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RECOGNITION OF THE DIRECT PRODUCTS OF SUZUKI GROUPS BY THEIR COMPLEX GROUP ALGEBRAS

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Abstract. Denote by $\widehat{p_n}$, the largest prime among the primitive prime divisors of $2^{2n+1}-1$ and $2^{2(4n+2)}-1$, where $n \in \mathbb{N}$. In this paper, we prove that if $q = 2^{2n+1} \geq 8$ and $\alpha \leq \widehat{p_n}$, then the direct product of α copies of $\mathrm{Sz}(q)$ is uniquely determined by its complex group algebra.

Keywords: Character degree, Order, Suzuki groups, Complex group algebra.
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1. Introduction and Preliminary Results

Let G be a finite group and $\mathrm{Irr}(G)$ be the set of irreducible characters of G. The set of prime divisors of $|G|$ is denoted by $\pi(G)$. A p-defect zero character of G is an irreducible complex character $\chi \in \mathrm{Irr}(G)$, where p does not divide $|G|/\chi(1)$. If n is a natural number, by G^n we mean the direct product of n copies of G.

In [6, Problem 2*], R. Brauer posed the following question: Let G and H be two finite groups. If for all fields $\mathbb{F}$, two group algebras $\mathbb{F}G$ and $\mathbb{F}H$ are isomorphic can we get that G and H are isomorphic? In [8], E. C. Dade showed that this is false in general. In [10], Huppert proposed the following conjecture:

Conjecture. *Let H be a finite non-abelian simple group and G be a group such that $\mathrm{cd}(G) \cong \mathrm{cd}(H)$. Then $G \cong H \times A$, where A is abelian.*

Also in [18], Tong-Viet posed the following question:

Question. *Which groups can be uniquely determined by the structure of their complex group algebras?*

It is proved that non-abelian simple groups, quasi-simple groups and symmetric groups are uniquely determined up to isomorphism by the structure of their complex group algebras (see [4, 14, 16, 19]).

One of the next natural groups to be considered is the characteristically simple groups (A group is called characteristically simple if it has no proper non-trivial subgroups which are invariant by all of its automorphisms). Khosravi et al. proved that $\mathrm{PSL}(2, p) \times \mathrm{PSL}(2, p)$ is uniquely determined by its complex

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group algebra, where $p \geq 5$ is a prime number (see [12]). In [1], Baniasad et al. proved that if M is a simple K_3-group, then $M \times M$ is uniquely determined by its order and some information on irreducible character degrees. In [2], Baniasad et al. proved that the direct product of non-isomorphic Suzuki groups is uniquely determined by its complex group algebra.

A prime is called a primitive prime divisor of $a^m - 1$ if it divides this number but does not divide $a^k - 1$ for $0 < k < m$. Denote by $\widehat{p_n}$ the largest prime among the primitive prime divisors of $2^{2n+1} - 1$ and $2^{2(4n+2)} - 1$. In this paper, we prove that the direct product $\mathrm{Sz}(2^{2n+1})^\alpha$, where $\alpha \leq \widehat{p_n}$, is uniquely determined by its complex group algebra. Also in special cases, we prove that these groups are characterizable by their orders and the existence of a p-defect zero character.

Given a natural number n, let $\mathrm{P}(n)$ denote the greatest prime divisor of n. For every integer a coprime to n, let $\mathrm{Ord}_n(a)$ denote the smallest positive integer e such that $a^e \equiv 1 \pmod{n}$. Let n_r, where r is a prime, denote the r-part of n, i.e., the largest power of r that divides n. For a prime number s, we write $s^k \| n$, if $s^k \mid n$ but $s^{k+1} \nmid n$.

Lemma 1.1. [20, Lemma 1] *Let G be a non-solvable group. Then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$.*

Lemma 1.2. *[22] Let q, k, l be natural numbers. Then*

$$(q^k - 1, q^l + 1) = \begin{cases} q^{(k,l)} + 1 & \text{if } \frac{k}{(k,l)} \text{ is even and } \frac{l}{(k,l)} \text{ is odd} \\ (2, q+1) & \text{otherwise} \end{cases}$$

Lemma 1.3. *[23]* **(Zsigmondy's Theorem)** *If $b > 1$, then $b^n - 1$ has at least one primitive prime divisor with the following two possible exceptions:*
i) $2^6 - 1$, *ii)* $n = 2$ *and* $b + 1$ *is a power of* 2.

