x \in (a, b)$

$$\tilde{f}'_{ii.gH}(x) = \big(\tilde{f}'_3(x), \tilde{f}'_2(x), \tilde{f}'_1(x)\big), \tag{4}$$

is a triangular fuzzy number.

Definition 2.11. [14] We say that a point $x_0 \in (a, b)$ is a switching point for differentiability of $\tilde{f}$, if in any neighborhood V of x_0 there exist points $x_1 < x_0 < x_2$ such that

type(I) at x_1, (3) holds while (4) does not hold and at x_2, (4) holds and (3) does not hold, or

type(II) at x_1, (4) holds while (3) does not hold and at x_2, (3) holds and (4) does not hold.

Definition 2.12. A fuzzy-valued function $\tilde{f}$ of two variables is a rule that assigns to each ordered pair of real numbers (x, t), in a set D, a unique fuzzy number denoted by $\tilde{f}(x, t)$. The set D is the domain of $\tilde{f}$ and its range is the set of values that $\tilde{f}$ takes on, that is, $\{\tilde{f}(x, t) | (x, t) \in D\}$.

Definition 2.13. [26] Let $(x_0, t_0) \in D$. Then the first generalized Hukuhara partial derivative ($[gH - p]$-derivative for short) of a fuzzy-valued function $\tilde{f}(x, t) : \mathbb{I} \longrightarrow \mathbb{R}_{\mathcal{F}}$ at (x_0, t_0), $w.r.t.$ x and t are the functions $\frac{\partial \tilde{f}_{gH}}{\partial x}(x_0, t_0)$ and $\frac{\partial \tilde{f}_{gH}}{\partial t}(x_0, t_0)$ given by

$$\frac{\partial \tilde{f}_{gH}}{\partial x}(x_0, t_0) = \lim_{h \longrightarrow 0} \frac{\tilde{f}(x_0 + h, t_0) \ominus_{gH} \tilde{f}(x_0, t_0)}{h},$$

and

$$\frac{\partial \tilde{f}_{gH}}{\partial t}(x_0, t_0) = \lim_{k \longrightarrow 0} \frac{\tilde{f}(x_0, t_0 + k) \ominus_{gH} \tilde{f}(x_0, t_0)}{k},$$

provided that $\frac{\partial \tilde{f}_{gH}}{\partial x}(x_0, t_0)$ and $\frac{\partial \tilde{f}_{gH}}{\partial t}(x_0, t_0) \in \mathbb{R}_{\mathcal{F}}$.

Definition 2.14. [27] A triangular fuzzy function $\tilde{u}(x,t) : D \subseteqq \mathbb{R}^2 \to \mathbb{R}_{\mathcal{F}}$, without any switching point on D, is called
• $[(i) - gH]$-differentiable *w.r.t.* x at (x_0, t_0) if

$$\frac{\partial \tilde{u}_{i.gH}}{\partial x}(x_0,t_0) = \Big(\frac{\partial \tilde{u}_1}{\partial x}(x_0,t_0), \frac{\partial \tilde{u}_2}{\partial x}(x_0,t_0), \frac{\partial \tilde{u}_3}{\partial x}(x_0,t_0)\Big),$$

• $[(ii) - gH]$-differentiable *w.r.t.* x at (x_0, t_0) if

$$\frac{\partial \tilde{u}_{ii.gH}}{\partial x}(x_0,t_0) = \Big(\frac{\partial \tilde{u}_3}{\partial x}(x_0,t_0), \frac{\partial \tilde{u}_2}{\partial x}(x_0,t_0), \frac{\partial \tilde{u}_1}{\partial x}(x_0,t_0)\Big).$$

Moreover, if $\frac{\partial \tilde{u}_{gH}}{\partial x}$ is $[gH - p]$-differentiable at (x_0, t_0) *w.r.t.* x without any switching point on D and, if the type of $[gH-p]$-differentiability of both $\tilde{u}(x,t)$ and $\frac{\partial \tilde{u}_{gH}}{\partial x}$ are the same, then $\frac{\partial \tilde{u}_{gH}}{\partial x}$ is $[(i) - p]$-differentiable *w.r.t.* x and

$$\frac{\partial^2 \tilde{u}_{i.gH}}{\partial x^2}(x_0,t_0) = \Big(\frac{\partial^2 \tilde{u}_1}{\partial x^2}(x_0,t_0), \frac{\partial^2 \tilde{u}_2}{\partial x^2}(x_0,t_0), \frac{\partial^2 \tilde{u}_3}{\partial x^2}(x_0,t_0)\Big),$$

if the type of $[gH - p]$-differentiability of both $\tilde{u}(x,t)$ and $\frac{\partial \tilde{u}_{gH}}{\partial x}$ are different, then $\frac{\partial \tilde{u}_{gH}}{\partial x}$ is $[(ii) - p]$-differentiable *w.r.t.* x and

$$\frac{\partial^2 \tilde{u}_{ii.gH}}{\partial x^2}(x_0,t_0) = \Big(\frac{\partial^2 \tilde{u}_3}{\partial x^2}(x_0,t_0), \frac{\partial^2 \tilde{u}_2}{\partial x^2}(x_0,t_0), \frac{\partial^2 \tilde{u}_1}{\partial x^2}(x_0,t_0)\Big).$$

Proposition 2.15. *[7] That is $[(i)-gH]$-derivative and $[(ii)-gH]$-derivative are additive operators, i.e., if $\tilde{f}$ and $\tilde{g}$ are both $[(i) - gH]$-differentiable or both $[(ii) - gH]$-differentiable then*

(i) $(\tilde{f} \oplus \tilde{g})'_{(i)-gH} = \tilde{f}'_{(i)-gH} \oplus \tilde{g}'_{(i)-gH}$,

(ii) $(\tilde{f} \oplus \tilde{g})'_{(ii)-gH} = \tilde{f}'_{(ii)-gH} \oplus \tilde{g}'_{(ii)-gH}$.

Remark 2.16. [7] From Proposition 2.15, it follows that $[(i) - gH]$-derivative and $[(ii) - gH]$-derivative are semi-linear operators (that is, additive homogeneous). They are not linear in general since we have

$$(k\tilde{f}_{gH})'_{(i)-gH} = k(\tilde{f}_{gH})'_{(ii)-gH},$$

if $k < 0$.

Lemma 2.17. *[22] Consider $g : [a,b] \to I \subseteq \mathbb{R}$ is real and differentiable function at x, and $\tilde{f} : I \to \mathbb{R}_{\mathcal{F}}$ is gH-differentiable at the point $g(x)$ without any switching points. Then type of gH-differentiability for $\tilde{f}(x)$ and $\tilde{f}(g(x))$ is the same if*

$$\big(\tilde{f}(g(x))\big)' = \begin{cases} \textit{If } g(x) \textit{ is an increasing function, then} \\ g'(x) \odot \tilde{f}'_{gH}(g(x)), \\ \textit{If } g(x) \textit{ is an increasing function, then} \\ \ominus_{gH}(-1)g'(x) \odot \tilde{f}'_{gH}(g(x)). \end{cases}$$

Theorem 2.18. *[23] Let $I \subseteq \mathbb{R}$ be an open interval and $x \in I$. Let $\tilde{f} : I \to \mathbb{R}_{\mathcal{F}}$ and $g : I \longrightarrow \mathbb{R}^+$. Suppose that $g(x)$ is differentiable at x and the fuzzy function $\tilde{f}(x)$ is gH-differentiable at x. Then*

$$(\tilde{f} \odot g)'_{gH}(x) = \tilde{f}'_{gH}(x) \odot g(x) \oplus \tilde{f}(x) \odot g'(x).$$

Theorem 2.19. *[23] Let $I \subseteq \mathbb{R}$ be an open interval and $x \in I$. Let $\tilde{f}, \tilde{g} : I \to \mathbb{R}_{\mathcal{F}}$ are gH-differentiable with the same type of gH-differentiability at x. Then $\tilde{f}(x) \ominus_{gH} \tilde{g}(x)$ is gH-differentiable, and*

$$(\tilde{f} \ominus_{gH} \tilde{g})'(x) = \tilde{f}'_{gH}(x) \ominus_{gH} \tilde{g}'_{gH}(x).$$

Theorem 2.20. *[23] Let I be an open interval in $\mathbb{R}$. Consider $g : I \to \zeta := g(I) \subseteq \mathbb{R}$ is differentiable at x, and $\tilde{f} : \zeta \to \mathbb{R}_{\mathcal{F}}$ is gH-differentiable at the point $g(x)$. Then we have the following conditions:*
If $g'(x) > 0$

$$(\tilde{f}og)_{i-gH}(x) = g'(x) \odot \tilde{f}'_{i-gH}(g(x)),$$
$$(\tilde{f}og)_{ii-gH}(x) = g'(x) \odot \tilde{f}'_{ii-gH}(g(x)),$$

If $g'(x) < 0$

$$(\tilde{f}og)_{i-gH}(x) = g'(x) \odot \tilde{f}'_{ii-gH}(g(x)),$$
$$(\tilde{f}og)_{ii-gH}(x) = g'(x) \odot \tilde{f}'_{i-gH}(g(x)).$$

Theorem 2.21. *[23] (The fuzzy Chain rule)*
Let $\tilde{u} := \tilde{U}(\xi(t), \eta(t))$ is a fuzzy-valued function, where $\xi(t)$ and $\eta(t)$ are differentiable real-valued functions of t. Then, $\tilde{U}$ is the gH-differentiable function of t, therefore we have

$$\frac{\partial \tilde{u}}{\partial t} = \frac{\partial \tilde{U}_{gH}}{\partial \xi} \odot \frac{\partial \xi}{\partial t} \oplus \frac{\partial \tilde{U}_{gH}}{\partial \eta} \odot \frac{\partial \eta}{\partial t}.$$

Theorem 2.22. *[22] Let $\tilde{u}(x,t) = \tilde{U}(\xi)$ be a fuzzy-valued function, where $\xi(x,t)$ is a differentiable real-valued function of x and t. Then, $\tilde{U}$ is the gH-differentiable function of ξ, and we have*

$$\frac{\partial \tilde{u}}{\partial t} = \frac{\partial \tilde{U}_{gH}}{\partial \xi} \odot \frac{\partial \xi}{\partial t}.$$

Also

(1). *If $\frac{\partial \xi}{\partial t} > 0$ and $\tilde{U}(\xi)$ is $[(i) - gH]$-differentiable, then $\tilde{u}(x,t)$ is $[(i) - p]$-differentiable w.r.t. t.*
$\tilde{U}(\xi)$ is $[(ii) - gH]$-differentiable then $\tilde{u}(x,t)$ is $[(ii) - p]$-differentiable w.r.t. t.

(2). *If $\frac{\partial \xi}{\partial t} < 0$ and $U(\xi)$ is $[(i) - gH]$-differentiable, then $\tilde{u}(x,t)$ is $[(ii) - p]$-differentiable w.r.t. t.*
$\tilde{U}(\xi)$ is $[(ii) - gH]$-differentiable then $\tilde{u}(x,t)$ is $[(i) - p]$-differentiable w.r.t. t.

Theorem 2.23. *[22] Let $\tilde{u}(x,t) := \tilde{U}(\xi(x,t),\eta(x,t))$ and $\tilde{u}(x,t)$ is a $[gH-p]$-differentiable function such that the second-order $[(gH)-p]$-derivatives w.r.t. t and x exists. Then*

$$\frac{\partial^2 \tilde{u}_{gH}}{\partial t^2} = \Big(\frac{\partial^2 \tilde{U}_{gH}}{\partial \xi^2} \odot \frac{\partial \xi}{\partial t} \oplus \frac{\partial}{\partial \eta}\Big(\frac{\partial \tilde{U}_{gH}}{\partial \xi}\Big) \odot \frac{\partial \eta}{\partial t}\Big) \odot \frac{\partial \xi}{\partial t} \oplus \frac{\partial \tilde{U}}{\partial \xi} \odot \frac{\partial^2 \xi}{\partial t^2} \oplus \Big(\frac{\partial}{\partial \xi}\Big(\frac{\partial}{U_{gH}} \partial \eta\Big) \odot \frac{\partial \xi}{\partial t} \oplus \frac{\partial^2 \tilde{U}_{gH}}{\partial \eta^2} \odot \frac{\partial \eta}{\partial t}\Big) \odot \frac{\partial \eta}{\partial t} \oplus \frac{\partial \tilde{U}}{\partial \eta} \odot \frac{\partial^2 \eta}{\partial t^2},$$

and

$$\frac{\partial^2 \tilde{u}_{gH}}{\partial x^2} = \Big(\frac{\partial^2 \tilde{U}_{gH}}{\partial \xi^2} \odot \frac{\partial \xi}{\partial x} \oplus \frac{\partial}{\partial \eta}\Big(\frac{\partial \tilde{U}_{gH}}{\partial \xi}\Big) \odot \frac{\partial \eta}{\partial x}\Big) \odot \frac{\partial \xi}{\partial x} \oplus \frac{\partial \tilde{U}}{\partial \xi} \odot \frac{\partial^2 \xi}{\partial x^2} \oplus \Big(\frac{\partial}{\partial \xi}\Big(\frac{\partial \tilde{U}_{gH}}{\partial \eta}\Big) \odot \frac{\partial \xi}{\partial x} \oplus \frac{\partial^2 \tilde{U}_{gH}}{\partial \eta^2} \odot \frac{\partial \eta}{\partial x}\Big) \odot \frac{\partial \eta}{\partial x} \oplus \frac{\partial \tilde{U}}{\partial \eta} \odot \frac{\partial^2 \eta}{\partial x^2}.$$

Definition 2.24. [29] Let $\tilde{f} : [a,b] \longrightarrow \mathbb{R}_{\mathcal{F}}$. We say that $\tilde{f}(x)$ is fuzzy Riemann integrable in $\mathbb{I} \in \mathbb{R}_{\mathcal{F}}$ if for any $\epsilon > 0$, there exists $\delta > 0$ such that for any division $P = \{[u,v];\xi\}$ with the norms $\Delta(P) < \delta$, we have

$$D\Big(\sum_{p}^{*}(v-u) \odot \tilde{f}(\xi), \mathbb{I}\Big) < \epsilon,$$

where $\sum_p^*$ denotes the fuzzy summation. We choose to write $\mathbb{I} := \int_a^b \tilde{f}(x)dx$.

