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# METALLIC STRUCTURES ON TANGENT BUNDLES OF LORENTZIAN PARA-SASAKIAN MANIFOLDS

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Abstract. Let $M$ be a Lorentzian para-Sasakian manifold with a Lorentzian para-Sasakian structure $(\phi, \eta, \xi, g)$. In this paper, we introduce some metallic structures on tangent bundle of the manifold $M$ using vertical, horizontal and complete lifts of the Lorentzian para-Sasakian structure $(\phi, \eta, \xi, g)$ and investigate their parallelity. We also consider fundamental 2-forms and try to find conditions under which these 2-forms are closed.

*Keywords*: Lorentzian para-Sasakian manifold, Tangent bundle, Complete lift.
*2020 MSC*: Primary 53C15, 53C25, 53C21.

## 1. Introduction

Let $(M, g)$ be a Riemannian manifold. Hrețcanu and Crasmareanu defined a metallic structure on $M$ as a $(1,1)-$tensor field $\varphi$ which satisfies $\varphi^2 = p\varphi + qI$, where $p, q$ are positive integers and $I$ is the identity map acting on vector fields of $M$ [7]. It is said that $g$ is $\varphi$-compatible if

$$g(\varphi X, Y) = g(X, \varphi Y), \ \forall X, Y \in \chi(M),$$

or equivalently

$$g(\varphi X, \varphi Y) = pg(X, \varphi Y) + qg(X, Y), \ \forall X, Y \in \chi(M).$$

In this case, $(M, g, \varphi)$ is called a metallic Riemannian manifold. Particular cases of these manifolds are listed below:

If $p = q = 1$, then $(M, g, \phi)$ is a golden Riemannian manifold.
If $p = 2, \ q = 1$, then $(M, g, \phi)$ is a silver Riemannian manifold.
If $p = 3, \ q = 1$, then $(M, g, \phi)$ is a bronze Riemannian manifold.
If $p = 4, \ q = 1$, then $(M, g, \phi)$ is a subtle Riemannian manifold.
If $p = 1, \ q = 2$, then $(M, g, \phi)$ is a copper Riemannian manifold.
If $p = 1, q = 3$, then $(M, g, \phi)$ is a nickel Riemannian manifold.

Metallic structures and their particular cases on various bundles of Riemannian manifolds have been of great interest and were investigated by many geometers (see for instance [1], [4], [6], [8], [9], [10], [11], [13], [14], [15], [16]).

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In addition to above studies, metallic structures on tangent bundles can be investigated by choosing a special manifold as the base manifold. According to this, in [4], Azami studied metallic structures on tangent bundles of para-Sasakian manifolds. He constructed these structures using some lifts of a para-Sasakian structure. He also investigated their integrability and parallelity.

In this paper, we construct metallic structures on tangent bundles of Lorentzian para-Sasakian manifolds by using vertical, horizontal and complete lifts of a Lorentzian para-Sasakian structure and study their parallelity. Throughout the paper, all manifolds, tensor fields and connections are always assumed to be differentiable of class $C^\infty$.

## 2. **Preliminaries**

In this section, we will present some basic facts about Lorentzian para-Sasakian manifolds and tangent bundles.

Lorentzian para-Sasakian manifolds were introduced by Matsumoto in 1989 [11]. Since then, a lot of physical and geometric properties have been studied intensively by many authors (for example see [2], [5], [12]). Curves on Lorentzian manifolds are also studied widely by many authors (see for example [18]).

**Definition 2.1.** [11] A quartet $(\phi, \eta, \xi, g)$ given on an $n-$dimensional manifold $M$, where $\phi$ is a $(1,1)-$tensor field, $\eta$ is a 1-form, $\xi$ is a timelike vector field and $g$ is the Lorentzian metric satisfying for every $X, Y \in \chi(M)$

$$\begin{aligned} \phi^2 X &= X + \eta(X)\xi, \ \eta(\xi) = -1, \\ \eta(\phi X) &= 0, \ \phi\xi = 0, \ g(X,\xi) = \eta(X), \ g(X,Y) = g(\phi X, \phi Y) - \eta(X)\eta(Y) \end{aligned}$$

is called a Lorentzian para-contact metric structure on $M$ and the manifold $M$ associated with the metric $g$ is said to be the Lorentzian para-contact metric manifold. Moreover, if the relations

$$\begin{aligned} (\nabla_X \phi)Y &= [g(X,Y) + \eta(X)\eta(Y)]\xi + [X + \eta(X)\xi]\eta(Y) \quad \text{and} \\ \nabla_X \xi &= \phi X \Leftrightarrow (\nabla_X \eta)(Y) = g(\phi X, Y) \end{aligned}$$

hold on $M$ for every $X, Y \in \chi(M)$ then the Lorentzian para-contact metric manifold $M$ is known as a Lorentzian para-Sasakian (briefly, LP-Sasakian) manifold. Here, $\nabla$ is the Levi-Civita connection of $M$.