Lemma 1.4. *If $n > 2$ and $a > b > 0$, then $n + 1 \leq \mathrm{P}(a^n - b^n)$.*

Proof. If $a^n - b^n \neq 2^6 - 1$, then by [5, Theorem V] there exists a primitive prime divisor p of $a^n - b^n$. Then $p \equiv 1 \pmod{n}$. Hence $n + 1 \leq p \leq \mathrm{P}(a^n - b^n)$. If $a^n - b^n = 2^6 - 1$, then $6 + 1 \leq 7$. □

Lemma 1.5. [15, Theorem 3.6] *Let p be an odd prime, and let $a \neq \pm 1$ be an integer not divisible by p. Let d be the order of a modulo p. Let k_0 be the largest integer such that $a^d \equiv 1 \pmod{p^{k_0}}$. Then the order of a modulo p^k is d for $k = 1, \ldots, k_0$ and dp^{k-k_0} for $k > k_0$.*

2. Characterization by complex group algebra

In representation theory, p-defect zero characters are the subject of key questions by Richard Brauer.

Definition 2.1. If G has a p-defect zero character for every prime divisor p of $|G|$, then we say that G satisfies the *pdz-condition*.

We note that by results of G. Michler and W. Willems, every simple group of Lie type satisfies the pdz-condition (see [13] and [21]). If G satisfies the pdz-condition, then obviously G^n satisfies the pdz-condition.

Lemma 2.2. *If G satisfies the pdz-condition, then*

(a) *every subnormal subgroup of G satisfies the pdz-condition;*
(b) *G is a non-solvable group;*
(c) *the only solvable subnormal subgroup of G is 1.*

Proof. **(a)** Let H be a subnormal subgroup of G, where $H = H_r \trianglelefteq \cdots \trianglelefteq H_1 \trianglelefteq H_0 = G$ for some $r > 0$ and for each $1 \leq i \leq r$, $H_i \neq H_{i-1}$.

We proceed by induction on r. If $r = 1$ and p is an arbitrary prime such that $p^\beta \| |H_1|$, then there exists $\chi \in \mathrm{Irr(G)}$ such that $\chi(1)_p = p^\alpha$, where $p^\alpha \| |G|$. Also there exists $\theta \in \mathrm{Irr(H_1)}$ such that $[\chi_{H_1}, \theta] \neq 0$. Using [11, Corollary 11.29], we get that $\chi(1)/\theta(1) \mid |G : H_1|$ and so $p^\beta \mid \theta(1)$. On the other hand, $\theta(1) \mid |H_1|$ and so $\theta(1)_p = |H_1|_p$, which implies that H_1 satisfies the pdz-condition.

By the inductive hypothesis, H_{r-1} satisfies the pdz-condition. Since $H_r \trianglelefteq H_{r-1}$, similarly to the above H satisfies the pdz-condition.

(b) Let G be a solvable group. Suppose that M is a minimal normal subgroup of G which is an elementary abelian p-subgroup for some prime divisor p of $|G|$. Using part (a), M satisfies the pdz-condition. So there exists $\theta \in \mathrm{Irr(M)}$ such that $1 = \theta(1)_p = |M|_p = |M|$, and this is a contradiction.

(c) This is an immediate consequence of (a) and (b). □

Remark 2.3. [7] Let $q = 2^{2n+1} \geq 8$. We note that $|\mathrm{Sz}(q)| = (q^2+1)q^2(q-1)$ and $|\mathrm{Out}(\mathrm{Sz}(q))| = 2n+1$.

Lemma 2.4. *(a) For every natural numbers m and n, we have* $|\mathrm{Sz}(2^{2m+1})| \mid |\mathrm{Sz}(2^{2n+1})|$ *if and only if* $|\mathrm{Out}(\mathrm{Sz}(2^{2m+1}))| \mid |\mathrm{Out}(\mathrm{Sz}(2^{2n+1}))|$.

(b) If p is a primitive prime divisor of $2^{2n+1}-1$ or $2^{2(4n+2)}-1$, then $p \nmid |\mathrm{Sz}(2^{2m+1})|$, where $1 \leq m < n$.