Definition 2.25. [4] Let $\tilde{f} : (a,b) \to \mathbb{R}_{\mathcal{F}}$ is a triangular fuzzy-valued function and $x_0 \in (a,b)$. Then

$$\int_a^b \tilde{f}(x)dx = \Big(\int_a^b \tilde{f}_1(x)dx, \int_a^b \tilde{f}_2(x)dx, \int_a^b \tilde{f}_3(x)dx\Big).$$

Lemma 2.26. *[23] $\int_b^a \tilde{u}(x,t)dx = \ominus_H \int_a^b \tilde{u}(x,t)dx$, where $\ominus_H$ denotes the H-difference and $\tilde{u}(x,t)$ be a fuzzy-valued function.*

Lemma 2.27. *[20] If $\tilde{f} : (a,b) \to \mathbb{R}_{\mathcal{F}}$ is integrable and $c \in (a,b)$, then*

$$\int_a^c \tilde{f}(t)dt \oplus \int_c^b \tilde{f}(t) = \int_a^b \tilde{f}(t)dt.$$

Theorem 2.28. *[7] If $\tilde{f}$ is gH-differentiable with no switching point in the interval $[a, b]$, then we have*

$$\int_a^b \tilde{f}'_{gH}(x)dx = \tilde{f}(b) \ominus_{gH} \tilde{f}(a).$$

Lemma 2.29. *[22] If $\tilde{f} : [a, b] \to \mathbb{R}_{\mathcal{F}}$ is a triangular fuzzy function with no switching point, then we have*

(1). *If $\tilde{f}(x)$ is $[i - gH]$-differentiable, then*

$$\int_a^b \tilde{f}'_{gH}(x)dx = \tilde{f}(b) \ominus_{gH} \tilde{f}(a).$$

(2). *If $\tilde{f}(x)$ is $[(ii) - gH]$-differentiable, then*

$$\int_a^b \tilde{f}'_{gH}(x)dx = (-1)\tilde{f}(a) \ominus_{gH} (-1)\tilde{f}(b).$$

Lemma 2.30. *[23] If $\tilde{f}(x, y)$ and $\tilde{g}(x, y)$ are the fuzzy-valued functions, then we have*

$$\int \tilde{f}(x, y)dx \ominus_{gH} \int \tilde{g}(x, y)dx = \int \big(\tilde{f}(x, y) \ominus_{gH} \tilde{g}(x, y)\big)dx.$$

3. The physical interpretation of fuzzy D'Alembert solutions for the fuzzy wave under generalized Hukuhara derivative

Consider that we have a linear fuzzy partial differential equation (FPDE) in the following form

$$\tilde{F}\Big(\tilde{u}, \frac{\partial \tilde{u}_{gH}}{\partial t}, \frac{\partial \tilde{u}_{gH}}{\partial x}, \frac{\partial^2 \tilde{u}_{gH}}{\partial t^2}, \frac{\partial^2 \tilde{u}_{gH}}{\partial x^2}\Big) = \tilde{0}, \tag{5}$$

where $\tilde{u} = \tilde{u}(x, t)$ is an unknown fuzzy function, $\tilde{F}$ is a polynomial in $\tilde{u}$ and its generalized Hukuhara derivatives.

With the change of variable, we can define the new coordinates

$$\xi(x, t) = x - ct, \;\; \eta(x, t) = x + ct, \tag{6}$$

and let

$$\tilde{u}(x, t) = \tilde{U}(\xi, \eta),$$

where $c \in \mathbb{R}^+$ is an arbitrary constant generally termed the wave velocity. In this paper, we consider $c > 0$, which means the profile $\tilde{U}(x - ct)$ at a later time t is moving to the positive x-direction by an amount ct with speed c and $\tilde{U}(x + ct)$ at a later time, t is moving to the negative x-direction by an amount ct with speed c. By differentiating (6) *w.r.t.* x and t yields, we have

$$\frac{\partial \xi}{\partial t} = -c, \frac{\partial \eta}{\partial t} = c, \frac{\partial \xi}{\partial x} = 1, \frac{\partial \eta}{\partial x} = 1,$$
$$\frac{\partial^2 \xi}{\partial x^2} = \frac{\partial^2 \eta}{\partial x^2} = \frac{\partial^2 \xi}{\partial t^2} = \frac{\partial^2 \eta}{\partial t^2} = 0.$$

Substituting the above equation in (5) and considering the type of gH-differentiability for $\tilde{U}$ and by using Theorems 2.21 and 2.23, we have

$$\begin{aligned}
&\tilde{F}\Big(\tilde{U},\ (-1)c\odot\frac{\partial\tilde{U}_{ii.gH}}{\partial\xi}\oplus c\odot\frac{\partial\tilde{U}_{i.gH}}{\partial\eta},\ \frac{\partial\tilde{U}_{i.gH}}{\partial\xi}\\
&\oplus\frac{\partial_{i.gH}\tilde{U}}{\partial\eta}, c^2\odot\frac{\partial^2\tilde{U}_{i.gH}}{\partial\xi^2}\oplus(-1)c^2\odot\frac{\partial}{\partial\eta}\Big(\frac{\partial\tilde{U}_{ii.gH}}{\partial\xi}\Big)\\
&\oplus(-1)c^2\odot\frac{\partial}{\partial\xi}\Big(\frac{\partial\tilde{U}_{i.gH}}{\partial\eta}\Big)\oplus c^2\odot\frac{\partial^2\tilde{U}_{i.gH}}{\partial\eta^2},\ \frac{\partial^2\tilde{U}_{i.gH}}{\partial\xi^2}\\
&\oplus\frac{\partial}{\partial\eta}\Big(\frac{\partial\tilde{U}_{i.gH}}{\partial\xi}\Big)\oplus\frac{\partial}{\partial\xi}\Big(\frac{\partial\tilde{U}_{i.gH}}{\partial\eta}\Big)\oplus\frac{\partial^2\tilde{U}_{i.gH}}{\partial\eta^2}\Big)=\tilde{0},
\end{aligned}$$

and

$$\begin{aligned}
&F\Big(\tilde{U},\ (-1)c\odot\frac{\partial\tilde{U}_{i.gH}}{\partial\xi}\oplus c\odot\frac{\partial\tilde{U}_{ii.gH}}{\partial\eta},\ \frac{\partial\tilde{U}_{ii.gH}}{\partial\xi}\\
&\oplus\frac{\partial\tilde{U}_{ii.gH}}{\partial\eta}, c^2\odot\frac{\partial^2\tilde{U}_{ii.gH}}{\partial\xi^2}\oplus(-1)c^2\odot\frac{\partial}{\partial\eta}\Big(\frac{\partial}{U_{\ i.gH}}\partial\xi\Big)\\
&\oplus(-1)c^2\odot\frac{\partial}{\partial\xi}\Big(\frac{\partial\tilde{U}_{ii.gH}}{\partial\eta}\Big)\oplus c^2\odot\frac{\partial^2\tilde{U}_{ii.gH}}{\partial\eta^2},\ \frac{\partial^2\tilde{U}_{ii.gH}}{\partial\xi^2}\\
&\oplus\frac{\partial}{\partial\eta}\Big(\frac{\partial\tilde{U}_{ii.gH}}{\partial\xi}\Big)\oplus\frac{\partial}{\partial\xi}\Big(\frac{\partial\tilde{U}_{ii.gH}}{\partial\eta}\Big)\oplus\frac{\partial^2\tilde{U}_{ii.gH}}{\partial\eta^2}\Big)=\tilde{0}.
\end{aligned}$$

Let us consider the homogeneous one-dimensional fuzzy wave equation

$$\text{FPDE}:\ \frac{\partial^2\tilde{u}_{gH}(x,t)}{\partial t^2}\ominus_{gH}c^2\odot\frac{\partial^2\tilde{u}_{gH}(x,t)}{\partial x^2}=\tilde{0},\qquad -\infty<x<\infty,\ t>0, \tag{7}$$

$$\text{FIC}_s:\ \begin{cases}\tilde{u}(x,0)=\tilde{\gamma}\odot f(x)=\tilde{f}(x),\\ \frac{\partial_{gH}}{\partial t}\tilde{u}(x,0)=\tilde{\beta}\odot g(x)=\tilde{g}(x),\end{cases} \tag{8}$$

where $\tilde{\gamma},\tilde{\beta}\in\mathbb{R}_{\mathcal{F}}$ and $f,g:\mathbb{R}\longrightarrow\mathbb{R}$, f of twice and g is once continuously differentiable and $c\in(0,+\infty)$, $c^2=\dfrac{\tau}{\rho}$ is the coefficient constant where the tension τ is force per unit length and ρ is the mass of the undeflected membrane per unit area. Now by considering the type of $[(gH)-p]$-differentiability for $\tilde{U}(\xi,\eta)$, the following cases are obtained.

Case(i). Let $\tilde{u}(x,t)$ and $\frac{\partial\tilde{u}}{\partial t}$ are $[(i)-p]$-differentiable fuzzy functions $w.r.t.$ x,t then

• If $\tilde{U}(\xi,\eta)$ is $[(i)-gH]$-differentiable $w.r.t.$ t and $\frac{\partial\tilde{U}}{\partial\xi}$ is $[(ii)-gH]$-differentiable $w.r.t.$ t and $\frac{\partial\tilde{U}}{\partial\eta}$ is $[(i)-gH]$-differentiable $w.r.t.$ t fuzzy function without any switching points then

$$\frac{\partial \tilde{u}_{i.gH}}{\partial t} = \frac{\partial \tilde{U}_{ii.gH}}{\partial \xi} \odot \frac{\partial \xi}{\partial t} \oplus \frac{\partial \tilde{U}_{i.gH}}{\partial \eta} \odot \frac{\partial \eta}{\partial t} = (-1)c \odot \frac{\partial \tilde{U}_{ii.gH}}{\partial \xi} \oplus c \odot \frac{\partial \tilde{U}_{i.gH}}{\partial \eta}.$$

• If $\tilde{U}(\xi,\eta)$ is $[(i)-gH]$-differentiable $w.r.t.$ x and $\frac{\partial \tilde{U}}{\partial \xi}$ and $\frac{\partial \tilde{U}}{\partial \eta}$ are $[(i)-gH]$-differentiable $w.r.t.$ x fuzzy functions, then

$$\frac{\partial \tilde{u}_{i.gH}}{\partial x} = \frac{\partial \tilde{U}_{i.gH}}{\partial \xi} \odot \frac{\partial \xi}{\partial x} \oplus \frac{\partial \tilde{U}_{i.gH}}{\partial \eta} \odot \frac{\partial \eta}{\partial x} = \frac{\partial \tilde{U}_{i.gH}}{\partial \xi} \oplus \frac{\partial \tilde{U}_{i.gH}}{\partial \eta}.$$

• If $\tilde{U}(\xi,\eta)$ is $[(i)-gH]$-differentiable $w.r.t.$ t and $\frac{\partial^2 \tilde{U}}{\partial \xi^2}$ and $\frac{\partial^2 \tilde{U}}{\partial \eta^2}$ are $[(i)-gH]$-differentiable $w.r.t.$ t fuzzy functions, then

$$\begin{aligned}\frac{\partial^2 \tilde{u}_{i.gH}}{\partial t^2} =& c^2 \odot \frac{\partial^2 \tilde{U}_{i.gH}}{\partial \xi^2} \oplus (-1)c^2 \odot \frac{\partial}{\partial \eta}\Big(\frac{\partial \tilde{U}_{ii.gH}}{\partial \xi}\Big) \oplus (-1)c^2 \odot \frac{\partial}{\partial \xi}\Big(\frac{\partial \tilde{U}_{i.gH}}{\partial \eta}\Big)\\ &\oplus c^2 \odot \frac{\partial^2 \tilde{U}_{i.gH}}{\partial \eta^2}.\end{aligned} \tag{9}$$

• If $\tilde{U}(\xi,\eta)$ is $[(i)-gH]$-differentiable $w.r.t.$ x and $\frac{\partial^2 \tilde{U}}{\partial \xi^2}$ and $\frac{\partial^2 \tilde{U}}{\partial \eta^2}$ are $[(i)-gH]$-differentiable $w.r.t.$ x fuzzy functions, then

$$\frac{\partial^2 \tilde{u}_{i.gH}}{\partial x^2} = \frac{\partial^2 \tilde{U}_{i.gH}}{\partial \xi^2} \oplus \frac{\partial}{\partial \eta}\Big(\frac{\partial \tilde{U}_{i.gH}}{\partial \xi}\Big) \oplus \frac{\partial}{\partial \xi}\Big(\frac{\partial \tilde{U}_{i.gH}}{\partial \eta}\Big) \oplus \frac{\partial^2 \tilde{U}_{i.gH}}{\partial \eta^2}. \tag{10}$$

Substituting derivatives (9) and (10) into the FPDE (7) yields therefore

$$-2c^2 \odot \frac{\partial}{\partial \xi}\Big(\frac{\partial \tilde{U}_{i.gH}}{\partial \eta}(\xi,\eta)\Big) = \tilde{0}.$$

By using Definition 2.8, we have

$$-2c^2 \odot \frac{\partial}{\partial \xi}\Big(\frac{\partial \tilde{U}_{i.gH}}{\partial \eta}(\xi,\eta)\Big) = 0, \tag{11}$$

integrating the form (11) $w.r.t.$ ξ gives

$$\int_0^{\xi} \frac{\partial}{\partial \xi}\Big(\frac{\partial_{i.gH} \tilde{U}}{\partial \eta}\Big)(s,\eta)ds = 0,$$

by using Lemma 2.29, we have

$$\frac{\partial \tilde{U}}{\partial \eta}(\xi,\eta) \ominus_{gH} \frac{\partial \tilde{U}}{\partial \eta}(0,\eta) = 0, \tag{12}$$

by integrating the form (12) $w.r.t.$ η gives

$$\int_0^{\eta} \frac{\partial \tilde{U}}{\partial \eta}(\xi,l)dl \ominus_{gH} \int_0^{\eta} \frac{\partial \tilde{U}}{\partial \eta}(0,l)dl = 0,$$

by using Lemma 2.29, we have

$$\Big(\tilde{U}(\xi,\eta) \ominus_{gH} \tilde{U}(\xi,0)\Big) \ominus_{gH} \Big(\tilde{U}(0,\eta) \ominus_{gH} \tilde{U}(0,0)\Big) = 0,$$

where $U(0,0) = 0\cdot$. Therefore, we have

$$\tilde{U}(\xi,\eta) \ominus_{gH} \left(\tilde{U}(\xi,0) \oplus \tilde{U}(0,\eta)\right) = 0.$$

By Definition 2.4, we have

$$\tilde{U}(\xi,\eta) = \tilde{\psi}(\xi) \oplus \tilde{\phi}(\eta).$$

Substituting for ξ and η from (6) and recalling that $\tilde{u}(x,t) = \tilde{U}(\xi,\eta)$ gives

$$\tilde{u}(x,t) = \tilde{U}(\xi,\eta) = \tilde{\psi}(x-ct) \oplus \tilde{\phi}(x+ct).$$

The forward wave is the function $\tilde{\psi}(x-ct)$ which represents a traveling wave in the positive x-direction with scaled velocity c. In physical coordinates, the function depends on $x-ct$ and the speed of the wave is $c = \sqrt{\tau/\rho}$. The shape of the wave is determined by the function $\tilde{\psi}(x)$ and the motion is governed by the line $x - ct = const$. The wave moves forward in time along the string. The backward wave is the function $\tilde{\phi}(x+ct)$ which represents a traveling wave in the negative X-direction with scaled speed c.