In the second part of this section, we only recall essentials about tangent bundles of differentiable manifolds to be used in the sequel (for more details see [19]). Let $(M, g)$ be an $n-$dimensional Riemannian manifold. The tangent bundle $TM$ of the manifold $M$ is the disjoint union of the tangent spaces at each point of $M$, i.e., $TM = \cup_{p\in M} T_p M$. The manifold $TM$ is of $2n-$dimension and the differentiable structure on $M$ induces a differentiable structure on the tangent bundle $TM$ turning it into a differentiable manifold.

Let $f, X, \omega$ and $F$ be a function, a vector field, a 1-form and a $(1,1)-$type tensor field on $M$, respectively. We will denote their vertical lifts by $f^v, X^v, \omega^v, F^v$, their complete lifts by $f^c, X^c, \omega^c, F^c$ and their horizontal lifts by $f^h, X^h, \omega^h, F^h$ to tangent bundle $TM$. The following relations are satisfied:

(1)

$$\begin{aligned}
(fX)^v &= f^v X^v,\ X^v f^v = 0,\ F^v(X^v) = 0, \omega^v(X^v) = 0,\\
(fX)^c &= f^c X^v + f^v X^c,\ X^c f^c = (Xf)^c,\ F^c(X^c) = (F(X))^c,\\
X^v(f^c) &= X^c(f^v) = (Xf)^v, F^c(X^v) = (F(X))^v, F^v(X^c) = (F(X))^v,\\
\omega^v(X^c) &= \omega^c(X^v) = (\omega(X))^v, \omega^c(X^c) = (\omega(X))^c,\\
\omega^h(X^h) &= 0, \omega^h(X^v) = (\omega(X))^v, F^h(X^h) = (F(X))^h, F^h(X^v) = (FX)^v.
\end{aligned}$$

Remark that if $P(x)$ is a polynomial in one variable $x$, then $P(F^c) = (P(F))^c$ and $P(F^h) = (P(F))^h$.

For the Lie brackets, we have

$$\begin{aligned}
[X^v, Y^v] &= 0,\ [X^v, Y^c] = [X, Y]^v, [X^c, Y^c] = [X, Y]^c,\\
[X^v, Y^h] &= -(\nabla_Y X)^v,\ [X^h, Y^h] = [X, Y]^h - \gamma R(X, Y),
\end{aligned}$$

where $R$ is the curvature tensor of the metric $g$ on $M$ and the vertical vector lift $\gamma R(X, Y)$ is defined by $\gamma(R(X, Y)) = (R(X, Y)u)^v$, where $u$ is the canonical vector field on $TM$.

On the other hand, it is known that the complete lift metric $g^c$ of the metric $g$ is a semi-Riemannian metric on $TM$ and it is defined by

$$\begin{cases}
g^c(X^v, Y^c) = g^c(X^c, Y^v) = (g(X, Y))^v,\\
g^c(X^v, Y^v) = 0,\\
g^c(X^c, Y^c) = (g(X, Y))^c,\ \forall X, Y \in \chi(M).
\end{cases}$$

The complete lift connection $\nabla^c$ of a linear connection $\nabla$ satisfies the following relations:

$$\begin{aligned}
&\nabla^c_{X^v} Y^v = (\nabla_X Y)^v, \nabla^c_{X^c} Y^c = (\nabla_X Y)^c, \nabla^c_{X^v} Y^c = \nabla^c_{X^c} Y^v = (\nabla_X Y)^v,\\
&\nabla^c F^v = (\nabla F)^v,\ \nabla^c F^c = (\nabla F)^c.
\end{aligned}$$

Similarly, the horizontal lift metric $g^h$ is a semi-Riemannian metric and it is defined by

$$\begin{cases}
g^h(X^v, Y^h) = g^h(X^h, Y^v) = (g(X, Y))^v,\\
g^h(X^v, Y^v) = 0,\\
g^h(X^h, Y^h) = 0, \forall X, Y \in \chi(M).
\end{cases}$$

Moreover, the horizontal lift connection $\nabla^h$ of a linear connection $\nabla$ satisfies the following relations:

$$\nabla^h_{X^v} Y^v = \nabla^h_{X^v} Y^h = 0, \nabla^h_{X^h} Y^v = (\nabla_X Y)^v, \nabla^h_{X^h} Y^h = (\nabla_X Y)^h.$$

Finally, a Riemannian metric $g^s$, which is known as the Sasaki metric, is defined by

(2)
$$\begin{cases}
g^s(X^h, Y^h) = (g(X, Y))^v,\\
g^s(X^h, Y^v) = 0,\\
g^s(X^v, Y^v) = (g(X, Y))^v, \forall X, Y \in \chi(M).
\end{cases}$$

## 3. Metallic structures on tangent bundles of LP-Sasakian manifolds

In the following, let $M$ be an $n-$dimensional LP-Sasakian manifold with the quartet $(\phi, \eta, \xi, g)$. We construct a metallic structure on $TM$ using complete lift and study its parallelity. We also define a 2-form and find a condition under which it is closed. We shall denote $\sigma_{p,q} = \frac{p+\sqrt{p^2+4q}}{2}$.

