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BLOW-UP AND GLOBAL EXISTENCE OF SOLUTIONS FOR HIGHER-ORDER KIRCHHOFF-TYPE EQUATIONS WITH VARIABLE EXPONENTS

F.M.D. Dohemeto ✉, G.A. Degla, and S.C. Dansou

Dedicated to sincere professor Mashaallah Mashinchi

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Abstract. This paper is concerned with the blow-up and global existence of solutions for Higher-Order Kirchhoff-Type Equations with variable exponents. Under suitable assumptions, we prove some finite time blow-up results for certain solutions with positive initial energy by using a concavity-type method. This work improves and generalizes several interesting recent blow-up results on wave equations in particular on Kichhoff-type equations. We also show the global existence of solutions under appropriate conditions.

Keywords: Blow-up, Global existence, Higher-Order Kirchhoff-Type Equation, Boundary value problem, Variable exponents, Positive initial energy.
2020 MSC: Primary 35B44, 35A01, 35L25.

1. Introduction

We consider initial boundary value problems for higher-order Kirchhoff-type equation with variable exponents as follows:

$$\tag{1} \begin{cases} u_{tt} + \mathcal{M}\left(\|D^m u\|_2^2\right)(-\Delta)^m u + \gamma |u_t|^{q(x)-2}u_t \\ \qquad\qquad = \beta |u|^{p(x)-2}u, \quad \text{in } \Omega \times (0,T), \\ u(0,x) = u_0(x), \quad x \in \Omega, \\ u_t(0,x) = u_1(x), \quad x \in \Omega, \\ \frac{\partial^i u}{\partial \nu^i}(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T), \ i = 0,1,...,m-1, \end{cases}$$

where Ω is a nonempty bounded domain in $\mathbb{R}^n (n \geq 1)$ with smooth boundary $\partial\Omega$, $\mathcal{M}(\tau) = a + b\tau^{\frac{\alpha}{2}-1}$ for $\tau \geq 0$, $a, \ b \geq 0$, $a + b > 0$, $\alpha \geq 2$, $m \geq 1$, and $\beta, \ \gamma > 0$. $\frac{\partial^i u}{\partial \nu^i}$ denotes the $i-$th order normal derivation of u and

$$(-\Delta)^m = (-1)^m D^{2m}, \text{ with } D^m = \overbrace{\nabla \cdot \nabla \ \cdots \ \nabla}^{m}.$$

✉ fortune.dohometo@imsp-uac.org, ORCID: 0000-0003-1439-8710
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The exponents $p(\cdot)$ and $q(\cdot)$ are given measurable functions on Ω satisfying the log-Hölder continuity condition:

$$|j(x) - j(y)| \leq -\frac{A}{\log|x-y|}, \text{ for a.e. } x,\ y \in \Omega, \quad \text{with } |x-y| < \delta, \tag{2}$$

for some $A > 0$ and for any $0 < \delta < 1$; and

$$\begin{cases} 2 \leq \min\{p^-,\ q^-\} \leq p(x), q(x) \leq \max\{p^+,\ q^+\} \leq \frac{2n}{n-2m}, & \text{if} \quad n > 2m \\ 2 \leq \min\{p^-,\ q^-\} \leq p(x), q(x) \leq \max\{p^+,\ q^+\} < +\infty, & \text{if} \quad n \leq 2m, \end{cases} \tag{3}$$

where

$$p^- := \operatorname*{ess\,inf}_{x\in\Omega} p(x), \qquad p^+ := \operatorname*{ess\,sup}_{x\in\Omega} p(x)$$

and

$$q^- := \operatorname*{ess\,inf}_{x\in\Omega} q(x), \qquad q^+ := \operatorname*{ess\,sup}_{x\in\Omega} q(x).$$

