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EQUALIZERS AND COEQUALIZERS IN THE CATEGORY OF TOPOLOGICAL MOLECULAR LATTICES

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Abstract. A completely distributive complete lattice is called a molecular lattice. It is well known that the category **TML** of all topological molecular lattices with generalized order homomorphisms in the sense of Wang, is both complete and cocomplete. In this note, we give an example which shows that the structure of equalizers introduced by Zhao need not be true, in general. In particular, we present the structures of equalizers, coequalizers, monomorphisms and epimorphisms in this category.

Keywords: Topological molecular lattice, Equalizer, Coequalizer.
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1. Introduction and preliminaries

Lattice theory and general topology are two related branches of mathematics which influence each other. For example, the well-known Wallman compactification theorem for T_1 spaces was proved by using lattice theory [10], and the famous representation theorem on Boolean algebras was proved by topological methods [9]. Topological lattice theory is a combination of topology and lattice theory. In 1992, Wang showed that the completely distributive complete lattices are suitable for establishing the pointwise topology [13]. He introduced his important theory called topological molecular lattices as a generalization of ordinary topological spaces, fuzzy topological spaces and L-fuzzy topological spaces in terms of closed elements, molecules, remote neighborhoods and generalized order homomorphisms. Then many authors characterized some topological notions in such spaces, such as convergence theories of molecular nets or ideals, separation axioms and other notions.

We first recall some basic results and definitions of topological molecular lattices. A pair (f, g) of order-preserving maps $f : P \to Q$ and $f : Q \to P$ between posets is called a Galois connections if for all $p \in P$ and $q \in Q$,

$$f(p) \leq_Q q \Longleftrightarrow p \leq_P g(q).$$

The map f is called the left adjoint of g and the map g the right adjoint of f. If $f : P \to Q$ is a mapping between complete lattices which preserves

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arbitrary joins, then f has a right adjoint, denoted by $\hat{f}$, which is defined by $\hat{f}(y) = \bigvee\{x \in P : f(x) \leq y\}$ for every $y \in Q$ [2].

Definition 1.1. [13]

1. A completely distributive complete lattice is called a molecular lattices.
2. A mapping $f : L_1 \to L_2$ between molecular lattices is called a generalized order homomorphism or an **ml**-map, if f preserves arbitrary joins and its right adjoint $\hat{f}$ is a complete homomorphism, i.e., it preserves arbitrary joins and arbitrary meets.

Remark 1.2. If (f, g) is a Galois connections between complete lattices, then f preserves arbitrary joins and g preserves arbitrary meets. So to prove a map f between molecular lattices which preserves arbitrary joins is an **ml**-map, it is enough to show that $\hat{f}$ preserves arbitrary joins.

Definition 1.3. [13] A topological molecular lattice (briefly, **tml**) is a pair (L, τ) such that L is a molecular lattice and $\tau \subseteq L$ is a cotopology, i.e., it is closed under arbitrary meets, finite joins and $0, 1 \in \tau$, where 0 and 1 are the smallest and the greatest elements of L, respectively.

Definition 1.4. [13] An **ml**-map $f : (L_1, \tau_1) \to (L_2, \tau_2)$ between **tml**s is said to be continuous if $b \in \tau_2$ implies $\hat{f}(b) \in \tau_1$.

Definition 1.5. [13] An element a of a complete lattice L is called coprime, if $a \leq b \vee c$ implies $a \leq b$ or $a \leq c$ for every $b, c \in L$; and it is called completely coprime if $a \leq \bigvee S$ and $S \subseteq L$ implies $a \leq s$ for some $s \in S$.

The set of all nonzero coprime elements and nonzero completely coprime elements of L is denoted by $M(L)$ and $\overline{M}(L)$, respectively. Nonzero coprime elements are also called molecules. If L is a molecular lattice, then L is $\vee$-generated by the set $M(L)$, i.e., every element of L is a join of some elements of $M(L)$.

In the following, we recall the definition of an extra order introduced by Li [6]. Extra orders are useful tools to construct molecular lattices and function spaces in topological molecular lattices.