**Proposition 3.1.** *The* $(1,1)-$*tensor field* $J$ *defined by*

$$J = \frac{p}{2}I + (\frac{2\sigma_{p,q} - p}{2})(\phi^c - \eta^v \otimes \xi^v - \eta^c \otimes \xi^c) \tag{3}$$

*is a metallic structure on the tangent bundle of a LP-Sasakian manifold with the quartet* $(\phi, \eta, \xi, g)$.

*Proof.* The following relations are occurred from the definition of LP-Sasakian manifolds and lifts to tangent bundles:

$$\begin{aligned} (\phi^c)^2 &= (\phi^2)^c = I + \eta^c \otimes \xi^v + \eta^v \otimes \xi^c, \\ \eta^v(\xi^c) &= \eta^c(\xi^v) = -1,\ \eta^v(\xi^v) = \eta^c(\xi^c) = 0, \\ \phi^c(\xi^v) &= \phi^c(\xi^c) = 0,\ \eta^v \circ \phi^c = \eta^c \circ \phi^c = 0. \end{aligned}$$

If we use these relations, we get

$$\begin{aligned} J(\xi^v) &= \frac{p}{2}I(\xi^v) + (\frac{2\sigma_{p,q} - p}{2})(\phi^c - \eta^v \otimes \xi^v - \eta^c \otimes \xi^c)(\xi^v) \\ &= \frac{p}{2}\xi^v + (\frac{2\sigma_{p,q} - p}{2})(\phi^c(\xi^v) - \eta^v(\xi^v)\xi^v - \eta^c(\xi^v)\xi^c) \\ &= \frac{p}{2}\xi^v + \frac{2\sigma_{p,q} - p}{2}\xi^c. \end{aligned}$$

By a similar way, we obtain

$$\begin{aligned} J(\xi^c) &= \frac{p}{2}\xi^c + \frac{2\sigma_{p,q} - p}{2}\xi^v, \\ J(\phi^c(\widetilde{X})) &= \frac{p}{2}\phi^c(\widetilde{X}) + \frac{2\sigma_{p,q} - p}{2}(\widetilde{X} + \eta^c(\widetilde{X})\xi^v + \eta^v(\widetilde{X})\xi^c). \end{aligned}$$

Therefore, we have

$$J(\widetilde{X}) = \frac{p}{2}\widetilde{X} + (\frac{2\sigma_{p,q} - p}{2})(\phi^c(\widetilde{X}) - \eta^v(\widetilde{X})\xi^v - \eta^c(\widetilde{X})\xi^c)$$

and

$$\begin{aligned}
J^2(\widetilde{X}) &= \frac{p}{2}J(\widetilde{X}) + (\frac{2\sigma_{p,q}-p}{2})(J(\phi^c(\widetilde{X})) - \eta^v(\widetilde{X})J(\xi^v) - \eta^c(\widetilde{X})J(\xi^c)) \\
&= \frac{p}{2}[\frac{p}{2}\widetilde{X} + (\frac{2\sigma_{p,q}-p}{2})(\phi^c(\widetilde{X}) - \eta^v(\widetilde{X})\xi^v - \eta^c(\widetilde{X})\xi^c)] \\
&\quad +(\frac{2\sigma_{p,q}-p}{2})[\frac{p}{2}\phi^c(\widetilde{X}) + \frac{2\sigma_{p,q}-p}{2}(\widetilde{X} + \eta^c(\widetilde{X})\xi^v + \eta^v(\widetilde{X})\xi^c \\
&\quad -\eta^v(\widetilde{X})(\frac{p}{2}\xi^v + \frac{2\sigma_{p,q}-p}{2}\xi^c) - \eta^c(\widetilde{X})(\frac{p}{2}\xi^c + \frac{2\sigma_{p,q}-p}{2}\xi^v)] \\
&= (\frac{p^2}{4} + \frac{p^2+4q}{4})\tilde{X} + p\frac{2\sigma_{p,q}-p}{2}(\phi^c(\widetilde{X}) - \eta^v(\widetilde{X})\xi^v - \eta^c(\widetilde{X})\xi^c) \\
&= pJ(\tilde{X}) + q\widetilde{X}.
\end{aligned}$$