The fuzzy solution to the wave equation is the fuzzy superposition of a forward wave $\tilde{\psi}(x-ct)$ and a backward wave $\tilde{\phi}(x+ct)$, both with c speed.

• If $\tilde{u}(x,t)$ is $[(i)-p]$-differentiable *w.r.t.* t, then $\tilde{\psi}$ and $\tilde{\phi}$ are $[(i)-gH]$-differentiable *w.r.t.* $(x-ct)$ and *w.r.t.* $(x+ct)$, respectively. In general, it follows that any solution to the fuzzy wave equation can be obtained as a superposition of two traveling waves

$$\tilde{u}(x,t) = \tilde{\psi}(x-ct) \oplus \tilde{\phi}(x+ct). \tag{13}$$

Now we would like to satisfy the initial conditions

$$\tilde{u}(x,0) = \tilde{f}(x), \tag{14}$$

$$\frac{\partial \tilde{u}}{\partial t}(x,0) = \tilde{g}(x). \tag{15}$$

Since equation (13) is a fuzzy solution for equation (7), it must apply to the fuzzy initial conditions of the equation (8), hence the fuzzy initial condition $\tilde{u}(x,0) = \tilde{f}(x)$ concludes

$$\tilde{u}(x,0) = \tilde{\psi}(x) \oplus \tilde{\phi}(x) = \tilde{f}(x). \tag{16}$$

By differentiating (13) *w.r.t.* t, we have

$$\frac{\partial_{i.gH}\tilde{u}}{\partial t}(x,t) = (-1)c \odot \frac{\partial_{ii.gH}\tilde{\psi}}{\partial t}(x-ct) \oplus c \odot \frac{\partial_{i.gH}\tilde{\phi}}{\partial t}(x+ct) = \tilde{g}(x),$$

so that at $t = 0$ by initial condition, we obtain

$$\begin{aligned} \frac{\partial \tilde{u}_{i.gH}}{\partial t}(x,0) &= (-1)c \odot \tilde{\psi}'_{ii.gH}(x) \oplus c \odot \tilde{\phi}'_{i.gH}(x) \\ &= \tilde{g}(x). \end{aligned} \tag{17}$$

By using Lemma 2.26, we have

$$\int_0^x (-1)\tilde{\psi}'_{ii.gH}(s)ds \oplus \int_0^x \tilde{\phi}'_{i.gH}(s)ds = \int_x^0 \tilde{\psi}'_{ii.gH}(s)ds \oplus \int_0^x \tilde{\phi}'_{i.gH}(s)ds$$
$$= \frac{1}{c}\int_0^x \tilde{g}(s)ds.$$

By using Lemma 2.29, we have

$$\Big((-1)\tilde{\psi}(x) \ominus_{gH} (-1)\tilde{\psi}(0)\Big) \oplus \Big(\tilde{\phi}(x) \ominus_{gH} \tilde{\phi}(0)\Big) = \frac{1}{c}\int_0^x \tilde{g}(s)ds,$$

thus

$$\underbrace{\Big((-1)\tilde{\psi}(x) \oplus \tilde{\phi}(x)\Big)}_{A} \ominus_{gH} \underbrace{\Big((-1)\tilde{\psi}(0) \oplus \tilde{\phi}(0)\Big)}_{B} = \underbrace{\frac{1}{c}\int_0^x \tilde{g}(s)ds}_{C},$$

by using Definition 2.4,

$$\begin{cases} (1)\ (-1)\tilde{\psi}(x) \oplus \tilde{\phi}(x) = \Big((-1)\tilde{\psi}(0) \oplus \tilde{\phi}(0)\Big) \oplus \frac{1}{c}\int_0^x \tilde{g}(s)ds, \\ or \\ (2)\ \big((-1)\tilde{\psi}(0) \oplus \tilde{\phi}(0)\big) = \big((-1)\tilde{\psi}(x) \oplus \tilde{\phi}(x)\big) \oplus (-1)\frac{1}{c}\int_0^x \tilde{g}(s)ds. \end{cases} \tag{18}$$

According to the definition of gH-differentiability, either case (1) or case (2) from (18) holds. Assume that case (1) holds, so if we achieve the solution in this case we will not consider the case (2) anymore. From equations (16) and the first equation (18), we have

$$\begin{cases} \tilde{\psi}(x) \oplus \tilde{\phi}(x) = \tilde{f}(x), \\ (-1)\underbrace{\tilde{\psi}(x)}_{C} \oplus \underbrace{\tilde{\phi}(x)}_{A} = \underbrace{(-1)\tilde{\psi}(0) \oplus \tilde{\phi}(0) \oplus \frac{1}{c}\int_0^x \tilde{g}(s)ds}_{B}. \end{cases} \tag{19}$$

From the second equation (19) we find $\tilde{\psi}(x)$, therefore we have

$$\tilde{\psi}(x) = \tilde{\phi}(x) \ominus_{gH} \Big((-1)\tilde{\psi}(0) \oplus \tilde{\phi}(0) \oplus \frac{1}{c}\int_0^x \tilde{g}(s)ds\Big). \tag{20}$$

Now, we substituting $\tilde{\psi}(x)$ in the first equation (19), we have

$$\underbrace{\tilde{\phi}(x)}_{A} \ominus_{gH} \underbrace{\frac{1}{2}\Big((-1)\tilde{\psi}(0) \oplus \tilde{\phi}(0) \oplus \frac{1}{c}\int_0^x \tilde{g}(s)ds\Big)}_{B} = \underbrace{\frac{1}{2}\tilde{f}(x)}_{C}.$$

By using Definition 2.4, we have

$$\begin{cases} (3)\ \tilde{\phi}(x) = \frac{1}{2}\Big((-1)\tilde{\psi}(0) \oplus \tilde{\phi}(0) \oplus \frac{1}{c}\int_0^x \tilde{g}(s)ds\Big) \oplus \frac{1}{2}\tilde{f}(x), \\ or \\ (4)\ \frac{1}{2}\Big((-1)\tilde{\psi}(0) \oplus \tilde{\phi}(0) \oplus \frac{1}{c}\int_0^x \tilde{g}(s)ds\Big) = \tilde{\phi}(x) \oplus (-1)\frac{1}{2}\tilde{f}(x). \end{cases} \tag{21}$$

By substituting $\tilde{\phi}(x)$ in the first equation (20), we have

$$(22)\quad \begin{cases} (5)\ \tilde{\psi}(x) = (-1)\frac{1}{2}\Big((-1)\tilde{\psi}(0) \oplus \tilde{\phi}(0)\Big) \oplus (-1)\frac{1}{2c}\int_0^x \tilde{g}(s)ds \oplus \frac{1}{2}\tilde{f}(x), \\ or \\ (6)\ \frac{1}{2}\tilde{f}(x) \ominus_{gH} \tilde{\psi}(x) = \frac{1}{2}\big((-1)\tilde{\psi}(0) \oplus \tilde{\phi}(0)\big) \oplus \frac{1}{2c}\int_0^x \tilde{g}(s)ds. \end{cases}$$

On the other hand, by using Lemma 2.26 in the first equation (22), we have

$$(23)\qquad \tilde{\psi}(x) = (-1)\Big(\frac{1}{2}\big((-1)\tilde{\psi}(0) \oplus \tilde{\phi}(0)\big) \oplus \frac{1}{2c}\int_x^0 \tilde{g}(s)ds \oplus \frac{1}{2}\tilde{f}(x).$$

By substituting $\tilde{\psi}(x)$ from equation (23) and $\tilde{\phi}(x)$ from the first equation (21) into equation (13), we have

$$(24)\qquad \tilde{u}(x,t) = \frac{1}{2}\Big(\tilde{f}(x-ct) \oplus \tilde{f}(x+ct)\Big) \oplus \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{g}(s)ds.$$

Suppose that $\tilde{u}(x,t) = \tilde{u}_1(x,t)$, therefore

$$(25)\qquad \tilde{u}_1(x,t) = \frac{1}{2}\Big(\tilde{f}(x-ct) \oplus \tilde{f}(x+ct)\Big) \oplus \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{g}(s)ds.$$

According to the definition of gH-differentiability, since the solution was achieved in cases (1), (3), and (5), we do not consider cases (2), (4), and (6). Furthermore, cases (4) and (6) do not provide explicit solutions in terms of $\tilde{\psi}(x)$ and $\tilde{\phi}(x)$.

Case(ii). Let $\tilde{u}(x,t)$ is $[(i)-p]$-differentiable *w.r.t.* t and $\frac{\partial \tilde{u}}{\partial t}$ are $[(ii)-p]$-differentiable *w.r.t.* t fuzzy function then,

• If $\tilde{U}(\xi,\eta)$ is $[(i)-gH]$-differentiable *w.r.t.* t and $\frac{\partial \tilde{U}}{\partial \xi}$ is $[(i)-gH]$-differentiable *w.r.t.* t and $\frac{\partial \tilde{U}}{\partial \eta}$ is $[(ii)-gH]$-differentiable *w.r.t.* t fuzzy function without any switching points, then

$$\frac{\partial \tilde{u}_{ii.gH}}{\partial t} = \frac{\partial \tilde{U}_{i.gH}}{\partial \xi} \odot \frac{\partial \xi}{\partial t} \oplus \frac{\partial \tilde{U}_{ii.gH}}{\partial \eta} \odot \frac{\partial \eta}{\partial t} = (-1)c \odot \frac{\partial \tilde{U}_{i.gH}}{\partial \xi} \oplus c \odot \frac{\partial \tilde{U}_{ii.gH}}{\partial \eta}.$$

• If $\tilde{U}(\xi,\eta)$ is $[(i)-gH]$-differentiable *w.r.t.* x and $\frac{\partial \tilde{U}}{\partial \xi}$ and $\frac{\partial \tilde{U}}{\partial \eta}$ are $[(ii)-gH]$-differentiable *w.r.t.* x fuzzy functions, then

$$\frac{\partial \tilde{u}_{ii.gH}}{\partial x} = \frac{\partial \tilde{U}_{ii.gH}}{\partial \xi} \odot \frac{\partial \xi}{\partial x} \oplus \frac{\partial \tilde{U}_{ii.gH}}{\partial \eta} \odot \frac{\partial \eta}{\partial x} = \frac{\partial \tilde{U}_{ii.gH}}{\partial \xi} \oplus \frac{\partial \tilde{U}_{ii.gH}}{\partial \eta}.$$

• If $\tilde{U}(\xi,\eta)$ is $[(i)-gH]$-differentiable *w.r.t.* t and $\frac{\partial^2 \tilde{U}}{\partial \xi^2}$ and $\frac{\partial^2 \tilde{U}}{\partial \eta^2}$ are $[(ii)-gH]$-differentiable *w.r.t.* t fuzzy functions then,

$$\begin{aligned} \frac{\partial^2 \tilde{u}_{ii.gH}}{\partial t^2} &= c^2 \odot \frac{\partial^2 \tilde{U}_{ii.gH}}{\partial \xi^2} \oplus (-1)c^2 \odot \frac{\partial}{\partial \eta}\Big(\frac{\partial \tilde{U}_{i.gH}}{\partial \xi}\Big) \\ (26)\qquad &\oplus (-1)c^2 \odot \frac{\partial}{\partial \xi}\Big(\frac{\partial \tilde{U}_{ii.gH}}{\partial \eta}\Big) \oplus c^2 \odot \frac{\partial^2 \tilde{U}_{ii.gH}}{\partial \eta^2}. \end{aligned}$$

• If $\tilde{U}(\xi,\eta)$ is $[(i)-gH]$-differentiable *w.r.t.* x and $\frac{\partial^2\tilde{U}}{\partial\xi^2}$ and $\frac{\partial^2\tilde{U}}{\partial\eta^2}$ are $[(ii)-gH]$-differentiable *w.r.t.* x fuzzy functions then,

$$\frac{\partial^2 \tilde{u}_{ii.gH}}{\partial x^2} = \frac{\partial^2 \tilde{U}_{ii.gH}}{\partial \xi^2} \oplus \frac{\partial}{\partial \eta}\Big(\frac{\partial \tilde{U}_{ii.gH}}{\partial \xi}\Big) \oplus \frac{\partial}{\partial \xi}\Big(\frac{\partial \tilde{U}_{ii.gH}}{\partial \eta}\Big) \oplus \frac{\partial^2 \tilde{U}_{ii.gH}}{\partial \eta^2}. \tag{27}$$