Definition 1.6. [6] Let P be a poset and $\prec$ be a binary relation on P.

a) $\prec$ is called an extra order, if it satisfies the following conditions for $x, y, u, v \in P$:

(i) $x \prec y \Rightarrow x \leq y$,

(ii) $u \leq x \prec y \leq v \Rightarrow u \prec v$.

b) $\prec$ satisfies the interpolation property (short by INT), if $x \prec y$ implies that there exists $z \in P$ such that $x \prec z \prec y$.

Remark 1.7. [6] If $\prec$ is an extra order on a poset P, then there exists a largest extra order $\prec^*$ over P contained in $\prec$ satisfying (INT).

Definition 1.8. [6] Let $\prec$ be an extra order satisfying (INT) on a poset P. A subset I in P is called a lower-Dedekind $\prec$-cut, if it satisfies the following conditions:

(i) I is a lower set, that is $\downarrow I = I$, where $\downarrow I = \{x \in P \mid \exists y \in I, x \leq_P y\}$.
(ii) If $x \in I$, then there exists $y \in I$ such that $x \prec y$.

The set of all lower-Dedekind $\prec$-cuts in P ordered by subset inclusion is denoted by $Low_{\prec}(P)$. The following important result is a construction of molecular lattices using extra order.

Theorem 1.9. *[6] If $\prec$ is an extra order over P satisfying (INT), then $Low_{\prec}(P)$ is a molecular lattice.*

Remark 1.10. [4,13] For a complete lattice L, an extra order $\lhd$ is defined by $a \lhd b$ if for every subset $S \subseteq L$, $b \leq \bigvee S$ implies $a \leq s$ for some $s \in S$. If L is a molecular lattice, then $\lhd$ satisfies the condition (INT). Also, a complete lattice L is a molecular lattice if and only if $b = \bigvee \beta_L(b)$ for every $b \in L$, where $\beta_L(b) := \{a \in L \mid a \lhd b\}$ and is called a minimal family of b. It is clear that $\beta_L(b)$ is a lower set with respect to $\leq_L$. If L is a molecular lattice, then $Low_{\lhd^*}(L) = Low_{\lhd}(L) \cong L$. Also, we have $\beta_L(\bigvee_{i\in I} a_i) = \bigcup_{i\in I} \beta_L(a_i)$.

Notice that $\bigvee \emptyset = 0$, so $0 \not\lhd 0$ and hence $\beta_L(0) = \emptyset$.

Example 1.11. *Consider the lattice $L = \{0, a, b, c, 1\}$, where $a < c$, $b < c$ and suppose a and b are incomparable. Then L is a molecular lattice, and $\beta_L(0) = \emptyset$, $\beta_L(a) = \{0, a\}$, $\beta_L(b) = \{0, b\}$, $\beta_L(c) = \{0, a, b\}$ and $\beta_L(1) = L$. By Remark 1.10, it follows that $Low_{\lhd^*}(L) = Low_{\lhd}(L) = \{\emptyset, \{0, a\}, \{0, b\}, \{0, a, b\}, L\} \cong L$.*

Lemma 1.12. *[4] Let $f : L_1 \to L_2$ be a mapping between molecular lattices. Then f is an* **ml***-map if and only if it preserves arbitrary joins and $\lhd$-relation, i.e., $x \lhd y$ implies $f(x) \lhd f(y)$.*

2. Equalizers and Monomorphisms

The category of all molecular lattices with **ml**-maps between them is denoted by **MOL**, and the category of all topological molecular lattices with continuous **ml**-maps between them is denoted by **TML**. It is well known that these categories are both complete and cocomplete, and some properties of them were introduced by many authors [3–5, 8, 11–15]. In the following, readers are suggested to refer to [1] for some categorical notions.

In this section, we first give an example which shows that the structure of equalizers in **TML** introduced by Zhao [15] need not be true, in general. We introduce the structure of equalizers in **TML** and show that equalizers are continuous embedding **ml**-maps. Thus every regular monomorphism is injective, but we show that monomorphisms need not be injective, in general.