Proof. (a) By assumption $2^{4m+2}(2^{4m+2}+1)(2^{2m+1}-1)$ divides $2^{4n+2}(2^{4n+2}+1)(2^{2n+1}-1)$. Since $2m+1$ is an odd integer, it follows that $(2m+1)/(2m+1, 4n+2)$ is an odd integer. Hence by Lemma 1.2, $(2^{2m+1}-1, 2^{4n+2}+1) = (2, 2+1) = 1$. Similarly, we obtain that $(2^{2m+1}-1, 2^{4m+2}+1) = (2^{2n+1}-1, 2^{4n+2}+1) = (2^{2n+1}-1, 2^{4m+2}+1) = 1$. Therefore $2^{2m+1}-1$ divides $2^{2n+1}-1$, which implies that $2m+1$ divides $2n+1$.
(b) The result is obtained from the definition of primitive prime divisor. □

Lemma 2.5. *Let p be a primitive prime divisor of $2^{2n+1}-1$ or $2^{2(4n+2)}-1$, and $H = \mathrm{Sz}(2^{2m+1})^t$, where $1 \leq m < n$ and t is a natural number. If $p \mid |\mathrm{Out}(H)|$, then $|H|_2 > |\mathrm{Sz}(2^{2n+1})|_2$.*

Proof. Since p is a primitive prime divisor of $2^{2n+1}-1$ or $2^{2(4n+2)}-1$, we get that $p \equiv 1 \pmod{2n+1}$ or $p \equiv 1 \pmod{8n+4}$, respectively. Hence $p > 2n+1$ and so $p \nmid |\mathrm{Out}(\mathrm{Sz}(2^{2n+1}))|$. Since $n > m$, we have $p > 2m+1$ and hence

$p \nmid |\mathrm{Out}(\mathrm{Sz}(2^{2m+1}))|$. Since $p \mid |\mathrm{Out}(H)|$ and $\mathrm{Out}(H) \cong \mathrm{Out}(\mathrm{Sz}(2^{2m+1})) \wr S_t$, we have $p \mid t!$. So $t \geq p$. Therefore

$$|H|_2 \geq (2^6)^t \geq 2^{6p} > 2^{6(2n+1)} > 2^{4n+2} = |\mathrm{Sz}(2^{2n+1})|_2.$$

Thus $|H|_2 > |\mathrm{Sz}(2^{2n+1})|_2$, as required. □

Lemma 2.6. *Let R and T be non-abelian simple groups, where $p_0 = \mathrm{P}(|R|)$ is a divisor of $|T|$. If G is an extension of R^m by T^n, where $m < p_0$, then $G \cong R^m \times T^n$.*

Proof. If R is an alternating group or a sporadic simple group, then $p_0 \nmid |\mathrm{Out}(R)|$. So let R be a simple group of Lie type over GF(q), where $q = p^f$. We prove that $p_0 \nmid |\mathrm{Out}(R)|$. By [7], the order of the graph automorphism of R is a divisor of 3!. Also if $R \ncong \mathrm{PSL}(l+1, q)$ and $R \ncong \mathrm{PSU}(l+1, q)$, then the order of the diagonal automorphism of R is less than or equal to 4 and otherwise the order of the diagonal automorphism of R is a divisor of $l+1$. Let k be the largest integer, where $(q^k - 1) \mid |R|$. By Lemma 1.4, $fk + 1 \leq \mathrm{P}(q^k - 1)$. Now in each case, using [7] we get that each prime divisor of $|\mathrm{Out}(R)|$ is less than $fk + 1 \leq \mathrm{P}(p^{kf} - 1) \leq p_0$, and so $p_0 \nmid |\mathrm{Out}(R)|$.

By assumptions, there exists a normal subgroup K of G, which is isomorphic to R^m. As R is a non-abelian simple group, $K \cap C_G(K) = 1$ and it follows that $KC_G(K) \cong R^m \times C_G(K)$ and $C_G(K) \cong KC_G(K)/K \trianglelefteq G/K \cong T^n$. On the other hand, $G/C_G(K) \hookrightarrow \mathrm{Aut}(K)$ and $\mathrm{Out}(K) \cong \mathrm{Out}(R) \wr S_m$. Since $m < p_0$, therefore

$$\begin{aligned} |\frac{G}{C_G(K)}| \big| |\mathrm{Aut}(K)| &\Rightarrow |\frac{G}{C_G(K)}|_{p_0} \big| |\mathrm{Aut}(K)|_{p_0} \\ &\Rightarrow \frac{|R|_{p_0}^m |T|_{p_0}^n}{|C_G(K)|_{p_0}} \big| |R|_{p_0}^m \Rightarrow |T|_{p_0}^n = |C_G(K)|_{p_0}. \end{aligned}$$

Hence $C_G(K) \cong T^n$ and so $G \cong R^m \times T^n$. □

The generalized Fitting subgroup of G is the subgroup $F^*(G) = E(G)F(G)$, where $E(G)$ is the subgroup of G generated by all components of G, i.e. quasisimple subnormal subgroups of G and $F(G)$ is the Fitting subgroup of G.