The problem (1) unifies several well known classical models with respect to specific conditions on the parameters. Indeed, when $b = 0$, (1) becomes a classical wave equation and when $b > 0$, (1) is often called a Kirchhoff-type wave equation introduced by Kirchhoff [12], in the case of constant exponents, in order to study the nonlinear vibrations of an elastic string; this model have regained interest nowadays. When p and q are constant, the existence and the blow-up of solutions of (1) have interested many mathematicians. Georgiev and Todorova [9] discuss the case where $\big(b = 0,\ m = 1\big)$ and prove the finite-time blow-up for the solutions with negative initial energy. Later, Messaoudi [17] studies the case $(b = 0,\ m = 2)$, and proves that the solution to (1) is global if $q \geq p$ and it blows up in finite time if $q < p$ and the initial energy is negative. In addition, Chen [7] considers the same problem as Messaoudi and establishes the blow-up result for some solutions with positive initial energy. Ono [23] considering the problem

$$\begin{cases} u_{tt} - \left(a + b\,\|\nabla u\|_2^{2\alpha}\right)\Delta u + |u_t|^q u_t = |u|^p u, & \text{in } \Omega \times (0,\ T) \\ u(0,x) = u_0(x), & x \in \Omega, \\ u_t(0,x) = u_1(x), & x \in \Omega, \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,\ T), \end{cases} \tag{4}$$

with $a \geq 0,\ b \geq 0,\ a + b > 0,\ \alpha \geq 1$, shows that the solution blows up in finite time if the initial energy is negative and $p > \max\{q,\ 2\alpha\}$ with $\left(p < \frac{2}{n-4} \text{ if } n \geq 5,\ \ p > 0 \text{ if } n \leq 4\right)$. Wu [30] establishes the same blow-up result for the solutions with positive initial energy of the following general Kirchhoff-type equations

$$u_{tt} - \mathcal{M}(\|\nabla u(t)\|_2^2)\Delta u + |u_t|^{q-2}u_t = |u|^{p-2}u, \tag{5}$$

where $\mathcal{M}$ is a non-negative locally Lipschitz function. Li [14] studies the higher order Kirchhoff-type equation

$$u_{tt} + \left(\int_\Omega |D^m u|^2 \, dx\right)^\alpha (-\Delta)^m u + |u_t|^{q-2} u_t = |u|^{p-2} u, \tag{6}$$

with the initial boundary conditions defined in (1). He has proved that the global solution exists if $p \leq q$; however if $p > \max\{q, 2\alpha + 2\}$ the solution with negative initial energy blows up in finite time. Later, Messaoudi [19] improves Li's results by modifying the proof and shows the same result when the initial energy is positive. Several results concerning blow-up and global existence have been established when p and q are constant; see in this regard [5, 6, 8, 10, 11, 16, 29, 31, 32] and the references therein.

Recently, much attention has been paid to the study of hyperbolic, parabolic and elliptic nonlinear models with variable exponents of nonlinearity. In fact, these equations model some physical phenomena such as electro-rheological fluid flows or fluids whose viscosity depends on temperature, filtration processes in a porous medium and image processing (see [3, 4, 27]). However, concerning hyperbolic problems with variable exponent nonlinearities, few works have been done. An important references in this regard is established by Antontsev [2] for the problem

$$u_{tt} - \operatorname{div}\left(a(x,t)|\nabla u|^{r(x,t)-2}\nabla u\right) - b\Delta u_t = \beta(x,t)|u|^{p(x,t)-2}u, \tag{7}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$, with Dirichlet boundary conditions, where $b > 0$ is a constant and $a,\ \beta,\ p,\ r$ are given functions. Under suitable conditions on $a, \beta,\ p$ and r, he proves the finite-time blow-up of some solutions with negative initial energy. In [28], Sun et al. consider the following equation

$$u_{tt} - \operatorname{div}(a(x,t)\nabla u) + \alpha(x,t)|u_t|^{q(x,t)-1}u_t = \beta(x,t)|u|^{p(x,t)-1}u, \tag{8}$$

in a bounded domain, and established a finite-time blow-up result for solutions with positive initial energy. Messaoudi et al. [20] studied

$$u_{tt} - \Delta u + a|u_t|^{q(x)-2} = b|u|^{p(x)-2}u, \tag{9}$$

where $a,\ b$ are positive constants. They prove the existence and uniqueness of a weak solution using the Faedo-Galerkin method under appropriate assumptions on the variable exponents m and p. They also prove the finite-time blow-up for solutions with negative initial energy and give a numerical example in two dimensions to illustrate the blow-up result. Recently, Pişkin in [25] studied