Lemma 2.1. *[2] Let $f : L_1 \to L_2$ be an* **ml***-map. Then the following statements hold.*

(i) *$f \circ \hat{f} \leq id$, $\hat{f} \circ f \geq id$, $f \circ \hat{f} \circ f = f$ and $\hat{f} \circ f \circ \hat{f} = \hat{f}$, where id denote the identity map.*
(ii) *$\hat{f}$ is unique, i.e., if $g \circ f \geq id$ and $f \circ g \leq id$, then $g = \hat{f}$.*
(iii) *f is injective if and only if $\hat{f} \circ f = id$ if and only if $\hat{f}$ is surjective.*
(iv) *f is surjective if and only if $f \circ \hat{f} = id$ if and only if $\hat{f}$ is injective.*

Lemma 2.2. *For an* **ml**-*map f, we have $\hat{f}(0) = 0$, $\hat{f}(1) = 1$, and $f(a) = 0$ if and only if $a = 0$.*

Proof. Since f preserves arbitrary joins and $\hat{f}$ preserves both arbitrary joins and arbitrary meets, we have $\hat{f}(0) = \hat{f}(\bigvee \phi) = \bigvee \hat{f}(\phi) = \bigvee \phi = 0$; $\hat{f}(1) = \hat{f}(\bigwedge \phi) = \bigwedge \hat{f}(\phi) = \bigwedge \phi = 1$ and similarly, $f(0) = 0$. Now, suppose $f(a) = 0$, then $a \leq \hat{f}(f(a)) = \hat{f}(0) = 0$ and hence $a = 0$. □

In [15], the equalizer of morphisms $(L_1, \tau_1) \overset{f}{\underset{g}{\rightrightarrows}} (L_2, \tau_2)$ in **TML** introduced by $((Low_{\lhd^*}(E_{fg}), \delta), e)$, where $E_{fg} := \{x \in L_1 \mid f(x) = g(x)\}$, $e : Low_{\lhd^*}(E_{fg}) \to L_1$ is defined by $e(I) = \bigvee I$ and δ is the smallest cotopology on $Low_{\lhd^*}(E_{fg})$ such that e is continuous. The following example shows that this structure need not be true in **TML**, in general.

Example 2.3. *Let $L_1 = L_2 = \{0, a, b, 1\}$, where a and b are incomparable and the parallel morphisms $L_1 \overset{f}{\underset{g}{\rightrightarrows}} L_2$ defined by $f = id$, and $g(0) = 0$, $g(a) = b$, $g(b) = a$, $g(1) = 1$. If $\tau_1 = \tau_2 = \{0, 1\}$, then by Lemma 1.12, f and g are* **TML**-*morphisms. On the other hand, $E_{fg} = \{0, 1\}$ which is a molecular lattice and hence $Low_{\lhd^*}(E_{fg}) \cong E_{fg}$, $\delta = \{0, 1\}$ and $e : E_{fg} \to L_1$ is the inclusion map. We have $1 \lhd 1$ in E_{fg} but $1 = e(1) \not\lhd e(1) = 1$ in L_1 because $1 = a \vee b$ and $1 \nleq a$, $1 \nleq b$. By Lemma 1.12, e is not an* **ml**-*map. Thus e is not an equalizer of f and g in* **TML**.

In the following, we give the structure of equalizers in **MOL** and **TML**.

Lemma 2.4. *Let $e : E \to L_1$ and $L_1 \overset{f}{\underset{g}{\rightrightarrows}} L_2$ be* **ml**-*maps such that $f \circ e = g \circ e$. Then:*

(i) *$e(E)$ is a complete join-sublattice of L_1, i.e., $e(E) \subseteq L_1$ and it is closed under arbitrary joins. Hence $e(E)$ is a complete lattice.*
(ii) *$M(e(E)) = M(L_1) \cap e(E)$.*
(iii) *$\beta_{e(E)}(x) = \beta_{L_1}(x) \cap e(E)$.*
(iv) *$e(E) \subseteq \{x \in L_1 \mid \bigvee(\beta_{L_1}(x) \cap E_{fg}) = x\}$.*

Proof.