Theorem 2.7. *Let $q = 2^{2n+1} \geq 8$. If G satisfies the pdz-condition and $|G| = |\mathrm{Sz}(q)|^{\alpha}$ where $\alpha \leq \widehat{p_n}$, then $G \cong \mathrm{Sz}(q)^{\alpha}$.*

Proof. Using Lemma 2.2, $F(G) = 1$ and so $F^*(G) = E(G)$. We know that, if no minimal normal subgroup of G is abelian, then $F^*(G) = E(G)$ is a direct product of non-abelian simple groups.

Since the Suzuki groups are the only non-abelian simple groups whose orders are prime to 3, $F^*(G) \cong \mathrm{Sz}(q)^{\beta} \times \prod_{i=1}^{m} \mathrm{Sz}(q_i)^{\alpha_i}$, where $q_i = 2^{2n_i+1}$, $n_i \neq n$ and $0 \leq \beta \leq \alpha$. On the contrary suppose that $\beta < \alpha$. By Lemma 1.2 and Zsigmondy's theorem, we get that $n_i < n$, for each $1 \leq i \leq m$.

We know that $F^*(G)$ is self-centralizing, i.e., $C_G(F^*(G)) \subseteq F^*(G)$ and so $C_G(F^*(G)) = 1$. So $G/F^*(G)$ is embedded in $\mathrm{Out}(F^*(G))$. Hence

$$|\frac{G}{F^*(G)}| \;\Big|\; |\mathrm{Out}(F^*(G))| = |\mathrm{Out}(\mathrm{Sz}(q)^{\beta} \times \prod_{i=1}^{m} \mathrm{Sz}(q_i)^{\alpha_i})|.$$

Therefore using [17, Theorem 3.3.20] we have

$$\widehat{p_n}^{\alpha-\beta} \;\Big|\; (2n+1)^{\beta}\beta! \prod_{i=1}^{m} (2n_i+1)^{\alpha_i}\alpha_i!.$$

Now, similarly to Lemma 2.5, we claim that $|F^*(G)|_2 > |G|_2$. Since $\widehat{p_n}$ is a primitive prime divisor of $2^{2n+1}-1$ or $2^{2(4n+2)}-1$, we get that $\widehat{p_n} \equiv 1 \pmod{2n{+}1}$ or $\widehat{p_n} \equiv 1 \pmod{8n{+}4}$, respectively. So $2n{+}1 < \widehat{p_n}$. Since $n_i < n$, $2n_i{+}1 < \widehat{p_n}$. Thus $\widehat{p_n} \nmid |\mathrm{Out}(\mathrm{Sz}(2^{2n+1}))| = 2n{+}1$ and $\widehat{p_n} \nmid |\mathrm{Out}(\mathrm{Sz}(2^{2n_i+1}))| = 2n_i+1$, for $1 \le i \le m$. Hence $\widehat{p_n}^{\alpha-\beta}$ divides $\prod_{i=1}^{m} \alpha_i! = \alpha_1!\alpha_2!\cdots\alpha_m!$.

Let $\alpha - \beta = t_1 + t_2 + \cdots + t_z$ such that $\widehat{p_n}^{t_j} \mid \alpha_{i_j}!$, where $1 \le j \le z$ and $\{\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_z}\} \subseteq \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$. Therefore $t_j \le \dfrac{\alpha_{i_j}}{\widehat{p_n}-1}$ and so $t_j(\widehat{p_n}-1) \le \alpha_{i_j}$. We have