Thus, we proved the proposition. □

**Proposition 3.2.** *Let $M$ be a LP-Sasakian manifold with the quartet $(\phi,\eta,\xi,g)$ and $J$ is given as (3). In this case, the following equation is satisfied*

$$g^c(J\widetilde{X}, J\widetilde{Y}) = pg^c(\widetilde{X}, J\widetilde{Y}) + qg^c(\tilde{X}, \tilde{Y}),\ \forall \tilde{X}, \tilde{Y} \in \chi(TM).$$

*Proof.* We have

$$\begin{aligned}
g^c(X^v, Y^v) &= 0,\ g^c(X^v, \xi^c) = (g(X,\xi))^v = (\eta(X))^v, \\
g^c((\phi X)^v, \xi^c) &= (g(\phi X, \xi))^v = (g(X, \phi\xi))^v = 0, \\
g^c(\xi^c, \xi^c) &= (g(\xi,\xi))^c = 0.
\end{aligned}$$

We also have

$$\begin{aligned}
JX^v &= \frac{p}{2}X^v + \frac{2\sigma_{p,q}-p}{2}(\phi^c(X^v) - \underbrace{\eta^v(X^v)}_{0}\xi^v - \eta^c(X^v)\xi^c) \\
&= \frac{p}{2}X^v + \frac{2\sigma_{p,q}-p}{2}((\phi(X))^v - (\eta(X))^v\xi^c), \qquad (4)
\end{aligned}$$

$$\begin{aligned}
JX^c &= \frac{p}{2}X^c + \frac{2\sigma_{p,q}-p}{2}(\phi^c(X^c) - \eta^v(X^c)\xi^v - \eta^c(X^c)\xi^c) \\
&= \frac{p}{2}X^c + \frac{2\sigma_{p,q}-p}{2}((\phi(X))^c - (\eta(X))^v\xi^v - (\eta(X))^c\xi^c). \qquad (5)
\end{aligned}$$

So,

$$\begin{aligned}
g^c(JX^v, JY^v) &= g^c(\frac{p}{2}X^v + \frac{2\sigma_{p,q}-p}{2}((\phi X)^v - (\eta X)^v\xi^c), \\
&\quad \frac{p}{2}Y^v + \frac{2\sigma_{p,q}-p}{2}((\phi Y)^v - (\eta Y)^v\xi^c)) \\
&= g^c(\frac{p}{2}X^v, -\frac{2\sigma_{p,q}-p}{2}(\eta Y)^v\xi^c) \\
&\quad -g^c(\frac{2\sigma_{p,q}-p}{2}(\phi X)^v, \frac{2\sigma_{p,q}-p}{2}(\eta Y)^v\xi^c) \\
&\quad -g^c(\frac{2\sigma_{p,q}-p}{2}(\eta X)^v\xi^c, \frac{p}{2}Y^v) \\
&\quad -g^c(\frac{2\sigma_{p,q}-p}{2}(\eta X)^v\xi^c, \frac{2\sigma_{p,q}-p}{2}(\phi Y)^v) \\
&\quad -g^c(\frac{2\sigma_{p,q}-p}{2}(\eta X)^v\xi^c, \frac{2\sigma_{p,q}-p}{2}(\eta Y)^v\xi^c) \\
&= -\frac{(2\sigma_{p,q}-p)p}{2}(\eta X)^v(\eta Y)^v,
\end{aligned}$$

$$\begin{aligned}
g^c(X^v, JY^v) &= g^c(X^v, \frac{p}{2}Y^v + \frac{2\sigma_{p,q}-p}{2}((\phi Y)^v - (\eta Y)^v\xi^c) \\
&= -\frac{2\sigma_{p,q}-p}{2}(\eta X)^v(\eta Y)^v.
\end{aligned}$$