(i) Let $x_i \in e(E)$ for $i \in I$. Then $x_i = e(a_i)$ for some $a_i \in E$, so $\bigvee_{i \in I} x_i = \bigvee_{i \in I} e(a_i) = e(\bigvee_{i \in I} a_i) \in e(E)$. Hence $e(E)$ is a complete join-sublattice of L_1.

(ii) Let $m \in M(e(E))$ and $m \leq b \vee c$ in L_1. Then for some $a \in E$, $m = e(a) \leq b \vee c$ and so $a \leq \hat{e}(b) \vee \hat{e}(c)$. Hence $m = e(a) \leq e(\hat{e}(b)) \vee e(\hat{e}(c))$ and consequently $m \leq e(\hat{e}(b)) \leq b$ or $m \leq e(\hat{e}(c)) \leq c$, this implies that $m \in M(L_1) \cap e(E)$. Conversely, let $m \in M(L_1) \cap e(E)$ and $m \leq b \vee c$ in $e(E)$. Then $m \leq b \vee c$ in L_1 and hence $m \leq b$ or $m \leq c$, this implies that $m \in M(e(E))$.

(iii) Let $y \in \beta_{e(E)}(x)$. Then $y \lhd x$ in $e(E)$. If $x \leq \bigvee_{i \in I} x_i$ in L_1, then for some $a \in E$, $x = e(a) \leq \bigvee_{i \in I} x_i$. Hence $a \leq \bigvee_{i \in I} \hat{e}(x_i)$ and so $x = e(a) \leq \bigvee_{i \in I} e(\hat{e}(x_i))$. By assumption $y \leq e(\hat{e}(x_{i_0})) \leq x_{i_0}$ for some $i_0 \in I$, this implies that $y \in \beta_{L_1}(x) \cap e(E)$. Conversely, since $e(E)$ is a complete join-sublattice of L_1, the result holds.

(iv) For any $x \in E$, we have $e(x) = \bigvee e(\beta_E(x)) \leq \bigvee \beta_{L_1}(e(x)) \cap e(E) \leq \bigvee \beta_{L_1}(e(x)) \cap E_{fg} \leq \bigvee \beta_{L_1}(e(x)) = e(x)$. Thus $e(x) = \bigvee \beta_{L_1}(e(x)) \cap E_{fg}$ and hence $e(x) \in \{x \in L_1 \mid \bigvee(\beta_{L_1}(x) \cap E_{fg}) = x\}$.

□

Theorem 2.5. *The equalizer of* $L_1 \overset{f}{\underset{g}{\rightrightarrows}} L_2$ *in* **MOL** *is a pair* (E, e)*, where* $E := \{x \in L_1 \mid \bigvee(\beta_{L_1}(x) \cap E_{fg}) = x\}$ *and* e *is the inclusion map.*

Proof. Let $\{x_i \mid i \in I\} \subseteq E$. Then we have
$\bigvee(\beta_{L_1}(\bigvee_{i \in I} x_i) \cap E_{fg}) = \bigvee[(\bigcup_{i \in I} \beta_{L_1}(x_i) \cap E_{fg}] = \bigvee \bigcup_{i \in I}(\beta_{L_1}(x_i) \cap E_{fg}) = \bigvee_{i \in I} \bigvee(\beta_{L_1}(x_i) \cap E_{fg}) = \bigvee_{i \in I} x_i$.

Thus $\bigvee_{i \in I} x_i \in E$ and hence E is a complete lattice. Since $\beta_{L_1}(x) \cap E_{fg}$ is a minimal family of x in E, by Remark 1.10, E is molecular. If $x \lhd y$ in E, then $y = \bigvee(\beta_{L_1}(y) \cap E_{fg})$. So there is $z \in \beta_{L_1}(y)$ such that $x \leq z$ and $z \lhd y$ in L_1, which shows that e is an **ml**-map. Let $h : N \to L_1$ be an **ml**-map such that $f \circ h = g \circ h$. By Lemma 2.4 for $x \in N$, we have $h(x) = \bigvee \beta_{L_1}(h(x)) \cap E_{fg}$ and hence $h(x) \in E$. Now, we define $r : N \to E$ by $r(x) = h(x)$. Then r is an **ml**-map such that $e \circ r = h$. Also, it is easy to check that r is unique. □

Theorem 2.6. $e : (L_3, \tau_3) \to (L_1, \tau_1)$ *is an equalizer of* $(L_1, \tau_1) \overset{f}{\underset{g}{\rightrightarrows}} (L_2, \tau_2)$ *in* **TML** *if and only if* e *is an equalizer in* **MOL** *and* $\tau_3 = \{\hat{e}(a) \mid a \in \tau_1\}$.