Substituting derivatives (26) and (27) into the FPDE (7) yields

$$-2c^2 \odot \frac{\partial}{\partial \xi}\Big(\frac{\partial \tilde{U}_{ii.gH}}{\partial \eta}(\xi,\eta)\Big) = \tilde{0},$$

by using Definition 2.8, we have

$$-2c^2 \odot \frac{\partial}{\partial \xi}\Big(\frac{\partial \tilde{U}_{ii.gH}}{\partial \eta}(\xi,\eta)\Big) = 0, \tag{28}$$

with integrating form equation (28) *w.r.t.* ξ gives

$$\int_0^\xi \frac{\partial}{\partial \xi}\Big(\frac{\partial \tilde{U}_{ii.gH}}{\partial \eta}\Big)(s,\eta)ds = 0,$$

by using Lemma 2.29, we have

$$\frac{\partial \tilde{U}}{\partial \eta}(\xi,\eta) \ominus_{gH} \frac{\partial \tilde{U}}{\partial \eta}(0,\eta) = 0, \tag{29}$$

integrating form equation (29) *w.r.t.* η gives

$$\int_0^\eta \frac{\partial \tilde{U}}{\partial \eta}(\xi,l)dl \ominus_{gH} \int_0^\eta \frac{\partial \tilde{U}}{\partial \eta}(0,l)dl = 0,$$

by using Lemma 2.29, we have

$$\Big((-1)\tilde{U}(\xi,0) \ominus_{gH} (-1)\tilde{U}(\xi,\eta)\Big) \ominus_{gH} \Big((-1)\tilde{U}(0,0) \ominus_{gH} (-1)(U(0,\eta)\Big) = 0,$$

where $U(0,0) = 0\cdot$ therefore, we have

$$\tilde{U}(\xi,\eta) \ominus_{gH} \Big(\tilde{U}(\xi,0) \oplus \tilde{U}(0,\eta)\Big) = 0.$$

By Definition 2.4, we have

$$\tilde{U}(\xi,\eta) = \tilde{\psi}(\xi) \oplus \tilde{\phi}(\eta).$$

Substituting for ξ and η from equation (6) and recalling that $\tilde{u}(x,t) = \tilde{U}(\xi,\eta)$ gives

$$\tilde{u}(x,t) = \tilde{U}(\xi,\eta) = \tilde{\psi}(x-ct) \oplus \tilde{\phi}(x+ct).$$

• If $\tilde{u}(x,t)$ is $[(i)-p]$-differentiable *w.r.t.* t, then $\tilde{\psi}$ and $\tilde{\phi}$ are $[(i)-gH]$-differentiable *w.r.t.* $(x-ct)$ and *w.r.t.* $(x+ct)$, respectively. In general, it follows that any solution to the fuzzy wave equation can be obtained as a superposition of two traveling waves that is called a fuzzy standing wave as in the following form:

$$\tilde{u}(x,t) = \tilde{\psi}(x-ct) \oplus \tilde{\phi}(x+ct). \tag{30}$$

Now we would like to satisfy the initial conditions, we have

$$\tilde{u}(x,0) = \tilde{f}(x), \ \frac{\partial u}{\partial t}(x,0) = \tilde{g}(x). \tag{31}$$

Since equation (13) is a fuzzy solution for equation (7), then it must apply to the fuzzy initial conditions of the equation (8), hence the fuzzy initial condition $\tilde{u}(x,0) = \tilde{f}(x)$ concludes

$$\tilde{u}(x,0) = \tilde{\psi}(x) \oplus \tilde{\phi}(x) = \tilde{f}(x). \tag{32}$$

Differentiating from (30) $w.r.t.$ t yields, we have

$$\frac{\partial \tilde{u}_{ii.gH}}{\partial t}(x,t) = (-1)c \odot \frac{\partial \tilde{\psi}_{i.gH}}{\partial t}(x-ct) \oplus c \odot \frac{\partial \tilde{\phi}_{ii.gH}}{\partial t}(x+ct) = \tilde{g}(x),$$

so that at $t=0$ by an initial condition, we obtain

$$\frac{\partial \tilde{u}_{ii.gH}}{\partial t}(x,0) = (-1)c \odot \tilde{\psi}'_{i.gH}(x) \oplus c \odot \tilde{\phi}'_{ii.gH}(x) = \tilde{g}(x). \tag{33}$$

Dividing this last equation by c and by using Lemma 2.26, we have

$$\begin{aligned} &\int_0^x (-1)\tilde{\psi}'_{i.gH}(s)ds \oplus \int_0^x \tilde{\phi}'_{ii.gH}(s)ds \\ &= (-1)\int_0^x \tilde{\psi}'_{i.gH}(s)ds \oplus \int_0^x \tilde{\phi}'_{ii.gH}(s)ds = \frac{1}{c}\int_0^x \tilde{g}(x)ds. \end{aligned}$$

By using Lemma 2.29, we have

$$\Big((-1)\tilde{\psi}(x) \ominus_{gH} (-1)\tilde{\psi}(0)\Big) \oplus \Big((-1)\tilde{\phi}(0) \ominus_{gH} (-1)\tilde{\phi}(x)\Big) = \frac{1}{c}\int_0^x \tilde{g}(x)ds.$$

Thus

$$\underbrace{\Big(\tilde{\psi}(0) \oplus (-1)\tilde{\phi}(0)\Big)}_{A} \ominus_{gH} \underbrace{\Big(\tilde{\psi}(x) \oplus (-1)\tilde{\phi}(x)\Big)}_{B} = \underbrace{\frac{1}{c}\int_0^x \tilde{g}(x)ds}_{C}.$$

By using Lemma 2.29, we have

$$\begin{cases} (1) \ \tilde{\psi}(x) \oplus (-1)\tilde{\phi}(x) = \tilde{\psi}(0) \oplus (-1)\tilde{\phi}(0) \oplus (-1)\frac{1}{c}\int_0^x \tilde{g}(x)ds, \\ or \\ (2) \ \big(\tilde{\psi}(0) \oplus (-1)\tilde{\phi}(0)\big) = \big(\tilde{\psi}(x) \oplus (-1)\tilde{\phi}(x)\big) \oplus \frac{1}{c}\int_0^x \tilde{g}(x)ds. \end{cases} \tag{34}$$

According to the definition of gH-differentiability, either case (1) or case (2) from equations (34) holds. Assume that case (1) holds, so if we achieve the

solution in this case we will not consider the case (2) anymore.
From equation (32) and the first equation (34), we have

$$\begin{cases} \tilde{\psi}(x) \oplus \tilde{\phi}(x) = \tilde{f}(x), \\ \underbrace{\tilde{\psi}(x)}_{A} \oplus (-1) \underbrace{\tilde{\phi}(x)}_{C} = \underbrace{\tilde{\psi}(0) \oplus (-1)\tilde{\phi}(0) \oplus (-1)\frac{1}{c}\int_0^x \tilde{g}(x)ds}_{B}. \end{cases} \tag{35}$$

From the second equation (35) we find $\phi(x)$, therefore we have

$$\tilde{\phi}(x) = \tilde{\psi}(x) \ominus_{gH} \Big(\tilde{\psi}(0) \oplus (-1)\tilde{\phi}(0) \oplus (-1)\frac{1}{c}\int_0^x \tilde{g}(x)ds\Big). \tag{36}$$

Now, we substituting $\tilde{\phi(x)}$ in the first equation (35), we have

$$\underbrace{\tilde{\psi}(x)}_{A} \ominus_{gH} \underbrace{\frac{1}{2}\Big(\tilde{\psi}(0) \oplus (-1)\tilde{\phi}(0)\Big) \oplus \Big(\frac{(-1)}{2c}\int_0^x \tilde{g}(x)ds\Big)}_{B} = \underbrace{\frac{1}{2}\tilde{f}(x)}_{C}.$$

By using Definition 2.4, we have

$$\begin{cases} (3)\ \tilde{\psi}(x) = \frac{1}{2}\Big(\tilde{\psi}(0) \oplus (-1)\tilde{\phi}(0)\Big) \oplus \Big(\frac{(-1)}{2c}\int_0^x \tilde{g}(x)ds\Big) \oplus \frac{1}{2}\tilde{f}(x), \\ or \\ (4)\ \frac{1}{2}\Big(\tilde{\psi}(0) \oplus (-1)\tilde{\phi}(0)\Big) \oplus \Big(\frac{(-1)}{2c}\int_0^x \tilde{g}(x)ds\Big) = \tilde{\psi}(x) \oplus (-1)\frac{1}{2}\tilde{f}(x). \end{cases} \tag{37}$$

By substituting $\tilde{\psi}(x)$ in the first equation (35), we have

$$\begin{cases} (5)\ \tilde{\phi}(x) = (-1)\frac{1}{2}\Big(\tilde{\psi}(0) \oplus (-1)\tilde{\phi}(0)\Big) \oplus (-1)\frac{1}{2c}\int_0^x \tilde{g}(x)ds \oplus \frac{1}{2}\tilde{f}(x), \\ or \\ (6)\ \frac{1}{2}\tilde{f}(x) \ominus_{gH} \tilde{\psi}(x) = \frac{1}{2}\big((-1)\tilde{\psi}(0) \oplus \tilde{\phi}(0)\big) \oplus \frac{1}{2c}\int_0^x \tilde{g}(x)ds. \end{cases} \tag{38}$$

On the other hand, by using Lemma 2.26 in the first equation of (37), we have

$$\tilde{\psi}(x) = \frac{1}{2}\Big(\big(\tilde{\psi}(0) \oplus (-1)\tilde{\phi}(0)\Big) \ominus_{gH} \frac{(-1)}{2c}\int_x^0 \tilde{g}(x)ds \oplus \frac{1}{2}\tilde{f}(x). \tag{39}$$

By substituting $\tilde{\psi}(x)$ from equation (39) and $\tilde{\phi}(x)$ from the first equation of (38) into equation (13), we have

$$\tilde{u}(x,t) = \frac{1}{2}\Big(\tilde{f}(x-ct) \oplus \tilde{f}(x+ct)\Big) \ominus_{gH} \frac{(-1)}{2c}\int_{x-ct}^{x+ct} \tilde{g}(x)ds. \tag{40}$$

Suppose that $u(x,t) = u_2(x,t)$, therefore

$$\tilde{u}_2(x,t) = \frac{1}{2}\Big(\tilde{f}(x-ct) \oplus \tilde{f}(x+ct)\Big) \ominus_{gH} \frac{(-1)}{2c}\int_{x-ct}^{x+ct} \tilde{g}(x)ds. \tag{41}$$

According to the definition of gH-differentiability, since the solution was achieved in cases (1), (3), and (5), we do not consider cases (2), (4), and (6). Furthermore, cases (4), and (6) do not provide explicit the solutions in terms of $\tilde{\psi}(x)$ and $\tilde{\phi}(x)$. In this section, we examine the solution of $\tilde{u}_1$ and $\tilde{u}_2$ by the use of the below Tables and according to the type of gH-differentiability for the functions of $\tilde{f}, \tilde{f}'$ and $\tilde{g}, \tilde{g}'$.

Table 1. The kind of $[(gH) - p]$-differentiability for $\tilde{U}(\xi, \eta)$

$\tilde{u}$	$\frac{\partial \tilde{u}}{\partial t}$	$\frac{\partial \tilde{u}}{\partial x}$	$\frac{\partial^2 \tilde{u}}{\partial t^2}$	$\frac{\partial^2 \tilde{u}}{\partial x^2}$	$\tilde{U}$	$\frac{\partial \tilde{U}_{gH}}{\partial \xi}$	$\frac{\partial \tilde{U}_{gH}}{\partial \eta}$	$\frac{\partial^2 \tilde{U}_{gH}}{\partial \xi^2}$	$\frac{\partial^2 \tilde{U}_{gH}}{\partial \eta^2}$
(i)	(i)	$-$	(i)	$-$	(i)	(ii)	(i)	(i)	(i)
(i)	$-$	(i)	$-$	(i)	(i)	(i)	(i)	(i)	(i)
(i)	(ii)	$-$	(ii)	$-$	(ii)	(i)	(ii)	(ii)	(ii)
(ii)	$-$	(ii)	$-$	(ii)	(ii)	(ii)	(ii)	(ii)	(ii)
(ii)	(i)	$-$	(ii)	$-$	(i)	(ii)	(i)	(i)	(i)
(ii)	(ii)	$-$	(ii)	$-$	(ii)	(ii)	(ii)	(ii)	(ii)
(ii)	(ii)	$-$	(i)	$-$	(ii)	(i)	(ii)	(ii)	(ii)
(i)	$-$	(i)	$-$	(i)	(i)	(i)	(i)	(i)	(i)

Now, by using Theorem 3.1, we show that according to the gH-differentiability of $\tilde{f}$ and $\tilde{g}$ that $\tilde{u}_1$ and $\tilde{u}_2$ are the solutions of FPDE 7.

Theorem 3.1. *Let $\tilde{f}, \tilde{g} : \mathbb{R}^2 \to \mathbb{R}_{\mathcal{F}}$ be two fuzzy functions such that $\tilde{g}$ is once continuously generalized differentiable function, $\tilde{f}$ twice and $c > 0$. Consider*

$$\tilde{u}_1(x,t) = \frac{1}{2}\big(\tilde{f}(x-ct) \oplus \tilde{f}(x+ct)\big) \oplus \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds,$$

and

$$\tilde{u}_2(x,t) = \frac{1}{2}\big(\tilde{f}(x-ct) \oplus \tilde{f}(x+ct)\big) \ominus_{gH} \frac{(-1)}{2c}\int_{x-ct}^{x+ct} g(s)ds,$$

provided for all $(x,t) \in \mathbb{R} \times (0,\infty)$ the above gH-difference exists.
1. If $\tilde{f}, \tilde{f}'$ and $\tilde{g}, \tilde{g}'$ are $[(i)-gH]$-differentiable or $\tilde{f}, \tilde{f}'$ and $\tilde{g}, \tilde{g}'$ are $[(ii)-gH]$-differentiable on $\mathbb{R}$, then $\tilde{u}_1$ is a solution of FPDE 7.
2. If $\tilde{f}$ is $[(i)-gH]$-differentiable and $\tilde{f}'$ is $[(ii)-gH]$-differentiable and $\tilde{g}, \tilde{g}'$ are $[(i)-gH]$-differentiable (or conversely) w.r.t. x and t, then $\tilde{u}_2$ is a solution of FPDE 7. The details of the current Theorem are given in Tabel 2.

In order to prove Theorem 3.1, it is required to state the following Lemmas.