$$u_{tt} - \mathcal{M}(\|\nabla u(t)\|_2^2)\Delta u + |u_t|^{q(x)-2}u_t = |u|^{p(x)-2}u, \tag{10}$$

and established the finite-time blow-up of solutions by using modified energy functional method. Very recently, Alkhalifa et al. [1] consider

$$u_{tt} - \mathcal{M}(\mathcal{N}u)\,\Delta_{r(x)}u + \gamma(x,t)|u_t|^{q(x)-2}u_t = \beta(x,t)|u|^{p(x)-2}u, \tag{11}$$

where

$$\mathcal{N}u = \int_\Omega \frac{|\nabla u|^{r(x)}}{r(x)} dx, \tag{12}$$

and prove under suitable conditions a finite-time blow-up result of certain solutions with positive initial energy.

Our aim in this work is to prove some general blow-up results including the work [8, 14, 15, 20, 23, 25], for some solutions with positive initial energy, using an adaptation of the concavity method. In fact, instead of requiring that the functions Q and P defined in (22) and (56) be concave as in [15], [21] and [1], we consider only the fact that they decrease from a certain value. We also show a global existence result. This paper consists of two sections in addition to the introduction. In Section 2, we recall some preliminaries and then in Section 3 we present the main results. The first establishes sufficient conditions for the Blow-up of solutions with positive initial energy, the second is a consequence of the first, the third is an extension of the first, and finally the last gives sufficient conditions for the global existence of solutions.

2. Preliminaries

In this section, we present some preliminary facts about Lebesgue and Sobolev spaces with variable-exponents (see Lars et al. [13]). Let Ω be a domain of $\mathbb{R}^n$ with $n \geq 1$ and $p : \Omega \to [1, +\infty]$ be a measurable function. The Lebesgue space $L^{p(\cdot)}(\Omega)$ with a variable exponent $p(\cdot)$ is defined by

$$L^{p(\cdot)}(\Omega) = \left\{ v : \Omega \to \mathbb{R}, \text{ measurable in } \Omega \text{ and } \varrho_{p(\cdot)}(\lambda v) < +\infty, \text{ for } \ \text{ some } \ \lambda > 0 \right\},$$

where

$$\varrho_{p(\cdot)}(v) = \int_\Omega |v(x)|^{p(x)}\, dx.$$

Equipped with the following Luxemburg-type norm

$$\|v\|_{p(\cdot)} := \inf \left\{ \lambda > 0 \ : \ \int_\Omega \left| \frac{v(x)}{\lambda} \right|^{p(x)} dx \ \leq \ 1 \right\}.$$

$L^{p(\cdot)}(\Omega)$ is a Banach space. $C_0^\infty(\Omega)$ is dense in $L^{p(\cdot)}(\Omega)$ and $L^{p(\cdot)}(\Omega)$ is separable, if p is bounded and is reflexive if

$$1 < p^- \leq p(x) \leq p^+ < +\infty. \tag{13}$$

Let $m \in \mathbb{N}$ and p be a measurable function on Ω. We define the space $W^{m,p(\cdot)}(\Omega)$

$$W^{m,p(\cdot)}(\Omega) = \left\{ v \in L^{p(\cdot)}(\Omega) \ : \ \partial_\alpha v \in L^{p(\cdot)}(\Omega), \quad \forall \ |\alpha| \leq m \right\}$$

and we define a semimodular on $W^{m,p(\cdot)}(\Omega)$ by

$$\varrho_{W^{m,p(\cdot)}}(v) = \sum_{0 \leq |\alpha| \leq m} \varrho_{p(\cdot)}(\partial_\alpha v).$$

This induces a norm given by

$$\|v\|_{W^{m,p(\cdot)}} := \inf\left\{\lambda > 0 \ : \ \varrho_{W^{m,p(\cdot)}}\left(\frac{v}{\lambda}\right) \leq 1\right\} := \sum_{0\leq|\alpha|\leq m} \|\partial_\alpha v\|_{p(\cdot)} .$$

The space $W^{m,p(\cdot)}(\Omega)$ endowed with $\|\cdot\|_{W^{m,p(\cdot)}}$ is a Banach space, which is separable if p is bounded, and reflexive if p satisfies (13). We define the space

$$W_0^{m,p(\cdot)}(\Omega) = \overline{\left\{v \in W^{m,p(\cdot)}(\Omega) \ : \ v = v_{\chi K} \ \text{ for a compact } K \subset \Omega\right\}}$$

and

$$H_0^{m,p(\cdot)}(\Omega) = \overline{C_0^\infty(\Omega)}^{W^{m,p(\cdot)}(\Omega)}.$$