Proof. It is clear that e is continuous. Let $h : (N, \tau) \to (L_1, \tau_1)$ be a continuous **ml**-map such that $f \circ h = g \circ h$. By Theorem 2.5, there exists a unique **ml**-map r such that $e \circ r = h$. If $x \in \tau_3$, then $x = \hat{e}(a)$ for some $a \in \tau_1$, and hence $\hat{r}(x) = \hat{r}(\hat{e}(a)) = \widehat{e \circ r}(a) = \hat{h}(a) \in \tau$. Thus r is continuous. □

Remark 2.7. [5] The category **TOP** of topological spaces is a reflective and coreflective full subcategory of **TML** via the embedding power functor $\mathcal{P}$: **TOP** $\to$ **TML** defined by $\mathcal{P}(X, \tau) = (\mathcal{P}(X), \tau^c)$, where $\mathcal{P}(X)$ denotes the power set of X, $\tau^c = \{A^c \mid A \in \tau\}$ and A^c denotes the complement set of A in X.

Example 2.8. *Let* $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ *be two arbitrary continuous functions. Then the equalizer of* $\mathcal{P}(X) \underset{\mathcal{P}(g)}{\overset{\mathcal{P}(f)}{\rightrightarrows}} \mathcal{P}(Y)$ *in* **TML** *is the pair* (E, e)*, where* E *is the complete join sublattice generated by the set* $\{\{x\} \mid x \in X, f(x) = g(x)\}$ *and* $e : E \to \mathcal{P}(X)$ *is the inclusion map. Thus* $E = \mathcal{P}(E_{fg})$*, where* $E_{fg} := \{x \in X \mid f(x) = g(x)\}$ *is the equalizer of* f *and* g *in the category* **TOP**. *This of course amounts to the familiar fact that the reflector* $\mathcal{P}$ *preserves limits.*

By Theorems 2.6, we have every regular monomorphism in **TML** is an embedding map, but the following example shows that monomorphisms need not be injective, in general.

Example 2.9. *A* **TML***-monomorphism* $f : L_1 \to L_2$ *need not be an injective map. For instance, consider* $L_1 = \{0, a, b, 1\}$*, where* a *and* b *are incomparable and* $L_2 = \{0, a, 1\}$ *with* $0 < a < 1$*. If* $\tau_1 = \tau_2 = \{0, 1\}$*, then the mapping* $f : L_1 \to L_2$ *defined by:* $f(0) = 0$*,* $f(a) = a$*,* $f(1) = f(b) = 1$ *is a* **TML***-morphism, but it is not injective. Now, we show that* f *is monomorphism. Let* $L_3 \underset{s}{\overset{r}{\rightrightarrows}} L_1$ *such that* $r \neq s$*. Then there exists* $m \in M(L_3)$ *such that* $r(m) \neq s(m)$*. Since* r *and* s *preserve coprimes, it follows that* $r(m)$ *and* $s(m) \in M(L_1) = \{a, b\}$*. Without loss of generality, suppose* $r(m) = a$ *and* $s(m) = b$*, then* $f(r(m)) = a \neq f(s(m)) = 1$*, this implies that* $f \circ r \neq f \circ s$*. Also, it is easy to check that the restriction map* $f\mid_{\overline{M}(L_1)}: \overline{M}(L_1) \to \overline{M}(L_2)$ *is injective.*

Theorem 2.10. *If* $f : L_1 \to L_2$ *is a* **TML***-monomorphism, then the restriction mapping* $f\mid_{\overline{M}(L_1)}: \overline{M}(L_1) \to \overline{M}(L_2)$ *is injective.*