Lemma 3.2. *Let $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}_{\mathcal{F}}$ be a gH-differentiable function w.r.t. x, t and c be a positive constant real number for all $(x,t) \in \mathbb{R} \times (0,\infty)$. Consider*

$$\tilde{H}(x,t) = \tilde{f}(x-ct) \oplus \tilde{f}(x+ct).$$

TABLE 2. The kind of $[(gH)-p]$-differentiability solutions of $\tilde{u}_1$ and $\tilde{u}_2$ with respect to x and t

$Kind\,of$ $solution$ $\tilde{u}_1, \tilde{u}_2$	$\tilde{f}$	$\tilde{f}'$	$\tilde{g}$	$\tilde{g}'$	$\tilde{u}$ $w.r.t.$ x	$\tilde{u}$ $w.r.t.$ t	$\frac{\partial \tilde{u}}{\partial x}$ $w.r.t.$ x	$\frac{\partial \tilde{u}}{\partial t}$ $w.r.t.$ t
$\tilde{u}_1$	(i)	(i)	(i)	(i)	(i)	(i)	(i)	(i)
$\tilde{u}_1$	(ii)	(ii)	(ii)	(ii)	(ii)	(i)	(ii)	(i)
$\tilde{u}_2$	(i)	(ii)	(ii)	(ii)	(i)	(ii)	(ii)	(i)
$\tilde{u}_2$	(ii)	(i)	(i)	(i)	(ii)	(ii)	(i)	(i)

Then
1. If $\tilde{f}$ is gH-differentiable w.r.t. x, then the type of gH-derivative $\tilde{f}$ and $\tilde{H}$ is the same w.r.t. x and $\tilde{H}$ is always gH-differentiable w.r.t. x.
2. If $\tilde{f}'$ is gH-differentiable w.r.t. t, then $\tilde{H}$ is gH-differentiable w.r.t. t on $\mathbb{R} \times (0, \infty)$.

The expression of $\frac{\partial \tilde{H}}{\partial x}$ and $\frac{\partial \tilde{H}}{\partial t}$ is given in Tables 3 and 4 for practical use.

TABLE 3. The kind of gH-differentiability of $\tilde{H}$ $w.r.t.$ x

$\tilde{f}$	$\tilde{f}'$	$\tilde{H}$ $w.r.t.$ x	$\frac{\partial \tilde{H}}{\partial x}$
(i)	$-$	(i)	$\tilde{f}'(x-ct) \oplus \tilde{f}'(x+ct)$
(ii)	$-$	(ii)	$\tilde{f}'(x-ct) \oplus \tilde{f}'(x+ct)$
(i)	(i)	(i)	$\tilde{f}'(x-ct) \oplus \tilde{f}'(x+ct)$
(ii)	(ii)	(ii)	$\tilde{f}'(x-ct) \oplus \tilde{f}'(x+ct)$
(i)	(ii)	(i)	$\tilde{f}'(x-ct) \oplus \tilde{f}'(x+ct)$
(ii)	(i)	(ii)	$\tilde{f}'(x-ct) \oplus \tilde{f}'(x+ct)$

Proof. We only demonstrate the case of 3 from Tables 3 and 4, that the rest of the table states can be easily expressed. Using Proposition 2.15, the function $\tilde{H}$ is $[(i)-gH]$-differentiable $w.r.t.$ x, therefore we have

$$\frac{\partial \tilde{H}}{\partial x}(x,t) = \tilde{f}'(x-ct) \oplus \tilde{f}'(x+ct), \forall (x,t) \in \mathbb{R} \times (0, \infty).$$

We claim $\tilde{H}(x,t)$ is $[(i)-gH]$-differentiable $w.r.t.$ t, and we have

$$\frac{\partial \tilde{H}}{\partial t}(x,t) = (-1)c \odot \tilde{f}'(x-ct) \oplus c \odot \tilde{f}'(x+ct).$$

Table 4. The kind of gH-differentiability of $\tilde{H}$ $w.r.t.$ t

$\tilde{f}$	$\tilde{f}'$	$\tilde{H}$ $w.r.t.$ t	$\frac{\partial \tilde{H}}{\partial t}$
(i)	$-$	$-$	$-$
(ii)	$-$	$-$	$-$
(i)	(i)	(i)	$c \odot \tilde{f}'(x-ct) \oplus c \odot \tilde{f}'(x+ct)$
(ii)	(ii)	(i)	$c \odot \tilde{f}'(x-ct) \oplus c\tilde{f}'(x+ct)$
(i)	(ii)	(ii)	$(-1)c \odot \tilde{f}'(x-ct) \oplus \tilde{f}'(x+ct)$
(ii)	(i)	(ii)	$(-1)c \odot \tilde{f}'(x-ct) \oplus \tilde{f}'(x+ct)$

Using Theorem 2.20, the function $\tilde{f}(x-ct)$ is $[(ii)-gH]$-differentiable and $\tilde{f}(x+ct)$ is $[(i)-gH]$-differentiable $w.r.t.$ t, also their derivatives are

$$\frac{\partial \tilde{f}(x-ct)}{\partial t} = (-1)c \odot \tilde{f}(x-ct),$$
$$\frac{\partial \tilde{f}(x+ct)}{\partial t} = c \odot \tilde{f}(x+ct).$$

Using Remark 2.16 and Proposition 2.15, we have

$$\frac{\partial \tilde{H}}{\partial t}(x,t) = c \odot \tilde{f}'(x-ct) \oplus c \odot \tilde{f}'(x+ct).$$

Therefore $\tilde{H}$ is $[(i)-gH]$-differentiable $w.r.t.$ t. □

Lemma 3.3. *Let $\tilde{g} : \mathbb{R}^2 \to \mathbb{R}_{\mathcal{F}}$ be a fuzzy function and c is a positive constant real number for $(x,t) \in \mathbb{R} \times (0,\infty)$ set*

$$\tilde{G}(x,t) = \int_{x-ct}^{x+ct} \tilde{g}(s)ds.$$

If the type of gH-derivative $\tilde{g}$ and $\tilde{g}'$ are the same $w.r.t.$ x, then the type of gH-derivative $\tilde{G}$ is also proportional to the type of gH-derivative $\tilde{g}$ and $\tilde{g}'$. Also, if the types of gH-derivative $\tilde{g}$ and $\tilde{g}'$ are different, then the gH-derivative $\tilde{G}$ is proportional to the gH-derivative of $\tilde{g}$.

Proof. Let $\tilde{g}, \tilde{g}'$ be $[(i)-gH]$-differentiable $w.r.t.$ x, by using Lemma 2.29 and Theorem 2.19, then we have

$$\frac{\partial \tilde{G}}{\partial x} = \tilde{g}(x+ct) \ominus_{gH} \tilde{g}(x-ct),$$

which indicates that function $\tilde{G}$ is $[(i)-gH]$-differentiable $w.r.t.$ x. Now if $\tilde{g}, \tilde{g}'$ are $[(ii)-gH]$-differentiable $w.r.t, x$, by using Lemma 2.29 and Theorem 2.19, then we have

$$\frac{\partial \tilde{G}}{\partial x} = (-1)\tilde{g}(x-ct) \ominus_{gH} (-1)\tilde{g}(x+ct),$$

which indicates that function $\tilde{G}$ is $[(ii) - gH]$-differentiable $w.r.t.\ x$.
The other case can be investigated similarly and we omit the details.

□

TABLE 5. The kind of gH-differentiability of $\tilde{G}$ $w.r.t.\ x$

$\tilde{g}$	$\tilde{g}'$	$\tilde{G}$ $w.r.t.\ x$	$\frac{\partial \tilde{G}}{\partial x}$
(i)	(i)	(i)	$\tilde{g}(x+ct) \ominus_{gH} \tilde{g}(x-ct)$
(ii)	(ii)	(ii)	$(-1)\tilde{g}(x-ct) \ominus_{gH} (-1)\tilde{g}(x+ct)$
(i)	(ii)	(i)	$\tilde{g}(x+ct) \ominus_{gH} \tilde{g}(x-ct)$
(ii)	(i)	(ii)	$(-1)\tilde{g}(x-ct) \ominus_{gH} (-1)\tilde{g}(x-ct)$

Now by using Lemmas 3.2 and 3.3, we prove Theorem 3.1.

Proof. Here we only show cases 1 and 2 from Theorem 3.1 are similarly proven. Using Lemma 3.3 and according to case 3 of Table 4, as well as, using Lemma 3.3 and case 1 from Table 5, and Proposition 2.15, $\tilde{u}_1$ is the differentiable $w.r.t.\ x$ and we have

$$\frac{\partial \tilde{u}_1}{\partial x}(x,t) = \frac{1}{2}\big(\tilde{f}'(x-ct) \oplus \tilde{f}'(x+ct)\big) \oplus \frac{1}{2c}\big(\tilde{g}(x+ct) \ominus_{gH} \tilde{g}(x-ct)\big).$$

On the other hand, we conclude from case 3 of Table 3 and Proposition 2.15 that $\frac{\partial \tilde{u}}{\partial x}$ is $[(i) - gH]$-differentiable $w.r.t.\ x$, and we have

(42)

$$\frac{\partial^2 \tilde{u}_1}{\partial x^2}(x,t) = \frac{1}{2}\big(\tilde{f}''(x-ct) \oplus \tilde{f}''(x+ct)\big) \oplus \frac{1}{2c}\big(\tilde{g}'(x+ct) \ominus_{gH} \tilde{g}'(x-ct)\big).$$

Also, we conclude case 3 from Table 4 and case 1 from Table 6 and Proposition 2.15 that $\tilde{u}_1$ is $[(i) - gH]$-differentiable $w.r.t.\ t$ and

$$\frac{\partial \tilde{u}_1}{\partial t}(x,t) = \frac{1}{2}c \odot \big(\tilde{f}'(x-ct) \oplus \tilde{f}'(x+ct)\big)$$
$$\oplus \frac{1}{2c}\big(c \odot \tilde{g}'(x-ct) \ominus_{gH} (-1)c \odot \tilde{g}'(x+ct)\big).$$

TABLE 6. The kind of generalized differentiability of $\tilde{G}$ $w.r.t.\ t$

$\tilde{g}$	$\tilde{g}'$	$\tilde{G}$ $w.r.t.\ t$	$\frac{\partial \tilde{G}}{\partial t}$
(i)	(i)	(i)	$c \odot \tilde{g}(x+ct) \ominus_{gH} (-1)c \odot \tilde{g}(x-ct)$
(ii)	(ii)	(ii)	$c \odot \tilde{g}(x-ct) \ominus_{gH} (-1)c \odot \tilde{g}(x+ct)$
(i)	(ii)	(i)	$c \odot \tilde{g}(x+ct) \ominus_{gH} (-1)c \odot \tilde{g}(x-ct)$
(ii)	(i)	(ii)	$c \odot \tilde{g}(x-ct) \ominus_{gH} (-1)c \odot \tilde{g}(x-ct)$

It is obvious from case 3 of Table 4 and Proposition 2.15 that $\frac{\partial \tilde{u}}{\partial t}$ is $[(i)-gH]$-differentiable *w.r.t.* t and

$$\begin{aligned}\frac{\partial^2 \tilde{u}_1}{\partial t^2}(x,t) &= \frac{1}{2}c^2 \odot \big(\tilde{f}''(x-ct) \oplus \tilde{f}''(x+ct)\big)\\ &\oplus \frac{1}{2c}\big(c^2 \odot \tilde{g}'(x+ct) \ominus_{gH} c^2 \odot \tilde{g}'(x-ct)\big).\end{aligned} \tag{43}$$

By substituting (42) and (43) in the equation (7), we conclude that $\tilde{u}_1$ satisfies FPDE 7. □

Example 3.4. *Let us consider the homogeneous one-dimensional fuzzy wave equation*

$$\frac{\partial^2 \tilde{u}_{gH}(x,t)}{\partial t^2} \ominus_{gH} 9 \odot \frac{\partial^2 \tilde{u}_{gH}(x,t)}{\partial x^2} = \tilde{0},$$

where the fuzzy initial conditions are as follows:

$$\begin{cases} \tilde{u}(x,0) = \tilde{f}(x) = \langle 1,2,5\rangle \odot e^x, \ -\infty < x < \infty, \\ \frac{\partial \tilde{u}_{gH}(x,0)}{\partial t} = \tilde{g}(x) = \langle 1,3,4\rangle \odot e^x. \end{cases}$$

The solution of the fuzzy wave equation $\tilde{u}_1$ *is*

$$\begin{aligned}\tilde{u}_1(x,t) &= \frac{1}{2}\Big(\gamma \odot f(x-ct) \oplus \gamma \odot f(x+ct)\Big) \oplus \frac{1}{2c}\int_{x-ct}^{x+ct} \beta \odot g(s)ds\\ &= \frac{1}{2}\Big(\langle 1,2,5\rangle \odot e^{x-3t} \oplus \langle 1,2,5\rangle \odot e^{x+3t}\Big) \oplus \frac{1}{6}\int_{x-3t}^{x+3t} \langle 1,3,4\rangle \odot e^s ds\\ &= \frac{1}{2}\langle 1,2,5\rangle \odot \Big(e^{x-3t} \oplus e^{x+3t}\Big) \oplus \frac{1}{6}\langle 1,3,4\rangle \odot \Big(e^{x+3t} \ominus_{gH} e^{x-3t}\Big).\end{aligned}$$

Also, the solution of $\tilde{u}_2$ *is*

$$\begin{aligned}u_2(x,t) &= \frac{1}{2}\Big(\gamma \odot f(x-ct) \oplus \gamma \odot f(x+ct)\Big) \ominus_{gH} \frac{(-1)}{2c}\int_{x-ct}^{x+ct} \beta \odot g(s)ds\\ &= \frac{1}{2}\Big(\langle 1,2,5\rangle \odot e^{x-3t} \oplus \langle 1,2,5\rangle \odot e^{x+3t}\Big) \ominus_{gH} \frac{(-1)}{6}\int_{x-3t}^{x+3t} \langle -4,-3,-1\rangle \odot e^s ds\\ &= \frac{1}{2}\langle 1,2,5\rangle \odot \Big(e^{x-3t} \oplus e^{x+3t}\Big) \ominus_{gH} \frac{1}{6}\langle 1,3,4\rangle \odot \Big((-1)e^{x-3t} \ominus_{gH} (-1)e^{x+3t}\Big).\end{aligned}$$