$$\begin{aligned}
|F^*(G)|_2 &\ge |\mathrm{Sz}(q)|_2^{\beta}(2^6)^{\alpha_{i_1}}(2^6)^{\alpha_{i_2}}\cdots(2^6)^{\alpha_{i_z}}\\
&\ge |\mathrm{Sz}(q)|_2^{\beta}(2^6)^{t_1(\widehat{p_n}-1)}(2^6)^{t_2(\widehat{p_n}-1)}\cdots(2^6)^{t_z(\widehat{p_n}-1)}\\
&= (2^{4n+2})^{\beta}(2^{6\widehat{p_n}-6})^{t_1+t_2+\cdots+t_z} = (2^{4n+2})^{\beta}(2^{6\widehat{p_n}-6})^{\alpha-\beta}\\
&\ge (2^{4n+2})^{\beta}(2^{6(2n+1)-6})^{\alpha-\beta} > (2^{4n+2})^{\beta}(2^{4n+2})^{\alpha-\beta} = (2^{4n+2})^{\alpha} = |G|_2,
\end{aligned}$$

which is a contradiction. Therefore $\alpha = \beta$ and consequently we get the result. □

As a consequence of the above theorem, by [3, Theorem 2.13] we have the following result which is a partial answer to the question arosed in [18].

Corollary 2.8. *Let $q = 2^{2n+1} \ge 8$ and $\alpha \le \widehat{p_n}$. Then $\mathbb{C}G \cong \mathbb{C}\mathrm{Sz}(q)^{\alpha}$ if and only if $G \cong \mathrm{Sz}(q)^{\alpha}$.*

3. Characterization by a p-defect zero character and order

It is a well-known fact that characters of a finite group can give important information about the structure of the group. In [20], it is proved that all simple K_3-groups are uniquely determined by their orders and one or both of their largest and second largest irreducible character degrees. In the sequel, we show that the direct product of some copies of $\mathrm{Sz}(q)$ are characterizable by order and the existence of a p-defect zero character.

Lemma 3.1. *Let G be a solvable group of order ${p_1}^{\alpha_1}{p_2}^{\alpha_2}\cdots{p_k}^{\alpha_k}$. If there exists $1 \le i \le k$ such that G has a p_i-defect zero character, then*

$$ {p_i}^{\alpha_i} \mid \prod_{\substack{j=1\\ j\neq i}}^{k} |\mathrm{GL}(\alpha_j, p_j)|. $$

Proof. Due to the fact that G has a p_i-defect zero character and considering [11, Corollary 11.29], it is obtained that $O_{p_i}(G) = 1$ and $\mathrm{Fit}(G) \cong \prod_{j=1, j\neq i}^{k} O_{p_j}(G) \neq 1$. Since G is a solvable group, $C_G(\mathrm{Fit}(G)) \le \mathrm{Fit}(G)$. Note that $G/C_G(\mathrm{Fit}(G)) \hookrightarrow \mathrm{Aut}(\mathrm{Fit}(G))$. Therefore $|G| \mid |\mathrm{Fit}(G)| \cdot |\mathrm{Aut}(\mathrm{Fit}(G))|$ and

$$ \mathrm{Aut}(\mathrm{Fit}(G)) \cong \mathrm{Aut}(\prod_{\substack{j=1\\ j\neq i}}^{k} O_{p_j}(G)) \cong \prod_{\substack{j=1\\ j\neq i}}^{k} \mathrm{Aut}(O_{p_j}(G)). $$

Also, by [9] we know that $|\mathrm{Aut}(O_{p_j}(G))| \mid |\mathrm{GL}(\alpha_j, p_j)|$ and we get the result. □

Lemma 3.2. *Let G be a finite group such that $\pi(G) = \pi(\mathrm{Sz}(2^{2n+1}))$, where $1 \le n \le 10$. Let G have a p-defect zero character χ such that $\chi(1)_p = p^x$. Let $l = (p-1)^2 x/p$. If G satisfies one of the following conditions, then G is not solvable:*