We have $H_0^{m,p(\cdot)}(\Omega) \subset W_0^{m,p(\cdot)}(\Omega)$ and if p is log-Hölder continuous on Ω, then

$$W_0^{m,p(\cdot)}(\Omega) = H_0^{m,p(\cdot)}(\Omega).$$

The space $W_0^{m,p(\cdot)}(\Omega)$ is a Banach space, which is measurable if p is bounded, and reflexive if p satisfies (13).

Lemma 2.1. *(Lars et al. [13]). Let p be a measurable function on Ω and $v \in L^{p(\cdot)}(\Omega)$. Then*

$$\|v\|_{p(\cdot)} \leq 1 \qquad \text{if and only if} \qquad \varrho_{p(\cdot)}(v) \leq 1.$$

Lemma 2.2. *(Lars et al. [13]). If p is a measurable function on Ω satisfying (13), then*

$$\min\left\{\|v\|_{p(\cdot)}^{p^-}, \ \|v\|_{p(\cdot)}^{p^+}\right\} \leq \varrho_{p(\cdot)}(v) \leq \max\left\{\|v\|_{p(\cdot)}^{p^-}, \ \|v\|_{p(\cdot)}^{p^+}\right\},$$

for any $v \in L^{p(\cdot)}(\Omega)$.

Lemma 2.3. *([18]) If p is a bounded measurable function on Ω, then*

$$\int_\Omega |v(x)|^{p(x)} dx \leq \|v\|_{p^-}^{p^-} + \|v\|_{p^+}^{p^+}, \qquad \forall \ v \in L^{p(\cdot)}(\Omega).$$

Lemma 2.4. *(Young's inequality [22]). Let p, q and s be measurable functions defined on Ω such that*

$$\frac{1}{s(y)} = \frac{1}{p(y)} + \frac{1}{q(y)}, \qquad \text{for a.e. } \ y \in \Omega.$$

Then for all $a, \ b \geq 0$,

$$\frac{(ab)^{s(\cdot)}}{s(\cdot)} \leq \frac{(a)^{p(\cdot)}}{p(\cdot)} + \frac{(b)^{q(\cdot)}}{q(\cdot)}.$$

By taking $s = 1$ and $1 < p, \ q < +\infty$, then we have for any $\varepsilon > 0$,

$$ab \leq \varepsilon a^p + C_\varepsilon b^q, \qquad \forall \ a, \ b \geq 0,$$

where $C_\varepsilon = \frac{1}{q(\varepsilon p)^{\frac{q}{p}}}$. *For* $p = q = 2$, *we have*

$$ab \leq \varepsilon a^2 + \frac{b^2}{4\varepsilon}.$$

Lemma 2.5. *(Hölder's Inequality [13]). Let* p, q *and* s *be measurable functions defined on* Ω *such that*

$$\frac{1}{s(y)} = \frac{1}{p(y)} + \frac{1}{q(y)},$$

for almost every $y \in \Omega$. *Then*

$$\begin{array}{rcl} \|fg\|_{s(\cdot)} & \leq & 2\,\|f\|_{p(\cdot)}\,\|g\|_{q(\cdot)}\,, \\ \varrho_{s(\cdot)}(fg) & \leq & \varrho_{p(\cdot)}(f) \;+\; \varrho_{q(\cdot)}(g), \end{array}$$

for all $f \in L^{p(\cdot)}(\Omega)$ *and* $g \in L^{q(\cdot)}(\Omega)$. *By taking* $p = q = 2$, *we have the Cauchy-Schwarz inequality.*