Thus we obtain

$$g^c(JX^v, JY^v) = pg^c(X^v, JY^v) + qg^c(X^v, Y^v).$$

On the other hand, since

$$\begin{aligned}
g^c(JX^v, JY^c) &= g^c(\frac{p}{2}X^v + \frac{2\sigma_{p,q}-p}{2}((\phi X)^v - (\eta X)^v\xi^c, \\
&\quad \frac{p}{2}Y^c + \frac{2\sigma_{p,q}-p}{2}((\phi Y)^c - (\eta Y)^v\xi^v - (\eta Y)^c\xi^c) \\
&= \frac{p^2}{4}(g(X,Y))^v + \frac{(2\sigma_{p,q}-p)p}{4}[(g(X,\phi Y))^v - (\eta X)^v(\eta Y)^c] \\
&\quad + \frac{(2\sigma_{p,q}-p)p}{4}(g(\phi X, Y))^v + \frac{(2\sigma_{p,q}-p)^2}{4}[(g(X,Y))^v \\
&\quad + (\eta X)^v(\eta Y)^v] - \frac{(2\sigma_{p,q}-p)p}{4}(\eta X)^v(\eta Y)^c - \frac{(2\sigma_{p,q}-p)^2}{4}(\eta X)^v(\eta Y)^v \\
&= (\frac{p^2}{2} + q)(g(X,Y))^v + \frac{(2\sigma_{p,q}-p)p}{2}[(g(X,\phi Y))^v - (\eta X)^v(\eta Y)^c],
\end{aligned}$$

and

$$\begin{aligned}
g^c(X^v, JY^c) &= g^c(X^v, \frac{p}{2}Y^c + \frac{(2\sigma_{p,q}-p)}{2}((\phi Y)^c - (\eta Y)^v\xi^v - (\eta Y)^c\xi^c) \\
&= \frac{p}{2}g(X,Y)^v + \frac{(2\sigma_{p,q}-p)}{2}[(g(X,\phi Y))^v - (\eta X)^v(\eta Y)^c),
\end{aligned}$$

we occur

$$g^c(JX^v, JY^c) = pg^c(X^v, JY^c) + qg^c(X^v, Y^c).$$

By the same way, we can prove the other cases. □

**Theorem 3.3.** *If $M$ is a LP-Sasakian manifold with the quartet $(\phi, \eta, \xi, g)$ and $J$ is given as (3), then the metallic structure $J$ cannot be parallel with respect to $\nabla^c$.*

*Proof.* We know that the 1-form $\eta$ in LP-Sasakian manifolds induces an $(n-1)-$dimensional distribution $D$ as

$$D_p = \{X \in T_pM : \eta(X) = 0\}, \ \forall p \in M. \tag{6}$$

Now, for all $X \in D$, we have

$$\begin{aligned}(\nabla^c_{X^c}J)\xi^c &= \nabla^c_{X^c}(J\xi^c) - J(\nabla^c_{X^c}\xi^c) \\ &= \nabla^c_{X^c}(\frac{p}{2}\xi^c + \frac{(2\sigma_{p,q}-p)}{2}((\phi\xi)^c - (\eta(\xi))^v\xi^v \\ &\quad -(\eta(\xi))^c\xi^c) - \frac{p}{2}(\nabla_X\xi)^c - \frac{(2\sigma_{p,q}-p)}{2}((\phi(\nabla_X\xi))^c \\ &\quad -(\eta(\nabla_X\xi))^v\xi^v - (\eta(\nabla_X\xi))^c\xi^c) \\ &= \frac{(2\sigma_{p,q}-p)}{2}[\nabla^c_{X^c}(-(\eta(\xi))^v\xi^v - (\underbrace{\eta(\xi))^c}_{0}\xi^c) - \phi(\nabla_X\xi)^c \\ &\quad +(\eta(\nabla_X\xi))^v\xi^v + (\eta(\nabla_X\xi))^c\xi^c].\end{aligned}$$

Since $\nabla_X\xi = \phi X$, we obtain

$$\begin{aligned}(\nabla^c_{X^c}J)\xi^c &= \frac{(2\sigma_{p,q}-p)}{2}[\nabla^c_{X^c}\xi^v - (X - \eta(X)\xi)^c] \ (\text{since } X \in D, \ \eta(X) = 0) \\ &= \frac{(2\sigma_{p,q}-p)}{2}((\nabla_X\xi)^v - X^c) \neq 0.\end{aligned}$$

This completes the proof. □

**Example 3.4.** *Consider a 3-dimensional manifold $M = \{(x, y, z) : (x, y, z) \in R^3\}$, where $(x, y, z)$ are standard coordinates in $R^3$. The linearly independent vector fields are selected as*

$$E_1 = -e^x\frac{\partial}{\partial y}, \ E_2 = e^x(\frac{\partial}{\partial z} - \frac{\partial}{\partial y}), \ E_3 = \frac{\partial}{\partial x}.$$

*The Lorentzian metric $g$ is defined by*

$$\begin{aligned}g(E_1, E_2) &= g(E_1, E_3) = g(E_2, E_3) = 0, \\ g(E_1, E_1) &= g(E_2, E_2) = 1, \ g(E_3, E_3) = -1.\end{aligned}$$