Proof. Let $x_1, x_2 \in \overline{M}(L_1)$ and $f(x_1) = f(x_2)$. Then we define $r, s : \{0, 1\} \to L_1$ as follows: $r(0) = s(0) = 0$, $r(1) = x_1$, $s(1) = x_2$. Since $x_1 \lhd x_1$ and $x_2 \lhd x_2$, by Lemma 1.12, it follows that r and s are continuous **ml**-maps with respect to the discrete cotopology on $\{0, 1\}$. Clearly, $f \circ r = f \circ s$ and so by hypothesis $r = s$. Thus $x_1 = r(1) = s(1) = x_2$. □

The converse of Theorem 2.10 need not be true, in general as shown below.

Example 2.11. *Consider the continuous* **ml***-map* $f : [0, 1] \to \{0, 1\}$ *defined by* $f(0) = 0$ *and* $f(x) = 1$ *for each* $x \in (0, 1]$*, with respect to the discrete cotopologies. Since* $\overline{M}([0, 1]) = \emptyset$*, it follows that* $f\mid_{\overline{M}([0,1])}$ *is injective, but* f *is not monomorphism. Because, consider* $r, s : [0, 1] \to [0, 1]$ *given by* $r(x) = x$ *and* $s(x) = \frac{1}{2}x$*. Then* r, s *are continuous* **ml***-maps and* $f \circ r = f \circ s$ *but* $r \neq s$*.*

Let L be a complete lattice. Then $\overline{M}(L)$ is a join generating base for L if and only if it is isomorphic to a complete ring of sets [7]. Clearly, $\overline{M}(L) \subseteq M(L)$. If $\overline{M}(L) = M(L)$, then L is a complete ring of sets. For example, if $L = \mathcal{P}(X)$ or L is a finite molecular lattice, then $\overline{M}(L) = M(L)$ and so L is a complete ring of sets. We denote by **FTML** and **CRSET** the full subcategories of **TML** of all finite molecular lattices and of all complete rings of sets, respectively.

Theorem 2.12. *An* **ml**-*map* $f : L_1 \to L_2$ *is a monomorphism in* **FTML** *and* **CRSET** *if and only if the restriction mapping* $f\mid_{\overline{M}(L_1)}: \overline{M}(L_1) \to \overline{M}(L_2)$ *is injective.*

Proof. Let $f\mid_{\overline{M}(L_1)}$ be an injective map, $r, s : L_3 \to L_1$ be **ml**-maps such that $f \circ r = f \circ s$; and $m \in \overline{M}(L_3)$. Then $f(r(m)) = f(s(m))$, this implies that $r = s$. Conversely, by Theorem 2.10, the result follows. □

3. Coequalizers and Epimorphisms

In this section, we introduce the structure of coequalizers in **TML** and show that epimorphisms are precisely the morphisms with surjective underlying functions.

Theorem 3.1. *An* **ml**-*map* $f : L_1 \to L_2$ *is a* **TML**-*epimorphism if and only if it is a surjective map.*

Proof. Let f be a surjective **ml**-map. Then we have $f \circ \hat{f} = id$ and hence f is an epimorphism. Conversely, let f be an epimorphism. By Lemma 2.1, it is enough to show that $\hat{f}$ is an injective map. Suppose that $\hat{f}(y_1) = \hat{f}(y_2)$. We consider two cases:
Case 1. Let $y_1 \neq 0$ and $y_2 \neq 0$. Now, let $L_3 = \{0, a, 1\}$ such that $0 < a < 1$ and $\tau_3 = \{0, 1\}$. We define the maps $r, s : L_2 \to L_3$ as follows:

$$r(y) = \begin{cases} 0, & \text{if } y = 0, \\ a, & \text{if } y \leq y_1,\ y \neq 0, \\ 1, & o.w. \end{cases}$$

$$s(y) = \begin{cases} 0, & \text{if } y = 0, \\ a, & \text{if } y \leq y_2,\ y \neq 0. \\ 1, & o.w. \end{cases}$$