Lemma 2.6. *([8,31]) Let* Ω *be a bounded domain of* $\mathbb{R}^n$ *with a smooth boundary and assume that* $p(\cdot)$ *satisfies (3). Then* $H_0^m(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ *and for all* $v \in H_0^m(\Omega)$

$$\|v\|_{p(\cdot)} \;\leq\; B_1\,\|D^m v\|_2\,, \tag{14}$$

where B_1 *is the positive optimal constant of the Sobolev embedding. This implies that the space* $H_0^m(\Omega)$ *has an equivalent norm given by*

$$\|u\|_{H_0^m(\Omega)} = \|D^m u\|_2\,.$$

Theorem 2.7. *(Local existence theorem). Let* $u_0 \in H_0^m(\Omega) \cap H^{2m}(\Omega)$, $u_1 \in H_0^m(\Omega)$ *and* $a > 0$ *or* $u_0 \neq 0$. *Assume that* $p(\cdot)$ *and* $q(\cdot)$ *satisfy (3) and that*

$$p^+ \;\leq\; \frac{2(n-m)}{n-2m},\ \text{if}\ n \;>\; 2m\ \text{and}\ p^+ \;<\; +\infty,\ \text{if}\ n \;\leq\; 2m. \tag{15}$$

Then the problem (1) has a unique local solution satisfying

$$\left\{\begin{array}{lcl} u & \in & C\left([0,T];\ H_0^m(\Omega)\right), \\ u_t & \in & C\big([0,T];\ L^2(\Omega)\big) \cap L^{q(x)}\big(\Omega \times (0,\ T)\big). \end{array}\right. \tag{16}$$

The proof of this theorem can be established using the Galerkin method as in the work of Ono [23] and Messaoudi [20] (see also [8,16,24,31]).

3. Main results

3.1. **First blow-up result.** In this section, we establish a blow-up result for some solutions with positive initial energy. We set for $a \geq 0,\ b \geq 0,\ a+b > 0$ and $v \in C\left([0,T];\ H_0^m(\Omega)\right)$

$$N_{a,b}v(t) \;=\; \Big[a\,\|D^m v(t)\|_2^2 + b\,\|D^m v(t)\|_2^{\alpha}\Big]^{\frac{1}{k}}, \qquad t \in [0,T], \tag{17}$$

where $k = \alpha$ if $b > 0$ and $k = 2$ if $b = 0$. In addition, assume that $p(\cdot)$ satisfies (3), and so we have

$$\|v\|_{p(\cdot)} \leq B_1 \|D^m v\|_2 \leq B N_{a,b} v(t), \tag{18}$$

where $B = \frac{B_1}{b}$ if $b > 0$ and $B = \frac{B_1}{a}$ if $b = 0$. We define the energy of the solution of problem (1) by

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{a}{2}\|D^m u\|_2^2 + \frac{b}{\alpha}\|D^m u\|_2^\alpha - \int_\Omega \frac{\beta}{p(x)} |u|^{p(x)}\, dx \tag{19}$$

and we set

$$H(t) := E_1 - E(t), \tag{20}$$

where

$$\begin{aligned} E_1 &= Q(\lambda_1) \\ &= \begin{cases} \frac{\lambda_1^k}{k} - \frac{\beta}{p^-} & \text{if} \quad \widetilde{y_1} < B^{-1} < y_1 \\ \left(\frac{1}{k} - \frac{1}{p^-}\right)\lambda_1^k & \text{if} \quad y_1 \leq B^{-1}, \ \ \widetilde{y_1} < B^{-1} \\ \left(\frac{1}{k} - \frac{1}{p^+}\right)\lambda_1^k & \text{if} \quad \widetilde{y_1} \geq B^{-1}, \ \ y_1 \neq B^{-1} \end{cases}, \end{aligned} \tag{21}$$

$$Q(y) = \frac{1}{k} y^k - \frac{\beta}{p^-} \max\left\{B^{p^-} y^{p^-}, B^{p^+} y^{p^+}\right\} \tag{22}$$

and

$$\lambda_1 = \begin{cases} B^{-1} & \text{if} \quad \widetilde{y_1} < B^{-1} < y_1 \\ y_1 & \text{if} \quad y_1 \leq B^{-1}, \ \ \widetilde{y_1} < B^{-1} \\ \widetilde{y_1} & \text{if} \quad \widetilde{y_1} \geq B^{-1}, \ \ y_1 \neq B^{-1} \end{cases} \tag{23}$$

with

$$y_1 = \left(\frac{B^{-p^-}}{\beta}\right)^{\frac{1}{p^- - k}}, \qquad \widetilde{y_1} = \left(\frac{p^- B^{-p^+}}{p^+ \beta}\right)^{\frac{1}{p^+ - k}}. \tag{24}$$