*The $1-$form $\eta$ is defined by $\eta(Z) = g(Z, E_3)$ for any vector field $Z$ on $M$. The $(1,1)-$tensor field $\phi$ is given by $\phi(E_1) = -E_1, \ \phi(E_2) = -E_2, \ \phi(E_3) = 0$. The* *linearity property of $\phi$ and $g$ gives*

$$\begin{aligned} \eta(E_3) &= -1,\ \phi^2(X) = X + \eta(X)E_3, \\ g(\phi X, \phi Y) &= g(X,Y) + \eta(X)\eta(Y), \end{aligned}$$

*for every vector fields $X, Y$ on $M$. Therefore for $E_3 = \xi$, the quartet $(\phi, \eta, \xi, g)$ determines a Lorentzian paracontact structure on $M$. Thus, $M$ is a 3-dimensional LP-Sasakian manifold [17]. It can be easily shown that the metallic structure $J$ is given as (3) is not parallel with respect to $\nabla^c$, since $(\nabla_X E_3)^v \neq X^c$ for all $X \in \chi(M)$.*

**Proposition 3.5.** *If $M$ is a LP-Sasakian manifold with the quartet $(\phi, \eta, \xi, g)$, $\nabla\phi = 0$ and $J$ is given as (3), then the fundamental 2-form $\Phi$ defined by*

$$\Phi(\tilde{X}, \tilde{Y}) = g^c(\tilde{X}, J\tilde{Y}) - \frac{p}{2} g^c(\tilde{X}, \tilde{Y}),\ \forall \tilde{X}, \tilde{Y} \in \chi(TM),$$

*is closed if and only if the following equation is satisfied:*

$$g(\nabla_Y X, \phi Z) + g(\nabla_Z Y, \phi X) + g(\nabla_X Z, \phi Y) = 0,\ \forall X, Y, Z \in \chi(M). \tag{7}$$

*Proof.* We use the following formula:

$$\begin{aligned} 3d\Phi(\tilde{X}, \tilde{Y}, \tilde{Z}) &= \tilde{X}\Phi(\tilde{Y}, \tilde{Z}) + \widetilde{Y}\Phi(\tilde{Z}, \tilde{X}) + \widetilde{Z}\Phi(\tilde{X}, \widetilde{Y}) \\ &\quad - \Phi([\widetilde{X}, \widetilde{Y}], \widetilde{Z}) - \Phi([\widetilde{Z}, \widetilde{X}], \widetilde{Y}) - \Phi([\widetilde{Y}, \widetilde{Z}], \widetilde{X}), \end{aligned}$$

$\forall \tilde{X}, \tilde{Y}, \tilde{Z} \in \chi(TM)$. Therefore, we write

$$\begin{aligned} 3d\Phi(X^c, Y^c, Z^v) &= X^c g^c(Y^c, JZ^v) - \frac{p}{2} X^c g^c(Y^c, Z^v) + Y^c g^c(Z^v, JX^c) \\ &\quad - \frac{p}{2} Y^c g^c(Z^v, X^c) + Z^v g^c(X^c, JY^c) - \frac{p}{2} Z^v g^c(X^c, Y^v) \\ &\quad - g^c([X,Y]^c, JZ^v) + \frac{p}{2} g^c([X,Y]^c, Z^v) - g^c([Z,X]^v, JY^c) \\ &\quad + \frac{p}{2} g^c([Z,X]^v, Y^c) - g^c([Y,Z]^v, JX^c) + \frac{p}{2} g^c([Y,Z]^v, X^c). \end{aligned}$$

From (4) and (5), we recall

$$\begin{aligned} J(Z^v) &= \frac{p}{2} Z^v + \frac{2\sigma_{p,q} - p}{2}((\phi Z)^v - (\eta Z)^v \xi^c), \\ J(X^c) &= \frac{p}{2} X^c + \frac{2\sigma_{p,q} - p}{2}((\phi X)^c - (\eta X)^v \xi^v - (\eta X)^c \xi^c). \end{aligned}$$

Thus

$$\begin{aligned} \frac{6}{2\sigma_{p,q} - p} d\Phi(X^c, Y^c, Z^v) &= \{Xg(Y, \phi Z) + Yg(Z, \phi X) + Zg(X, \phi Y) \\ &\quad - g([X,Y], \phi Z) - g([Z,X], \phi Y) - g([Y,Z], \phi X)\}^v \\ &\quad - X^c[(\eta(Z))^v(\eta(Y))^c] - Y^c[(\eta(Z))^v(\eta(X))^c] \\ &\quad - Z^v[(\eta(Y))^v(\eta(X))^v] - Z^v[(\eta(Y))^c(\eta(X))^c] \\ &\quad - (\eta(Z))^v g([X,Y], \xi)^c - (\eta(Y))^c g([Z,X]), \xi)^v \\ &\quad - (\eta(X))^c g([Y,Z], \xi)^v. \end{aligned}$$