(a) $|G| = 2^a 5^b 7^c 13^x$, *and* $a + 3b + c < l$,
(b) $|G| = 2^a 5^b 31^c 41^x$ *and* $2(a + b + 2c) < l$,
(c) $|G| = 2^a 5^b 29^c 113^x 127^d$ *and* $4a + b + c + 4d < l$,
(d) $|G| = 2^a 5^b 7^c 13^d 37^e 73^f 109^x$ *and* $3a + 4b + 4c + d + e + 4f < l$,
(e) $|G| = 2^a 5^b 23^c 89^d 397^e 2113^x$ *and* $48a + b + c + 11d + e < l$,
(f) $|G| = 2^a 5^b 53^c 157^d 1613^x 8191^e$ *and* $31a + b + c + d + 52e < l$,
(g) $|G| = 2^a 5^b 7^c 13^d 31^e 41^f 61^g 151^h 1321^x$ *and* $22a + 2b + 5c + d + 8e + 2f + 3g + h < l$,
(h) $|G| = 2^a 5^b 137^c 953^d 26317^x 131071^e$ *and* $387a + 9b + c + d + 612e < l$,
(i) $|G| = 2^a 5^b 229^c 457^d 524287^e 525313^x$ *and* $6912a + b + c + 2d + 6912e < l$,
(j) $|G| = 2^a 5^b 7^c 13^d 29^e 113^f 127^g 337^h 1429^i 14449^x$ *and* $172a + 24b + 28c + 21d + 6e + 24f + 12g + 84h + 3i < l$.

Proof. (a) On the contrary, assume that G is solvable. By Lemma 1.5, we obtain that $\mathrm{Ord}_{13^k}(2) = \mathrm{Ord}_{13}(2) \cdot 13^{k-1}$, $\mathrm{Ord}_{13^k}(5) = \mathrm{Ord}_{13}(5) \cdot 13^{k-1}$ and $\mathrm{Ord}_{13^k}(7) = \mathrm{Ord}_{13}(7) \cdot 13^{k-1}$, for every $k \in \mathbb{N}$. Using Lemma 3.1, we have

$13^x \, \big| |\mathrm{GL}(a,2)| \cdot |\mathrm{GL}(b,5)| \cdot |\mathrm{GL}(c,7)|$. By calculating the power of 13, we have

$$\begin{aligned} x &\le [\frac{a}{\mathrm{Ord}_{13}(2)}] + [\frac{a}{\mathrm{Ord}_{13^2}(2)}] + [\frac{a}{\mathrm{Ord}_{13^3}(2)}] + \cdots \\ &+ [\frac{b}{\mathrm{Ord}_{13}(5)}] + [\frac{b}{\mathrm{Ord}_{13^2}(5)}] + [\frac{b}{\mathrm{Ord}_{13^3}(5)}] + \cdots \\ &+ [\frac{c}{\mathrm{Ord}_{13}(7)}] + [\frac{c}{\mathrm{Ord}_{13^2}(7)}] + [\frac{c}{\mathrm{Ord}_{13^3}(7)}] + \cdots \\ &\le [\frac{a}{12}] + [\frac{a}{12 \cdot 13}] + \cdots + [\frac{b}{4}] + [\frac{b}{4 \cdot 13}] + \cdots + [\frac{c}{12}] + [\frac{c}{12 \cdot 13}] + \cdots \\ &\le \frac{a+3b+c}{12} \cdot \frac{13}{12} < x, \end{aligned}$$

which is a contradiction.

The proof of other cases are similar. □

Theorem 3.3. *Let $q = 2^{2n+1} \ge 8$, where $n = 1, \ldots, 10$ and p be the largest primitive prime divisor of $2^{2(4n+2)} - 1$. If G has a p-defect zero character and $|G| = |\mathrm{Sz}(q)|^{\alpha}$, where $\alpha < p$, then $G \cong \mathrm{Sz}(q)^{\alpha}$.*

Proof. We put $H_0 := G$. By Lemma 3.2, it follows that G is not solvable. According to Lemma 1.1, $G = H_0$ has a normal series $1 \trianglelefteq H_1 \trianglelefteq K_1 \trianglelefteq H_0 = G$ such that K_1/H_1 is a direct product of isomorphic non-abelian simple groups and $|H_0/K_1| \, \big| \, |\mathrm{Out}(K_1/H_1)|$. If H_1 is not a solvable group, similarly to the proof of Theorem 2.7, we continue this process and finally we have a subnormal series of G as follows

(1) $\quad 1 \trianglelefteq H_m \trianglelefteq K_m \trianglelefteq H_{m-1} \trianglelefteq K_{m-1} \trianglelefteq \cdots \trianglelefteq H_2 \trianglelefteq K_2 \trianglelefteq H_1 \trianglelefteq K_1 \trianglelefteq H_0 = G,$

where m is the smallest number such that H_m is solvable. Hence

$$|G| = \prod_{i=1}^{m} |K_i/H_i| \cdot \prod_{i=1}^{m} |H_{i-1}/K_i| \cdot |H_m|.$$

We know that K_i/H_i is a direct product of α_i copies of a non-abelian simple group $S_i \cong \mathrm{Sz}(q_i)$, where $q_i = 2^{2n_i+1}$ such that $|H_{i-1}/K_i| \, \big| \, |\mathrm{Out}(K_i/H_i)|$. We note that for $1 \le n \le 10$, $|\mathrm{Sz}(2^{2n+1})|_p = p$.