4. The Fuzzy D'Alembert Solutions for the fuzzy Wave on an infinite string under the generalized derivative

$$\begin{aligned}\tilde{u} &= \frac{\partial_g^2 \tilde{y}}{\partial t^2}(x,t), \ \tilde{v} = c^2 \odot \frac{\partial_g^2 \tilde{y}}{\partial x^2}(x,t),\\ \tilde{w} &= \frac{\partial_g^2 \tilde{y}}{\partial t^2}(x,t) \ominus_g c^2 \odot \frac{\partial_g^2 \tilde{y}}{\partial x^2}(x,t) = 0,\end{aligned}$$

by using Proposition 2.6 for any $r \in [0,1]$, we have

$$\begin{aligned}
&\Big[\frac{\partial_g^2 \tilde{y}}{\partial t^2}(x,t) \ominus_g c^2 \cdot \frac{\partial_g^2 \tilde{y}}{\partial x^2}(x,t)\Big]_r \\
&= \Big[\inf_{\lambda \geq r} \min\Big\{\frac{\partial^2 \underline{y}_\lambda}{\partial t^2} - c^2 \cdot \frac{\partial^2 \underline{y}_\lambda}{\partial x^2}, \frac{\partial^2 \overline{y}_\lambda}{\partial t^2} - c^2 \cdot \frac{\partial^2 \overline{y}_\lambda}{\partial x^2}\Big\} \\
&, \sup_{\lambda \geq r} \max\Big\{\frac{\partial_g^2 \underline{y}_\lambda}{\partial t^2} - c^2 \cdot \frac{\partial^2 \underline{y}_\lambda}{\partial x^2}, \frac{\partial^2 \overline{y}_\lambda}{\partial t^2} - c^2 \cdot \frac{\partial^2 \overline{y}_\lambda}{\partial x^2}\Big\}\Big] \\
&= [\underline{0}, \overline{0}].
\end{aligned} \tag{44}$$

Therefore

$$\left[\frac{\partial_g^2 \tilde{y}}{\partial t^2}(x,t)\right]_\lambda = \left[\inf_{\alpha \geq \lambda} \min\left\{\frac{\partial^2 \underline{y}_\alpha}{\partial t^2}, \frac{\partial^2 \overline{y}_\alpha}{\partial t^2}\right\}, \sup_{\alpha \geq \lambda} \max\left\{\frac{\partial^2 \underline{y}_\alpha}{\partial t^2}, \frac{\partial^2 \overline{y}_\alpha}{\partial t^2}\right\}\right],$$

$$\left[\frac{\partial_g^2 \tilde{y}}{\partial x^2}(x,t)\right]_\lambda = \left[\inf_{\alpha \geq \lambda} \min\left\{\frac{\partial^2 \underline{y}_\alpha}{\partial x^2}, \frac{\partial^2 \overline{y}_\alpha}{\partial x^2}\right\}, \sup_{\alpha \geq \lambda} \max\left\{\frac{\partial^2 \underline{y}_\alpha}{\partial x^2}, \frac{\partial^2 \overline{y}_\alpha}{\partial t^2}\right\}\right].$$

Thus for any $\alpha \in [0,1]$, we have

$$\frac{\partial_g^2 \underline{y}_\lambda}{\partial t^2}(x,t) = \inf_{\alpha \geq \lambda} \min\left\{\frac{\partial^2 \underline{y}_\alpha}{\partial t^2}, \frac{\partial^2 \overline{y}_\alpha}{\partial t^2}\right\}, \tag{45}$$

$$\frac{\partial_g^2 \overline{y}_\lambda}{\partial t^2}(x,t) = \sup_{\alpha \geq \lambda} \max\left\{\frac{\partial^2 \underline{y}_\alpha}{\partial t^2}, \frac{\partial^2 \overline{y}_\alpha}{\partial t^2}\right\}, \tag{46}$$

$$\frac{\partial_g^2 \underline{y}_\lambda}{\partial x^2}(x,t) = \inf_{\alpha \geq \lambda} \min\left\{\frac{\partial^2 \underline{y}_\alpha}{\partial x^2}, \frac{\partial^2 \overline{y}_\alpha}{\partial x^2}\right\}, \tag{47}$$

$$\frac{\partial_g^2 \overline{y}_\lambda}{\partial x^2}(x,t) = \sup_{\alpha \geq \lambda} \max\left\{\frac{\partial^2 \underline{y}_\alpha}{\partial x^2}, \frac{\partial^2 \overline{y}_\alpha}{\partial x^2}\right\}, \tag{48}$$

with substituting derivatives (45), (46), (47), and (48) into (44) yields

$$\begin{aligned}
&\left[\frac{\partial_g^2 \tilde{y}}{\partial t^2}(x,t) \ominus_g c^2 \odot \frac{\partial_g^2 \tilde{y}}{\partial x^2}(x,t)\right]_r = \\
&\left[\inf_{\lambda \geq r} \min\left\{\inf_{\alpha \geq \lambda} \min\left\{\frac{\partial^2 \underline{y}_\alpha}{\partial t^2} - c^2 \frac{\partial^2 \underline{y}_\alpha}{\partial x^2}, \frac{\partial^2 \overline{y}_\alpha}{\partial t^2} - c^2 \frac{\partial^2 \overline{y}_\alpha}{\partial x^2}\right\}\right.\right., \\
&\left.\left.\sup_{\lambda \geq r} \max\left\{\sup_{\alpha \geq \lambda} \max\left\{\frac{\partial^2 \underline{y}_\alpha}{\partial t^2} - c^2 \frac{\partial^2 \underline{y}_\alpha}{\partial x^2}, \frac{\partial^2 \overline{y}_\alpha}{\partial t^2} - c^2 \frac{\partial^2 \overline{y}_\alpha}{\partial x^2}\right\}\right.\right] \\
&= [\underline{0}, \overline{0}].
\end{aligned}$$

Since $\alpha \geq \lambda$ and $\lambda \geq r$, then $\alpha \geq r$ therefore, we have

$$\left[\frac{\partial^2_{gH}\tilde{y}}{\partial t^2}(x,t) \ominus_g c^2 \odot \frac{\partial^2_{gH}\tilde{y}}{\partial x^2}(x,t)\right]_r =$$
$$\left[\inf_{\alpha \geq r} \min\left\{\frac{\partial^2 \underline{y}_\alpha}{\partial t^2} - c^2 \frac{\partial^2 \underline{y}_\alpha}{\partial x^2}, \frac{\partial^2 \overline{y}_\alpha}{\partial t^2} - c^2 \frac{\partial^2 \overline{y}_\alpha}{\partial x^2}\right\},\right.$$
$$\left.\sup_{\alpha \geq r} \max\left\{\frac{\partial^2 \underline{y}_\alpha}{\partial t^2} - c^2 \frac{\partial^2 \underline{y}_\alpha}{\partial x^2}, \frac{\partial^2 \overline{y}_\alpha}{\partial t^2} - c^2 \frac{\partial^2 \overline{y}_\alpha}{\partial x^2}\right\}\right]$$
$$= [\underline{0}, \overline{0}].$$

Now, let

$$\underline{w}_\alpha = \frac{\partial^2 \underline{y}_\alpha}{\partial t^2} - c^2 \frac{\partial^2 \underline{y}_\alpha}{\partial x^2},\ \overline{w}_\alpha = \frac{\partial^2 \overline{y}}{\partial t^2} - c^2 \frac{\partial^2 \overline{y}}{\partial x^2}.$$

We assume the following cases:
1) If $len(\underline{w}_\alpha) \leq len(\overline{w}_\alpha)$, then

$$\inf_{\alpha \geq r} \min\{\underline{w}_\alpha, \overline{w}_\alpha\} = \inf_{\alpha \geq r} \underline{w}_\alpha = \underline{0},$$
$$\sup_{\alpha \geq r} \max\{\underline{w}_\alpha, \overline{w}_\alpha\} = \sup_{\alpha \geq r} \overline{w}_\alpha = \overline{0}.$$

2) If $len(\underline{w}_\alpha) \geq len(\overline{w}_\alpha)$, then

$$\inf_{\alpha \geq r} \min\{\underline{w}_\alpha, \overline{w}_\alpha\} = \inf_{\alpha \geq r} \overline{w}_\alpha = \underline{0},$$
$$\sup_{\alpha \geq r} \max\{\underline{w}_\alpha, \overline{w}_\alpha\} = \sup_{\alpha \geq r} \underline{w}_\alpha = \overline{0}.$$

Suppose that case 1 holds, thus we have

$$\inf_{\alpha \geq r} \underline{w}_\alpha = \inf_{\alpha \geq r}\left\{\frac{\partial^2 \underline{y}_\alpha}{\partial t^2} - c^2 \frac{\partial^2 \underline{y}_\alpha}{\partial x^2}\right\} = \underline{0},$$
$$\sup_{\alpha \geq r} \overline{w}_\alpha = \sup_{\alpha \geq r}\left\{\frac{\partial^2 \overline{y}_\alpha}{\partial t^2} - c^2 \frac{\partial^2 \overline{y}_\alpha}{\partial x^2}\right\} = \overline{0}.$$

On the other hand, from (24) we have

$$\underline{y}_r(x,t) = \inf_{\alpha \geq r}\left\{\frac{1}{2}\big(\underline{\gamma}_r.f(x-ct) + \underline{\gamma}_r.f(x+ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \underline{\gamma}_r.g(s)ds\right\},$$

and

$$\overline{y}_r(x,t) = \sup_{\alpha \geq r}\left\{\frac{1}{2}\big(\overline{\gamma}_r.f(x-ct) + \overline{\gamma}_r.f(x+ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \overline{\gamma}_r.g(s)ds\right\}.$$

Now, suppose that case 2 holds, thus we have

$$\inf_{\alpha\geq r}\overline{w}_\alpha=\inf_{\alpha\geq r}\left\{\frac{\partial^2\overline{y}_\alpha}{\partial t^2}-c^2\frac{\partial^2\overline{y}_\alpha}{\partial x^2}\right\}=\underline{0},$$
$$\sup_{\alpha\geq r}\underline{w}_\alpha=\sup_{\alpha\geq r}\left\{\frac{\partial^2\underline{y}_\alpha}{\partial t^2}-c^2\frac{\partial^2\underline{y}_\alpha}{\partial x^2}\right\}=\overline{0}.$$

Thus, we have

$$\overline{y}_r(x,t)=\inf_{\alpha\geq r}\left\{\frac{1}{2}\big(\overline{\gamma}_r.f(x-ct)+\overline{\gamma}_r.f(x+ct)\big)+\frac{1}{2c}\int_{x-ct}^{x+ct}\overline{\gamma}_r.g(s)ds\right\}$$

and

$$\underline{y}_r(x,t)=\sup_{\alpha\geq r}\left\{\frac{1}{2}\big(\underline{\gamma}_r.f(x-ct)+\underline{\gamma}_r.f(x+ct)\big)+\frac{1}{2c}\int_{x-ct}^{x+ct}\underline{\gamma}_r.g(s)ds\right\}.$$

5. The fuzzy solutions and domain of dependence

At a given position $x=x_0$ on the string, the fuzzy solution in $case(i)$ at time $t=t_0$ is

$$\tilde{u}(x_0,t_0)=\frac{1}{2}\Big(\tilde{f}(x_0-ct_0)\oplus\tilde{f}(x_0+ct_0)\Big)\oplus\frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0}\tilde{g}(s)ds,$$

and the fuzzy solution in case(ii) at time $t=t_0$ is

$$\tilde{u}(x_0,t_0)=\frac{1}{2}\Big(\tilde{f}(x_0-ct_0)\oplus\tilde{f}(x_0+ct_0)\Big)\ominus\frac{(-1)}{2c}\int_{x_0-ct_0}^{x_0+ct_0}\tilde{g}(s)ds.$$

In other words, the fuzzy solutions are found by tracing backward and forward in time along with the characteristics $x-ct=x_0-ct_0$ and $x+ct=x_0+ct_0$ to the initial state $(\tilde{f}(x),\tilde{g}(x))$, then applying (24) and (41) to compute $\tilde{u}(x_0,t_0)$ from the initial state. Information from the initial state from the interval $x_0-ct_0\leq x\leq x_0+ct_0$ is all that is needed to find $\tilde{u}(x_0,t_0)$. In the tx-plane, we can think of a triangle opening backward and forward in time from (x_0,t_0) to the line $t=0$.

In general, the function $\tilde{f}(x)$ and $\tilde{g}(x)$ are case functions. We need to determine various regions by plotting the salient characteristics $x\pm ct=const$. The regions determine where $x-ct$ and $x+ct$ are relative to the cases for the functions $\tilde{f}(x)$ and $\tilde{g}(x)$ and tells us what part of the case functions should be used in each region.

1. Let $\tilde{U}(\xi,\eta)$ be a $[(i)-gH]$-differentiable fuzzy function. Then $\tilde{u}(x,t),\partial_t\tilde{u}(x,t)$ and $\partial_x\tilde{u}(x,t)$ are $[(i)-p]$-differentiable $w.r.t.$ t and x, also $\tilde{\psi}$ and $\tilde{\phi}$ are $[(i)-gH]$-differentiable $w.r.t.$ $(x-ct)$ and $(x+ct)$, respectively. For example, consider $\tilde{f}(x)$ and $\tilde{g}(x)$ of the following form

$$\tilde{f}(x)=\begin{cases}\tilde{\phi}(x), & |x|\leq l,\\ 0, & |x|>l,\end{cases}\quad \tilde{g}(x)=\begin{cases}\tilde{\psi}(x), & |x|\leq l,\\ 0, & |x|>l.\end{cases}\tag{49}$$

Here, according to Definition 2.8, we have considered zero singleton fuzzy numbers.

Step 1. Write down the fuzzy D'Alembert solution, from (24) we have

$$\tilde{u}(x,t) = \frac{1}{2}\Big(\tilde{f}(x-ct) \oplus \tilde{f}(x+ct)\Big) \oplus \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{g}(s)ds.$$

Step 2. Identify the regions.

The functions $\tilde{f}(x)$ and $\tilde{g}(x)$ are equal to functions $\tilde{\phi}(x)$ and $\tilde{\psi}(x)$, respectively, for $|x| \leq l$ and are zero for $|x| > l$. Thus, the regions of interest are found by plotting the four characteristics $x \pm ct = \pm l$. The regions are identified in the plot and are given mathematically by

$$\begin{aligned}
&R_1 = \big\{(x,t) : -l \leq x - ct \leq l \ and \ -l \leq x + ct \leq l\big\},\\
&R_2 = \big\{(x,t) : -l \leq x - ct \leq l \ and \ x + ct \geq l\big\},\\
&R_3 = \big\{(x,t) : x - ct \leq -l \ and \ -l \leq x + t \leq l\big\},\\
&R_4 = \big\{(x,t) : x - ct \leq -l \ and \ x + ct \geq l\big\},\\
&R_5 = \big\{(x,t) : x + ct \leq -l\big\},\\
&R_6 = \big\{(x,t) : x - ct \geq l\big\}.
\end{aligned} \tag{50}$$

The regions determine where $x - ct$ and $x + ct$ are relative to $\pm l$, which tells us what part of the case functions $\tilde{f}(x)$ and $\tilde{g}(x)$ should be used. It is helpful to define the lines.