We also set for $\frac{k}{p^-} < \mu < 1$

$$\lambda_\mu = \begin{cases} \lambda_\mu^{(0)} = \lambda_1 \left[\dfrac{\frac{p^-}{k} - \beta B^k}{\frac{p^-}{k} - \frac{1}{\mu}}\right]^{\frac{1}{k}} & \text{if} \quad \widetilde{y_1} < B^{-1} < y_1 \\ \lambda_\mu^{(1)} = \lambda_1 \left[\dfrac{\frac{p^-}{k} - 1}{\frac{p^-}{k} - \frac{1}{\mu}}\right]^{\frac{1}{k}} & \text{if} \quad y_1 \leq B^{-1}, \quad \widetilde{y_1} < B^{-1} \\ \lambda_\mu^{(2)} = \lambda_1 \left[\dfrac{\frac{p^-}{k} - \frac{p^-}{p^+}}{\frac{p^-}{k} - \frac{1}{\mu}}\right]^{\frac{1}{k}} & \text{if} \quad \widetilde{y_1} \geq B^{-1}, \quad y_1 \neq B^{-1} \end{cases} \tag{25}$$

and

$$\overline{\lambda}_1 = \lim_{\mu \to 1^-} \lambda_\mu. \tag{26}$$

Remark 3.1. We notice that $\lambda_\mu > \lambda_1$ for all $\mu \in \left(\frac{k}{p^-},\ 1\right)$ and $\lim\limits_{\mu \to 1^-} \lambda_\mu^{(1)} = \lambda_1$.

To prove the first main result, we establish the following lemmas inspired by the work of [20].

Lemma 3.2. *Let u be a solution to the problem* (1)*. Then E is a non-increasing function.*

Proof. We have

$$E'(t) = -\gamma \int_\Omega |u_t|^{q(x)} dx \leq 0. \tag{27}$$

for almost every t in $[0, T)$. Since $E(t)$ is absolutely continuous (see [9]), we have E is decreasing. □

Lemma 3.3. *Assume the conditions of Theorem 2.7 hold. Let u be a solution of* (1) *with initial data satisfying*

$$E(0) < E_1 \quad \text{and} \quad N_{a,b} u_0 > \lambda_1. \tag{28}$$

Then there exists a constant $\lambda_2 > \lambda_1 > 0$ such that

$$N_{a,b} u(t) \geq \lambda_2 > \lambda_1, \quad \forall\, t \geq 0. \tag{29}$$

Proof. From (17) and (19) we have

$$E(t) \geq \frac{1}{k}\left[a \|D^m u\|_2^2 + b \|D^m u\|_2^\alpha\right] - \frac{\beta}{p^-} \varrho_{p(\cdot)}(u),$$

and since

$$\begin{aligned} \varrho_{p(\cdot)}(u) &\leq \max\left\{\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\right\} \\ &\leq \max\left\{B^{p^-} N_{a,b}^{p^-} u\ ,\ B^{p^+} N_{a,b}^{p^+} u\right\}, \end{aligned}$$

we have

$$E(t) \geq \frac{1}{k} N_{a,b}^k u - \frac{\beta}{p^-} \max\left\{B^{p^-} N_{a,b}^{p^-} u\ ,\ B^{p^+} N_{a,b}^{p^+} u\right\} = Q\left(N_{a,b} u\right),$$

where

$$\begin{aligned} Q(y) &= \frac{1}{k} y^k - \frac{\beta}{p^-} \max\left\{B^{p^-} y^{p^-}, B^{p^+} y^{p^+}\right\} \\ &= \begin{cases} \frac{1}{k} y^k - \frac{\beta B^{p^-}}{p^-} y^{p^-} & \text{if} \quad 0 \leq y \leq B^{-1} \\ \frac{1}{k} y^k - \frac{\beta B^{p^+}}{p^-} y^{p^+} & \text{if} \quad y \geq B^{-1} \end{cases} \end{aligned}$$

and by derivation of Q on $[0, B^{-1}[$ and $]B^{-1},\ +\infty[$, we obtain

$$Q'(y) \;=\; \begin{cases} y^{k-1} - \beta B^{p^-} y^{p^- -1} & if \quad 0 \le y < B^{-1} \\ \\ y^{k-1} - \frac{p^+ \beta B^{p^+}}{p^-} y^{p^+ -1} & if \quad y > B^{-1} \end{cases}. \tag{30}$$