Similar to Proposition 2.5 in [3], we can see the existence of following relations in LP-Sasakian manifolds by using the equations $\nabla\phi = 0$, $g(X, \phi Y) = g(\phi X, Y)$ and $\phi X = \nabla_X \xi$:

$$\begin{aligned} Xg(Y, \phi Z) - g([X, Y], \phi Z) &= g(\nabla_Y X, \phi Z) + g(\nabla_X Z, \phi Y), \\ g([X, Y], \xi) &= X(\eta(Y)) - Y(\eta(X)). \end{aligned}$$

So,

$$\begin{aligned} \frac{6}{2\sigma_{p,q} - p} d\Phi(X^c, Y^c, Z^v) &= 2\{g(\nabla_Y X, \phi Z) + g(\nabla_X Z, \phi Y) + g(\nabla_Z Y, \phi X)\} \\ &\quad +2\{(\eta(Y))^c (Z\eta(X))^v + (\eta(Z))^v (X\eta(Y))^c \\ &\quad +(\eta(X))^c (Y\eta(Z))^v\}. \end{aligned}$$

In case of $X, Y, Z$ are the vector fields of the distribution $D$ in (6), we occur $3d\Phi(\tilde{X}, \tilde{Y}, \tilde{Z}) = 0$ if and only if (7) is fulfilled. The other cases are similar. □

Now, we construct a metallic structure $F$ on $TM$ using horizontal lift and investigate its parallelity.

**Proposition 3.6.** *The* $(1,1)-$*tensor field* $F$ *defined by*

$$F = \frac{p}{2} I + (\frac{2\sigma_{p,q} - p}{2})(\phi^h - \eta^h \otimes \xi^h - \eta^v \otimes \xi^v) \tag{8}$$

*is a metallic structure on the tangent bundle of a LP-Sasakian manifold with the quartet* $(\phi, \eta, \xi, g)$.

*Proof.* Using the properties of lifts, we have

$$\begin{aligned} (\phi^h)^2 &= (\phi^2)^h = I + \eta^h \otimes \xi^v + \eta^v \otimes \xi^h, \\ \eta^v(\xi^h) &= \eta^h(\xi^v) = -1, \ \eta^v(\xi^v) = \eta^h(\xi^h) = 0, \\ \phi^h(\xi^v) &= \phi^h(\xi^h) = 0, \ \eta^v \circ \phi^h = \eta^h \circ \phi^h = 0. \end{aligned}$$

So, we obtain

$$\begin{aligned} F(\xi^v) &= \frac{p}{2}\xi^v + \frac{2\sigma_{p,q} - p}{2}\xi^h, \ F(\xi^h) = \frac{p}{2}\xi^h - \frac{2\sigma_{p,q} - p}{2}\xi^v, \\ F(\phi^h \widetilde{X}) &= \frac{p}{2}\phi^h \widetilde{X} + \frac{2\sigma_{p,q} - p}{2}(\widetilde{X} + \eta^h(\widetilde{X})\xi^v + \eta^v(\widetilde{X})\xi^h). \end{aligned}$$

Thus, we conclude

$$F(\widetilde{X}) = \frac{p}{2}\widetilde{X} + (\frac{2\sigma_{p,q} - p}{2})(\phi^h \widetilde{X} - \eta^v(\widetilde{X})\xi^v - \eta^h(\widetilde{X})\xi^h)$$

and

$$\begin{aligned} F^2(\widetilde{X}) &= \frac{p}{2}F(\widetilde{X}) + (\frac{2\sigma_{p,q} - p}{2})(F(\phi^h \widetilde{X}) - \eta^v(\widetilde{X})F(\xi^v) - \eta^h(\widetilde{X})F(\xi^h)) \\ &= pF(\tilde{X}) + q\widetilde{X}. \end{aligned}$$

This ends the proof. □

**Proposition 3.7.** *Let $M$ be a LP-Sasakian manifold with the quartet $(\phi, \eta, \xi, g)$ and $F$ is given as (8). In this case, the following equation is satisfied*

$$g^s(F\widetilde{X}, F\widetilde{Y}) = pg^s(\widetilde{X}, F\widetilde{Y}) + qg^s(\tilde{X}, \tilde{Y}), \ \forall \tilde{X}, \tilde{Y} \in \chi(TM).$$

*Proof.* Using (1), we have

$$FX^v = \frac{p}{2}X^v + \frac{2\sigma_{p,q} - p}{2}(\phi X)^v.$$

From the definition of the Sasaki metric (8), we write

$$\begin{aligned} g^s(FX^v, FY^v) &= \{(q + \frac{p^2}{2})g(X,Y) + \frac{2\sigma_{p,q} - p}{2}pg(X, \phi Y)\}^v, \\ g^s(X^v, FY^v) &= \{\frac{p}{2}g(X,Y) + \frac{2\sigma_{p,q} - p}{2}g(X, \phi Y)\}^v. \end{aligned}$$