$x_A(t) = -l - ct$, $x_B(t) = -l + ct$, $x_C(t) = l - ct$, $x_D(t) = l + ct$.

Step 3. Consider the fuzzy solution in each region.

In R_1 region, we have $|x \pm ct| \leq l$, so that (24) implies

$$\tilde{f}(x-ct) = \tilde{\phi}(x-ct), \tilde{f}(x+ct) = \tilde{\phi}(x+ct),$$

and

$$\int_{x-ct}^{x+ct} \tilde{g}(s)ds = \int_{x-ct}^{x+ct} \tilde{\psi}(s)ds,$$

and hence

$$\tilde{u}(x,t) = \frac{1}{2}\Big(\tilde{\phi}(x-ct) \oplus \tilde{\phi}(x+ct)\Big) \oplus \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{\psi}(s)ds.$$

In R_2 region, we have $-l \leq x - ct \leq l \ and \ x + ct \geq l$, and so

$$\tilde{f}(x+ct) = 0, \tilde{f}(x-ct) = \tilde{\phi}(x-ct),$$

and by using Lemma 2.26 and Definition 2.8, we have

$$\int_{x-ct}^{x+ct} \tilde{g}(s)ds = \int_{x-ct}^{l} \tilde{g}(s)ds \oplus \int_{l}^{x+ct} \tilde{g}(s)ds = \int_{x-ct}^{l} \tilde{\psi}(s)ds,$$

and hence

$$\tilde{u}(x,t) = \frac{\tilde{\phi}(x-ct)}{2} \oplus \frac{1}{2}\int_{x-ct}^{l} \tilde{\psi}(s)ds.$$

In R_3 region, we have $x - ct \leq -l \ and \ -l \leq x + ct \leq l$ so that $f(x + ct) = \tilde{\phi}(x + ct)$, $\tilde{f}(x - ct) = 0$ and by using Lemma 2.26, and Definition 2.8, we have

$$\int_{x-ct}^{x+ct} \tilde{g}(s)ds = \int_{-l}^{x+ct} \tilde{\psi}(s)ds,$$

and hence

$$\tilde{u}(x,t) = \frac{\tilde{\phi}(x+ct)}{2} \oplus \frac{1}{2}\int_{-l}^{x+ct} \tilde{\psi}(s)ds.$$

In R_4 region, $x - ct \leq -l \ and \ x + ct \geq l$, so that

$$\tilde{f}(x+ct) = 0, \tilde{f}(x-ct) = 0,$$

and by using Lemma 2.26, and Definition 2.8, we have

$$\int_{x-ct}^{x+ct} \tilde{g}(s)ds = \int_{x-ct}^{l} \tilde{g}(s)ds \oplus \int_{-l}^{l} \tilde{g}(s)ds \oplus \int_{l}^{x+ct} \tilde{g}(s)ds = \int_{-l}^{l} \tilde{\psi}(s)ds,$$

and hence

$$\tilde{u}(x,t) = \frac{1}{2}\int_{x-ct}^{x+ct} \tilde{g}(s)ds = \frac{1}{2}\int_{-l}^{l} \tilde{\psi}(s)ds.$$

In R_5 and R_6 regions, $\tilde{f}(x+ct) = 0 = \tilde{f}(x-ct)$ and $\tilde{g}(s) = 0$ for $s \in [x - ct, x + ct]$, hence $u = 0$. To summarize

(51)

$$\tilde{u}(x,t) = \begin{cases} \frac{1}{2}\Big(\tilde{\phi}(x-ct) \oplus \tilde{\phi}(x+ct)\Big) \oplus \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{\psi}(s)ds, & (x,t) \in R_1, \\ \frac{\tilde{\phi}(x-ct)}{2} \oplus \frac{1}{2}\int_{x-ct}^{l} \tilde{\psi}(s)ds, & (x,t) \in R_2, \\ \frac{\tilde{\phi}(x+ct)}{2} \oplus \frac{1}{2}\int_{-l}^{x+ct} \tilde{\psi}(s)ds, & (x,t) \in R_3, \\ \frac{1}{2}\int_{-l}^{l} \tilde{\psi}(s)ds, & (x,t) \in R_4, \\ 0, & (x,t) \in R_5, R_6. \end{cases}$$

Step 4. For each specific time t_0, write the x-intervals corresponding to the intersection of the sets R_n with the line $t = t_0$.

At time 0, we use Table 7 to find the x intervals R_n corresponding to the intersection of R'_n with the line $t = 0$

$$R'_5 = (-\infty, -l], R'_1 = (-l, l], R'_6 = [l, \infty).$$

In R_1 region, we have (recall that $t = 0$)

$$\tilde{u}(x,t) = \frac{1}{2}\Big(\tilde{\phi}(x-0) \oplus \tilde{\phi}(x+0)\Big) \oplus \frac{1}{2c}\int_{x-0}^{x+0} \tilde{\psi}(s)ds = \tilde{\phi}(x).$$

TABLE 7. This amounts to computing the values of $x_A(t), x_B(t), x_C(t)$ and $x_D(t)$ for each time

t	0	$\frac{1}{2}$	1	2
$x_A(t)$	$-l$	$-l-\frac{1}{2}c$	$-l-c$	$-l-2c$
$x_B(t)$	$-l$	$-l+\frac{1}{2}c$	$-l+c$	$-l+2c$
$x_C(t)$	l	$l-\frac{1}{2}c$	$l-c$	$l-2c$
$x_D(t)$	l	$l+\frac{1}{2}c$	$l+c$	$l+2c$

Similarly, we can check that in the other regions, $\tilde{u}=0$, so that (51) becomes

$$\tilde{u}(x,0)=\begin{cases}\tilde{\phi}(x), & x\in R_1'=[-l,l],\\ 0, & x\in R_5'\cup R_6',\end{cases}$$
$$=\begin{cases}\tilde{\phi}(x), & |x|\le l\\ 0, & |x|>l\end{cases}=\tilde{f}(x).$$

At time 1/2, we use Table 7 to find the x intervals R_n' corresponding to the intersection of R_n with the line $t=\frac{1}{2}$

$$\begin{aligned}&R_5=\big(-\infty,-l-\frac{1}{2}c\big],R_1=\big(-l+\frac{1}{2}c,l-ct\big],\\ &R_6=\big[l+\frac{1}{2}c,\infty\big),R_3=\big(-l-\frac{1}{2}c,-l+\frac{1}{2}c\big],\\ &R_2=\big(l-\frac{1}{2}c,l+\frac{1}{2}c\big].\end{aligned}\tag{52}$$

At time 1, we use Table 7 to find the x intervals R_n' corresponding to the intersection of R_n with the line $t=1$

$$\begin{aligned}&R_5=\big(-\infty,-l-c\big],R_3=\big(-l-c,-l+c\big],\\ &R_2=\big[l-c,l+c\big],R_6=\big[l+c,\infty\big].\end{aligned}\tag{53}$$

At time 2, we use Table 7 to find the x intervals R_n' to the intersection of R_n with the line $t=2$

$$\begin{aligned}&R_5=\big(-\infty,-l-2c\big],R_3=\big[-l-2c,-l+2c\big],\\ &R_4=\big[-l+2c,l-2c\big],R_2=\big[l-2c,l+2c\big],\\ &R_6=\big[l+2c,\infty\big).\end{aligned}\tag{54}$$

Here, to illustrate the ability and reliability of the aforementioned concepts we have solved some application examples.

Example 5.1. *For an infinitely long string, consider giving the fuzzy string zero initial displacement $\tilde{u}(x,0)=0$ and fuzzy initial velocity $\partial_t\tilde{u}(x,0)=\tilde{g}(x)$.*

Suppose that

$$\tilde{g}(x)=\begin{cases}\tilde{\gamma}\odot cos^2(\frac{\pi}{2}x), & -1\leq x\leq 1,\\ 0, & otherwise.\end{cases}$$

The FICs have the form considered above for $\tilde{\phi}(x)=0$ *and* $\tilde{\psi}(x)=\tilde{\gamma}\odot cos^2(\frac{\pi}{2}x)$.
Step 1. *The fuzzy D'Alembert solution of* (24) *becomes*

$$\tilde{u}(x,t)=\frac{1}{2}\int_{x-t}^{x+t}\tilde{g}(s)ds.$$

Step 2. *By* (50) *we determine the regions.*
Step 3. *Determine* $u(x,t)$ *in each region. From* (51) *by consider* $c=1, l=1, \beta=\langle 1,3,5\rangle$, *we have*

$$\tilde{u}(x,t)=\begin{cases}\frac{1}{2}\int_{x-t}^{x+t}\langle 1,3,5\rangle\odot\psi(s)ds, & (x,t)\in R_1,\\ \frac{1}{2}\int_{x-t}^{1}\langle 1,3,5\rangle\odot\psi(s)ds, & (x,t)\in R_2,\\ \frac{1}{2}\int_{-1}^{x+t}\langle 1,3,5\rangle\odot\psi(s)ds, & (x,t)\in R_3,\\ \frac{1}{2}\int_{-1}^{1}\langle 1,3,5\rangle\odot\psi(s)ds, & (x,t)\in R_4,\\ 0, & (x,t)\in R_5, R_6.\end{cases}$$

Note that

R_1*:*

$$\frac{1}{2}\int_{x-t}^{x+t}\langle 1,3,5\rangle\odot\psi(s)ds=\langle 1,3,5\rangle\odot\Big(\frac{t}{2}\oplus\frac{1}{4\pi}sin(\pi(x+t))$$
$$\ominus_{gH}\frac{1}{4\pi}sin(\pi(x-t))\Big),$$

R_2*:*

$$\frac{1}{2}\int_{x-t}^{1}\langle 1,3,5\rangle\odot\psi(s)ds$$
$$=\langle 1,3,5\rangle\odot\Big(\frac{1}{4}\ominus_{gH}\frac{1}{4}(x-t)\Big)\ominus_{gH}\Big(\frac{1}{4\pi}sin(\pi(x-t)\Big),$$

R_3*:*

$$\int_{-1}^{x+t}\langle 1,3,5\rangle\odot\psi(s)ds$$
$$=\langle 1,3,5\rangle\odot\Big(\frac{1}{4}(x+t)\ominus_{gH}\frac{1}{4}\Big)\oplus\frac{1}{4\pi}sin(\pi(x+t)\Big),$$

R_4*:*

$$\frac{1}{2}\int_{-1}^{1}\langle 1,3,5\rangle\odot\psi(s)ds=\langle 1,3,5\rangle\odot\frac{1}{2},$$

R_5 *and* R_6*:*

$$\tilde{u}(x,t)=0.$$

Thus

(55)

$$\tilde{u}(x,t)=\begin{cases}\langle 1,3,5\rangle\odot\Big(\frac{t}{2}\oplus\frac{1}{4\pi}sin(\pi(x+t))\ominus_{gH}\frac{1}{4\pi}sin(\pi(x-t))\Big), & (x,t)\in R_1,\\ \langle 1,3,5\rangle\odot\Big(\frac{1}{4}\ominus_{gH}\frac{1}{4}(x-t)\Big)\ominus_{gH}\Big(\frac{1}{4\pi}sin(\pi(x-t)\Big), & (x,t)\in R_2,\\ \langle 1,3,5\rangle\odot\Big(\frac{1}{4}(x+t)\ominus_{gH}\frac{1}{4}\Big)\oplus\frac{1}{4\pi}sin(\pi(x+t)\Big), & (x,t)\in R_3,\\ \langle 1,3,5\rangle\odot\frac{1}{2}, & (x,t)\in R_4,\\ 0, & (x,t)\in R_5,R_6.\end{cases}$$

Step 4. *We early consider, intermediate and later times* $t=1/2,1,2$. *At* $t=1/2$, *the* R_n *regions are given by* (52), *and* (55) *becomes*

$$\tilde{u}(x,\frac{1}{2})=\begin{cases}\langle 1,3,5\rangle\odot\Big(\frac{t}{4}\oplus\frac{1}{2\pi}cos\pi x\Big), & -\frac{1}{2}\leq x\leq\frac{1}{2},\\ \langle 1,3,5\rangle\odot\Big(\frac{1}{4}\ominus_{gH}\frac{1}{4}(x-\frac{1}{2})\oplus\frac{cos\pi x}{4\pi}\Big), & \frac{1}{2}\leq x\leq\frac{3}{2},\\ \langle 1,3,5\rangle\odot\Big(\frac{1}{4}(x+t)\ominus_{gH}\frac{1}{4}\Big)\oplus\frac{1}{4\pi}sin(\pi(x+\frac{1}{2})\Big), & -\frac{3}{2}\leq x\leq-\frac{1}{2},\\ 0, & x\geq\frac{3}{2}.\end{cases}$$

At time 1, *the* R_n *regions are given by* (53), *and* (55) *becomes*

$$\tilde{u}(x,1)=\begin{cases}\langle 1,3,5\rangle\odot\Big(\frac{1}{2}\ominus_{gH}\frac{1}{4}\Big)\ominus_{gH}\frac{1}{4\pi}sin\pi x, & 0\leq x\leq 2,\\ \langle 1,3,5\rangle\odot\big(\frac{1}{4}x-\frac{1}{4\pi}sin\pi x\big), & -2\leq x\leq 0,\\ 0, & |x|\geq 2.\end{cases}$$

At time 2, *the* R_n *regions are given by* (54), *and* (55) *becomes*

$$\tilde{u}(x,2)=\begin{cases}\langle 1,3,5\rangle\odot\Big(\frac{1}{4}\ominus_{gH}\frac{1}{4}(x-2)\Big)\ominus_{gH}sin\pi x, & 1\leq x\leq 3,\\ \langle 1,3,5\rangle\odot\Big(\frac{1}{4}(x+2)\ominus_{gH}\frac{1}{4}\Big)\oplus\frac{1}{4\pi}sin\pi x, & -3\leq x\leq -1,\\ \frac{1}{2}\odot\langle 1,3,5\rangle, & -1\leq x\leq 1,\\ 0, & |x|\geq 3.\end{cases}$$