- Q is continuous on $\mathbb{R}_+$ and $Q(y) \to -\infty$ as $y \to +\infty$ and so Q has a maximal value.
- Q increases in $[0,\ \lambda_1]$ if $\lambda_1 \in \{B^{-1},\ y_1\}$ and decreases in $[\lambda_1,\ +\infty)$ for all value of λ_1.
- In particular, for $\lambda_1 = \widetilde{y_1}$, Q has at most three variations on $[0, \lambda_1]$ depending on the position of y_1. We have the following two cases:
  (a) If $y_1 > B^{-1}$, then Q is increasing on $[0,\ \lambda_1]$.
  (b) If $y_1 < B^{-1}$, then Q is increasing on $[0,\ y_1]$ and $[B^{-1},\ \lambda_1]$ and decreasing on $[y_1,\ B^{-1}]$.

From the above it follows that E_1 is a local maximum of Q reached in λ_1 and Q is decreasing on $[\lambda_1,\ +\infty)$. Furthermore $E(0) < E_1$, then there exists $\lambda_2 > \lambda_1$ such that $Q(\lambda_2) = E(0)$ and

$$Q\left(N_{a,b} u_0\right) \;\le\; E(0) \;=\; Q(\lambda_1).$$

Therefore $N_{a,b} u_0 \ \ge\ \lambda_2 \ >\ \lambda_1$. To establish $N_{a,b} u(t) \ge \lambda_2,\ \forall\ t \ge 0$, we suppose by contradiction that there exists $t_1 > 0$, such that $N_{a,b} u(t_1) < \lambda_2$. Since $N_{a,b} u(\cdot)$ is continuous, we can choose t_1 such that $\lambda_1 < N_{a,b} u(t_1) < \lambda_2$. It follows that

$$E(t_1) \;\ge\; Q(N_{a,b} u(t_1)) \;>\; Q(\lambda_2) \;=\; E(0).$$

This is a contradiction since from Lemma 3.2, E is decreasing. Hence (29) is established. □

Lemma 3.4. *([21]) Let the assumptions of Lemma 3.3 hold. Then the solution of (1) satisfies for some $c_0 > 0$,*

$$\varrho_{p(\cdot)}(u) \ge\ c_0 \, \|u\|_{p^-}^{p^-}. \tag{31}$$

Proof.

$$\varrho_{p(\cdot)}(u) \;=\; \int_\Omega |u|^{p(x)} dx \;=\; \int_{\Omega+} |u|^{p(x)} dx \;+\; \int_{\Omega-} |u|^{p(x)} dx,$$

where

$$\Omega_+ = \{x \in \Omega \ :\ |u(x,t)| \ge 1\} \qquad \text{and} \qquad \Omega_- = \{x \in \Omega \ :\ |u(x,t)| < 1\}.$$

Then

$$\varrho_{p(\cdot)}(u) \;\ge\; \int_{\Omega_+} |u|^{p^-} dx \;+\; \int_{\Omega_-} |u|^{p^+} dx \;\ge\; \int_{\Omega+} |u|^{p^-} dx \;+\; c_1 \left(\int_{\Omega-} |u|^{p^-} \right)^{\frac{p^+}{p^-}} dx.$$

It follows that

$$c_2(\varrho_{p(\cdot)}(u))^{\frac{p^-}{p^+}} \geq \int_{\Omega_-} |u|^{p^-} dx \quad \text{and} \quad \varrho_{p(\cdot)}(u) \geq \int_{\Omega_+} |u|^{p^-} dx,$$

and hence

$$c_2(\varrho_{p(\cdot)}(u))^{\frac{p^-}{p^+}} + \varrho_{p(\cdot)}(u) \geq \|u\|_{p^-}^{p^-}.$$