It is clear that

$$g^s(FX^v, FY^v) = pg^s(X^v, FY^v) + qg^s(X^v, Y^v).$$

The other cases can be proved similarly. □

**Proposition 3.8.** *If $M$ is a LP-Sasakian manifold with the quartet $(\phi, \eta, \xi, g)$ and $F$ is given as (8), then the metallic structure $F$ cannot be parallel with respect to $\nabla^h$.*

*Proof.* Using $\nabla_X \xi = \phi X$ we obtain

$$\begin{aligned} (\nabla^h_{X^h}F)\xi^h &= \nabla^h_{X^h}(F\xi^h) - F(\nabla^h_{X^h}\xi^h) \\ &= \nabla^h_{X^h}(\frac{p}{2}\xi^h + \frac{(2\sigma_{p,q} - p)}{2}((\underbrace{\phi\xi}_{0})^h - \underbrace{(\eta(\xi))^h}_{0}\xi^h \\ &\quad -(\eta(\xi))^v\xi^v) - \frac{p}{2}(\nabla_X\xi)^h - \frac{(2\sigma_{p,q} - p)}{2}((\phi(\nabla_X\xi))^h \\ &\quad -(\eta(\nabla_X\xi))^v\xi^v) - \underbrace{(\eta(\nabla_X\xi))^h}_{0}\xi^h \\ &= \frac{(2\sigma_{p,q} - p)}{2}[(\phi^2X)^h - (\phi X)^v]. \end{aligned}$$

For every $X \in D$ it is clear that $(\nabla^h_{X^h}F)\xi^h \neq 0$. Therefore the proof is complete. □

**Proposition 3.9.** *If $M$ is a LP-Sasakian manifold with the quartet $(\phi, \eta, \xi, g)$ and $F$ is given as (8), then the fundamental 2-form $\Phi'$ defined by*

$$\Phi'(\tilde{X}, \tilde{Y}) = g^s(\tilde{X}, J\tilde{Y}) - \frac{p}{2}g^s(\tilde{X}, \tilde{Y}), \ \forall \tilde{X}, \tilde{Y} \in \chi(TM)$$

*cannot be closed.*

*Proof.* We will use the following relations:

$$\begin{aligned} F(X^v) &= \frac{p}{2}X^v + \frac{(2\sigma_{p,q}-p)}{2}((\phi X)^v - (\eta(X))^v\xi^h), \\ F(\xi^v) &= \frac{p}{2}\xi^v + \xi^h, \\ F(X^h) &= \frac{p}{2}X^h + \frac{(2\sigma_{p,q}-p)}{2}((\phi X)^h - (\eta(X))^v\xi^v). \end{aligned}$$

If $X$ is a spacelike or timelike vector field in the distribution $D$ (6), then from the definition of Sasaki metric (2)

$$\begin{aligned} 3d\Phi'(X^h, X^v, \xi^v) &= X^h\Phi'(X^v,\xi^v) + X^v\Phi'(\xi^v, X^h) + \xi^v\Phi'(X^h, X^v) \\ &\quad -\Phi'([X^h, X^v], \xi^v) - \Phi'([\xi^v, X^h], X^v) - \Phi'(\underbrace{[X^v,\xi^v]}_{0}, X^h) \\ &= X^h(g^s(X^v, F\xi^v) - \frac{p}{2}g^s(X^v,\xi^v)) \\ &\quad +X^v(g^s(\xi^v, FX^h) - \frac{p}{2}\underbrace{g^s(\xi^v, X^h)}_{0}) \\ &\quad +\xi^v(g^s(X^h, FX^v) - \frac{p}{2}\underbrace{g^s(X^h, X^v)}_{0}) \\ &\quad -g^s((\nabla_X X)^v, F\xi^v) + \frac{p}{2}g^s((\nabla_X X)^v, \xi^v) \\ &\quad +g^s((\nabla_X \xi)^v, FX^v) - \frac{p}{2}g^s((\nabla_X\xi)^v, X^v) \\ &= \frac{(2\sigma_{p,q}-p)}{2}g^s((\nabla_X\xi)^v, (\phi X)^v) \\ &= \frac{(2\sigma_{p,q}-p)}{2}g^s((\phi X)^v, (\phi X)^v) \\ &= \frac{(2\sigma_{p,q}-p)}{2}g(X,X) \neq 0. \end{aligned}$$

So, we proved the proposition. □

## 4. **Conclusion**

In this paper, we define some metallic structures on tangent bundles of Lorentzian para-Sasakian manifolds and study their parallelity. We determine some differences in equations from the Riemannian case which is studied in [3].

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