2. *Let* $\tilde{U}(\xi,\eta)$ *is a* $[(ii)-gH]$*-differentiable fuzzy function, then* $\tilde{u}(x,t),\partial_t\tilde{u}(x,t)$ *and* $\partial_x\tilde{u}(x,t)$ *are* $[(i)-p]$*-differentiable w.r.t.* t *and* x, *also* $\tilde{\psi}$ *and* $\tilde{\phi}$ *are* $[(i)-gH]$*-differentiable w.r.t.* $(x-ct)$ *and* $(x+ct)$, *respectively. For example, consider* $\tilde{f}(x)$ *and* $\tilde{g}(x)$ *of the following form*

$$\tilde{f}(x)=\begin{cases}\tilde{\phi}(x), & |x|\leq l,\\ 0, & |x|>l,\end{cases}\quad\tilde{g}(x)=\begin{cases}\tilde{\psi}(x), & |x|\leq l,\\ 0, & |x|>l.\end{cases}\tag{56}$$

Step 1. *Write down fuzzy D'Alembert solution*

$$\tilde{u}(x,t) = \frac{1}{2}\Big(\tilde{f}(x-ct) \oplus \tilde{f}(x+ct)\Big) \ominus_{gH} \frac{(-1)}{2c} \int_{x-ct}^{x+ct} \tilde{g}(s)ds.$$

Step 2. *Identify the regions. (similarly, as in the previous)*

Step 3. *Determine the fuzzy solution in each region. In R_1 region, we have $|x \pm ct| \leq l$, so that (41) implies*

$$\tilde{f}(x-ct) = \tilde{\phi}(x-ct), \tilde{f}(x+ct) = \tilde{\phi}(x+ct),$$

and

$$\int_{x-ct}^{x+ct} \tilde{g}(s)ds = \int_{x-ct}^{x+ct} \tilde{\psi}(s)ds,$$

and hence

$$\tilde{u}(x,t) = \frac{1}{2}\Big(\tilde{\phi}(x-ct) \oplus \tilde{\phi}(x+ct)\Big) \ominus_{gH} \frac{(-1)}{2c} \int_{x+ct}^{x-ct} \tilde{\psi}(s)ds.$$

In R_2 region, we have $-l \leq x-ct \leq l$ and $x+ct \geq l$ so that

$$\tilde{f}(x+ct) = 0, \tilde{f}(x-ct) = \tilde{\phi}(x-ct),$$

and

$$\int_{x-ct}^{x+ct} \tilde{g}(s)ds = \int_{x-ct}^{l} \tilde{g}(s)ds \oplus \int_{l}^{x+ct} \tilde{g}(s)ds = \int_{x-ct}^{l} \tilde{\psi}(s)ds,$$

and hence

$$\tilde{u}(x,t) = \frac{\tilde{\phi}(x-ct)}{2} \ominus_{gH} \frac{(-1)}{2} \int_{x-ct}^{1} \tilde{\psi}(s)ds.$$

In R_3 region, we have $x-ct \leq -l$ and $-l \leq x+ct \leq l$ so that $\tilde{f}(x+ct) = \tilde{\phi}(x+ct)$, $\tilde{f}(x-ct) = 0$,

$$\int_{x-ct}^{x+ct} \tilde{g}(s)ds = \int_{-l}^{x+ct} \tilde{\psi}(s)ds,$$

and hence

$$\tilde{u}(x,t) = \frac{\tilde{\phi}(x+ct)}{2} \ominus_{gH} \frac{(-1)}{2} \int_{-l}^{x+ct} \tilde{\psi}(s)ds.$$

In R_4 region, $x-ct \leq -l$ and $x+ct \geq l$, so that

$$\tilde{f}(x+ct) = 0, \ \tilde{f}(x-ct) = 0,$$

$$\int_{x-ct}^{x+ct} \tilde{g}(s)ds = \int_{x-ct}^{-l} \tilde{g}(s)d \oplus \int_{-l}^{l} \tilde{g}(s)ds \int_{l}^{x+ct} \tilde{g}(s)ds = \int_{-l}^{l} \tilde{\psi}(s)ds,$$

and hence

$$\tilde{u}(x,t) = \ominus_{gH} \frac{(-1)}{2} \int_{x-ct}^{x+ct} \tilde{g}(s)d = \ominus_{gH} \frac{(-1)}{2} \int_{-l}^{l} \tilde{\psi}(s)ds.$$

In R_5 and R_6 regions, $\tilde{f}(x+ct) = 0 = \tilde{f}(x-ct)$ and $\tilde{g}(s) = 0$ for $s \in [x-ct, x+ct]$, hence $\tilde{u} = 0$. To summarize
(57)

$$\tilde{u}(x,t) = \begin{cases} \frac{1}{2}\Big(\tilde{\phi}(x-ct) \oplus \tilde{\phi}(x+ct)\Big) \ominus_{gH} \frac{(-1)}{2c} \int_{x-ct}^{x+ct} \tilde{\psi}(s)ds, & (x,t) \in R_1, \\ \frac{\tilde{\phi}(x-ct)}{2} \ominus_{gH} \frac{(-1)}{2} \int_{x-ct}^{l} \tilde{\psi}(s)ds, & (x,t) \in R_2, \\ \frac{\tilde{\phi}(x+ct)}{2} \ominus_{gH} \frac{(-1)}{2} \int_{-l}^{x+ct} \tilde{\psi}(s)ds, & (x,t) \in R_3, \\ \ominus_{gH} \frac{(-1)}{2} \int_{-l}^{l} \tilde{\psi}(s)ds, & (x,t) \in R_4, \\ 0, & (x,t) \in R_5, R_6. \end{cases}$$

Step 4. *For each specific time t_0, write the x-intervals corresponding to the intersection of the sets R_n with the line $t = t_0$. (similarly, as in the previous step 4)*

Example 5.2. *For an infinitely long string, consider giving the string zero initial displacement $\tilde{u}(x,0) = 0$ and initial velocity $\partial_t \tilde{u}(x,0) = \tilde{g}(x)$. Suppose that*

$$\tilde{g}(x) = \begin{cases} \tilde{\gamma} \odot cos^2(\frac{\pi}{2}x), & -1 \le x \le 1, \\ 0, & otherwise. \end{cases}$$

The ICs have the form considered above for $\tilde{\phi}(x) = 0$ and $\tilde{\psi}(x) = \tilde{\gamma} \odot cos^2(\frac{\pi}{2}x)$.
Step 1. *The wave D'Alembert solution of* (41) *becomes*

$$\tilde{u}(x,t) = \ominus \frac{(-1)}{2} \int_{x-t}^{x+t} \tilde{g}(s)ds.$$

Step 2. *By* (50) *we determine the regions.*
Step 3. *Determine $u(x,t)$ in each region. From* (57) *by consider $c = 1, l = 1, \beta = \langle 1,3,5 \rangle$, we have*

$$\tilde{u}(x,t) = \begin{cases} \ominus_{gH} \frac{(-1)}{2} \int_{x-t}^{x+t} \langle 1,3,5 \rangle \odot \psi(s)ds, & (x,t) \in R_1, \\ \ominus_{gH} \frac{(-1)}{2} \int_{x-t}^{1} \langle 1,3,5 \rangle \odot \psi(s)ds, & (x,t) \in R_2, \\ \ominus_{gH} \frac{(-1)}{2} \int_{-1}^{x+t} \langle 1,3,5 \rangle \odot \psi(s)ds, & (x,t) \in R_3, \\ \ominus_{gH} \frac{(-1)}{2} \int_{-1}^{1} \langle 1,3,5 \rangle \odot \psi(s)ds, & (x,t) \in R_4, \\ 0, & (x,t) \in R_5, R_6. \end{cases}$$

Note that
R_1*:*

$$\begin{aligned} &\ominus_{gH} \frac{(-1)}{2} \int_{x-t}^{x+t} \langle 1,3,5 \rangle \odot \psi(s)ds \\ &= \ominus_{gH} \langle 1,3,5 \rangle \odot \Big(\frac{(-1)t}{2} \oplus \frac{1}{4\pi} sin(\pi(x-t)) \ominus_{gH} \frac{1}{4\pi} sin(\pi(x+t)) \Big), \end{aligned}$$

R_2:

$$\ominus_{gH} \frac{(-1)}{2} \int_{x-t}^{1} \langle 1,3,5\rangle \odot \psi(s)ds$$
$$= \ominus_{gH} \langle 1,3,5\rangle \odot \Big(\frac{1}{4}(x-t) \ominus_{gH} \frac{1}{4} \oplus \frac{1}{4\pi} sin(\pi(x-t)\Big),$$

R_3:

$$\ominus_{gH} \frac{(-1)}{2} \int_{-1}^{x+t} \langle 1,3,5\rangle \odot \psi(s)ds =$$
$$\ominus_{gH} \langle 1,3,5\rangle \odot \Big(\frac{1}{4} \ominus_{gH} \frac{1}{4}(x+t)\Big) \ominus_{gH} \frac{(-1)}{4\pi} sin(\pi(x+t)\Big),$$

R_4:

$$\ominus_{gH} \frac{(-1)}{2} \int_{-1}^{1} \langle 1,3,5\rangle \odot \psi(s)ds = \ominus_{gH} \langle 1,3,5\rangle \odot \frac{(-1)}{2},$$

R_5 and R_6:

$$\tilde{u}(x,t) = 0.$$

Thus

(58)
$$\tilde{u}(x,t) = \begin{cases} \ominus_{gH}\langle 1,3,5\rangle \odot \Big(\frac{(-1)}{2}t \oplus \frac{1}{4\pi} sin(\pi(x-t)) \ominus_{gH} \frac{1}{4\pi} sin(\pi(x+t))\Big), \\ (x,t) \in R_1, \\ \ominus_{gH}\langle 1,3,5\rangle \odot \Big(\frac{1}{4}(x-t) \ominus_{gH} \frac{1}{4} \oplus \frac{1}{4\pi} sin(\pi(x-t)\Big), \\ (x,t) \in R_2, \\ \ominus_{gH}\langle 1,3,5\rangle \odot \Big(\frac{1}{4} \ominus_{gH} \frac{1}{4}(x+t)\Big) \ominus_{gH} \frac{(-1)}{4\pi} sin(\pi(x+t)\Big), \\ (x,t) \in R_3, \\ \ominus_{gH}\langle 1,3,5\rangle \odot \frac{(-1)}{2}, (x,t) \in R_4, \\ 0, (x,t) \in R_5, R_6. \end{cases}$$

Step 4. *We early consider, intermediate and later times $t = 1/2, 1, 2$. At $t = 1/2$, the R_n regions are given by (52), and (58) becomes*

$$\tilde{u}(x,\frac{1}{2}) = \begin{cases} \ominus_{gH} \langle 1,3,5\rangle \odot \Big(\frac{(-1)}{4} - \frac{1}{2\pi} cos\pi x\Big), & -\frac{1}{2} \leq x \leq \frac{1}{2}, \\ \ominus_{gH} \langle 1,3,5\rangle \odot \Big(\frac{1}{4}(x-\frac{1}{2}) \ominus_{gH} \frac{1}{4} \oplus \frac{1}{4\pi} cos\pi x\Big), & \frac{1}{2} \leq x \leq \frac{3}{2} \\ \ominus_{gH} \langle 1,3,5\rangle \odot \Big(\frac{1}{4} \ominus_{gH} \frac{1}{4}(x+\frac{1}{2})\Big) \ominus_{gH} \frac{(-1)}{4\pi} cos\pi x\Big), & -\frac{3}{2} \leq x \leq -\frac{1}{2}, \\ 0, & x \geq \frac{3}{2}. \end{cases}$$

At time 1*, the* R_n *regions are given by* (53)*, and* (58) *becomes*

$$\tilde{u}(x,1)=\begin{cases}\ominus_{gH}\langle 1,3,5\rangle\odot\Big(\frac{1}{4}(x-1)\ominus_{gH}\frac{1}{4}\Big)\oplus\frac{(-1)}{4\pi}sin\pi x, & 0\leq x\leq 2,\\ \ominus_{gH}\langle 1,3,5\rangle\odot\big(\frac{1}{4}\ominus_{gH}\frac{1}{4}(x+1)\ominus_{gH}\frac{1}{4\pi}sin\pi x\big), & -2\leq x\leq 0,\\ 0, & |x|\geq 2.\end{cases}$$

At time 2*, the* R_n *regions are given by* (54)*, and* (58) *becomes*

$$\tilde{u}(x,2)=\begin{cases}\ominus_{gH}\langle 1,3,5\rangle\odot\Big(\frac{1}{4}(x-2)\ominus_{gH}\frac{1}{4}\oplus\frac{1}{4\pi}sin\pi x\Big), & 1\leq x\leq 3,\\ \ominus_{gH}\langle 1,3,5\rangle\odot\Big(\frac{1}{4}\ominus_{gH}\frac{1}{4}(x+2)\ominus_{gH}\frac{(-1)}{4\pi}sin\pi x\Big), & -3\leq x\leq -1,\\ \ominus_{gH}\langle 1,3,5\rangle\odot\frac{(-1)}{2}, & -1\leq x\leq 1,\\ 0, & |x|\geq 3.\end{cases}$$

6. Conclusion

The physics phenomena can be expressed by connecting fuzzy space and fuzzy derivatives with time, such as wave equations in electromagnetic. In this paper, the one-dimensional fuzzy wave and telegraph equations have been solved by an analytical technique under generalized Hukuhara derivatives. The basic concepts of the fuzzy theory have been presented. Considering physical interpretations, we have discussed fuzzy wave solutions obtained in detail using position variable restriction. Then, the fuzzy wave solutions have been clarified by providing examples using the fuzzy D'Alembert method.

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M. Miri Karbasaki
Orcid number: 0000-0003-3326-3707
Department of Mathematics
Islamic Azad University
Kerman, Iran
Email address: mohsenmiri52@gmail.com

M. R. Balooch shahryari
Orcid number: 0000-0002-9599-8090
Department of Mathematics
Islamic Azad University
Kerman, Iran
Email address: baloochshahryari.1980@gmail.com

O. Sedaghatfar
Orcid number: 0000-0002-2188-3776
Department of Mathematics
Islamic Azad University
Tehran, Iran
Email address: sedaghatfar@yahoo.com