• If $2n+1$ is a product of two prime numbers p_1 and p_2 (not necessarily distinct), then K_i/H_i is isomorphic to $\mathrm{Sz}(q)^{\alpha_i}$, $\mathrm{Sz}(2^{p_1})^{\alpha_i}$ or $\mathrm{Sz}(2^{p_2})^{\alpha_i}$. If there exists i such that $p \big| |H_{i-1}/K_i|$, then $p \big| \, |\mathrm{Out}(K_i/H_i)| = |\mathrm{Out}(S)|^{\alpha_i}(\alpha_i!)$, where $S \cong \mathrm{Sz}(q), \mathrm{Sz}(2^{p_1})$ or $\mathrm{Sz}(2^{p_2})$. Using Lemma 2.4, we have $p \nmid |\mathrm{Out}(S)|$, therefore $\alpha_i > p$, where $S \cong \mathrm{Sz}(q), \mathrm{Sz}(2^{p_1})$ or $\mathrm{Sz}(2^{p_2})$. In each case, we can find a prime r such that $r^p \nmid |G|$ but $r^p \mid |K_i/H_i|$, which is a contradiction. So $p \nmid |H_{i-1}/K_i|$.

We claim that $p \nmid |H_m|$. Otherwise, if $p \mid |H_m|$, let $p^{\beta} \, \| \, |H_m|$. Then

$$p^{\beta} \| |H_m| \cdot \prod_{i=1}^{m} |H_{i-1}/K_i| = \frac{|G|}{\prod_{i=1}^{m} |K_i/H_i|}.$$

Let $p^\gamma \parallel \prod_{i=1}^m |K_i/H_i|$. Hence $\alpha = \beta + \gamma$. We know that each K_i/H_i is a direct product of α_i copies of $\mathrm{Sz}(q)$, $\mathrm{Sz}(2^{p_1})$ or $\mathrm{Sz}(2^{p_2})$. Since $p \mid |\mathrm{Sz}(q)|$ and $p \nmid |\mathrm{Sz}(2^{p_1})||\mathrm{Sz}(2^{p_2})|$, we get that $|\mathrm{Sz}(q)|^\gamma \mid \prod_{i=1}^m |K_i/H_i|$. Therefore, $|H_m| \bigm| |\mathrm{Sz}(q)|^\beta$ and by [11, Corollary 11.29], H_m has a p-defect zero character of degree p^β. By Lemma 3.2, H_m is not solvable, which is a contradiction. Thus $p \nmid |H_m|$. Hence $p^\alpha \parallel \prod_{i=1}^m |K_i/H_i|$. Therefore as we mentioned above, $H_m = 1$, and for each $1 \leq i \leq m$, $H_{i-1} = K_i$ and $K_i/H_i \cong \mathrm{Sz}^{\alpha_i}(q)$, where $\alpha_1 + \cdots + \alpha_m = \alpha$.

• If $2n+1$ is a prime, then $K_i/H_i \cong \mathrm{Sz}(q)^{\alpha_i}$ and similarly to the above we have $p \nmid |H_{i-1}/K_i|$ and $H_m = 1$.

Hence for each possibility of $2n+1$, we get the following series

$$1 = H_m \trianglelefteq H_{m-1} \trianglelefteq H_{m-2} \trianglelefteq \cdots \trianglelefteq H_2 \trianglelefteq H_1 \trianglelefteq H_0 = G,$$

such that $H_{i-1}/H_i \cong \mathrm{Sz}^{\alpha_i}(q)$. Considering Lemma 2.6, we obtain that $H_{m-1} \cong \mathrm{Sz}^{\alpha_m}(q)$, $H_{m-2} \cong \mathrm{Sz}^{\alpha_m+\alpha_{m-1}}(q)$ and finally $G \cong \mathrm{Sz}^{\alpha_m+\alpha_{m-1}+\cdots+\alpha_1}(q) = \mathrm{Sz}^\alpha(q)$ and the proof is complete. □

4. Aknowledgement

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