By Lemma 1.12, it is easy to check that r and s are continuous **ml**-maps. Then we have: $\hat{r}(0) = \hat{s}(0) = 0$, $\hat{s}(1) = \hat{r}(1) = 1$, $\hat{r}(a) = y_1$, $\hat{s}(a) = y_2$, and hence $\hat{f} \circ \hat{r} = \hat{f} \circ \hat{s}$. Thus $\widehat{r \circ f} = \widehat{s \circ f}$ and so by Lemma 2.1, $r \circ f = s \circ f$. By assumption, $r = s$, and hence $y_1 = \hat{r}(a) = \hat{s}(a) = y_2$.
Case 2. Let $y_1 = 0$. Then $\hat{f}(y_2) = \hat{f}(0) = 0$. Now, we show that $y_2 = 0$. Suppose that $y_2 \neq 0$. We define $r : L_2 \to L_3$ by $r(0) = 0$ and $r(y) = 1$ for each $0 \neq y \in L_2$. Let $s : L_2 \to L_3$ be the **ml**-map defined in case 1. Then we have $\hat{r}(0) = \hat{r}(a) = 0$ and $\hat{r}(1) = 1$, and hence $\widehat{r \circ f} = \widehat{s \circ f}$. This implies that $r \circ f = s \circ f$, and by assumption, $r = s$. Thus $1 = r(y_2) = s(y_2) = a$, which is a contradiction. □

Lemma 3.2. *[1] Let* $g : L_2 \to L_1$ *be a complete homomorphism between molecular lattices. Then* g *has a left adjoint* $f : L_1 \to L_2$ *and hence* f *is an* **ml**-*map.*

Let $(L_1,\tau)\overset{f}{\underset{g}{\rightrightarrows}}(L_2,\eta)$ be a pair of **TML**-morphisms. Then $\hat{f}$ and $\hat{g}$ are complete homomorphisms. Thus the set $Q_{fg} := \{a \in L_2 \mid \hat{f}(a) = \hat{g}(a)\}$ is a complete sublattice of L_2 and the inclusion map $e : Q_{fg} \to L_2$ is a complete homomorphism. By Lemma 3.2, e has a left adjoint denoted by $q : L_2 \to Q_{fg}$.

Theorem 3.3. *The coequalizer of* $(L_1,\tau)\overset{f}{\underset{g}{\rightrightarrows}}(L_2,\eta)$ *in* **TML** *is* (Q_{fg}, q, δ)*, where* $q : (L_2,\eta) \to (Q_{fg},\delta)$ *is the left adjoint of the inclusion map* $e : Q_{fg} \to L_2$ *and* $\delta = \{a \mid \hat{q}(a) \in \eta\}$.

Proof. Since $\hat{f} \circ e = \hat{g} \circ e$, it follows that $q \circ f = q \circ g$. Let $h : (L_2,\eta) \to (N,\tau')$ be a continuous **ml**-map such that $h \circ f = h \circ g$. Then $\hat{g} \circ \hat{h} = \hat{f} \circ \hat{h}$, so $\hat{h}(a) \in Q_{fg}$ for every $a \in N$. So the map $\alpha : N \to Q_{fg}$ defined by $\alpha(a) := \hat{h}(a)$ is a complete homomorphism and $e \circ \alpha = \hat{h}$. By Lemma 3.2, α has a left adjoint $r : Q_{fg} \to N$ such that $\hat{q} \circ \hat{r} = \hat{h}$ and consequently $r \circ q = h$. If $x \in \tau'$, then $\hat{q}(\hat{r}(x)) = \hat{h}(x) \in \eta$. Thus $\hat{r}(x) \in \delta$, which shows that r is continuous. It is easy to check that r is unique. □

4. Conclusion

It is well known that the category **TML** of topological molecular lattices with generalized order homomorphisms is both complete and cocomplete, and some categorical properties of it were introduced by many authors. In this paper, we have introduced the structures of equalizers, coequalizers, monomorphisms and epimorphisms in this category. We have proved that equalizers are continuous embedding generalized order homomorphisms, so every regular monomorphism is injective, but shown that the converse need not be true. Also, we have proved that epimorphisms are precisely the morphisms with surjective underlying functions.

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