Since E is decreasing, H creasing and

$$0 < H(0) \leq H(t).$$

In addition

$$\begin{aligned} H(t) &\leq \int_\Omega \frac{\beta}{p(x)} |u|^{p(x)} dx + E_1 - \frac{a}{2} \|D^m u\|_2^2 - \frac{b}{\alpha} \|D^m u\|_2^\alpha \\ &\leq \int_\Omega \frac{\beta}{p(x)} |u|^{p(x)} dx + E_1 - \frac{1}{k} N_{a,b}^k u \\ &\leq \frac{\beta}{p^-} \int_\Omega |u|^{p(x)} dx + E_1 - \frac{1}{k} \lambda_1^k. \end{aligned}$$

From (21) we have in all case $E_1 - \frac{1}{k}\lambda_1^k < 0$, and so

$$0 < H(0) \leq H(t) \leq \frac{\beta}{p^-} \varrho_{p(\cdot)}(u), \tag{32}$$

then from (32) we get

$$\varrho_{p(\cdot)}(u) \left[1 + c_2 \left(\frac{p^-}{\beta} H(0) \right)^{\frac{p^-}{p^+} - 1} \right] \geq \|u\|_{p^-}^{p^-}.$$

Thus, (4.7) follows. □

Lemma 3.5. *([21]) Let the assumptions of Lemma 3.4 hold. Then the solution u of (1) satisfies*

$$\int_\Omega |u|^{q(x)} dx \leq C \left(\left(\varrho_{p(\cdot)}(u)\right)^{\frac{q^-}{p^-}} + \left(\varrho_{p(\cdot)}(u)\right)^{\frac{q^+}{p^-}} \right). \tag{33}$$

Proof.

$$\begin{aligned} \int_\Omega |u|^{q(x)} dx &\leq \int_{\Omega_-} |u|^{q^-} dx + \int_{\Omega_+} |u|^{q^+} dx \\ &\leq C \left[\left(\int_{\Omega_-} |u|^{p^-} dx \right)^{\frac{q^-}{p^-}} + \left(\int_{\Omega_+} |u|^{p^-} dx \right)^{\frac{q^+}{p^-}} \right] \\ &\leq C \left(\|u\|_{p^-}^{q^-} + \|u\|_{p^-}^{q^+} \right) \\ &\leq C \left((\varrho_{p(\cdot)}(u))^{\frac{q^-}{p^-}} + (\varrho_{p(\cdot)}(u))^{\frac{q^+}{p^-}} \right), \end{aligned}$$

by Lemma 3.4. $\square$

Lemma 3.6. *Let the conditions of Lemma 3.3 be fulfilled. We set*

$$\lambda_{sup} = \inf_{t\geq 0} N_{a,b}u(t). \tag{34}$$

If $\lambda_{sup} > \overline{\lambda}_1$, *then there exists* $\delta_1 \in \left(0,\ 1 - \frac{k}{p^-}\right)$ *such that for all* $\mu \in (1-\delta_1,\ 1)$,

$$\begin{cases} \lambda_{sup} > \lambda_\mu > \lambda_1 \\ \left(\frac{\mu p^-}{k} - 1\right) \lambda_\mu^k \frac{N_{a,b}^k u(t)}{\lambda_{sup}^k} \ - \ \mu p^- E_1 \geq 0,\ \forall\ t \geq 0. \end{cases} \tag{35}$$

Proof. According to Lemma 3.3,

$$\lambda_1 \ < \ \lambda_{sup} \ = \ \inf_{t\geq 0} N_{a,b}u(t) \ \leq \ N_{a,b}u(0) \ = \ N_{a,b}u_0 \ < \ +\infty, \tag{36}$$

and so $\lambda_1 < \lambda_{sup} < +\infty$. Now we are going to prove (35). To this end, let us set $\epsilon = \lambda_{sup} - \overline{\lambda}_1$. There exists $\delta_1 \in \left(0,\ 1 - \frac{k}{p^-}\right)$ such that:

$$\begin{aligned} 1 - \delta_1 < \mu < 1 \quad &\Longrightarrow \quad \left|\lambda_\mu - \bar{\lambda}_1\right| < \epsilon \\ &\Longrightarrow \quad \lambda_1 < \lambda_\mu < \lambda_{sup}, \end{aligned}$$

since $\lambda_1 < \lambda_\mu,\ \forall\ \mu \in (1-\delta_1,\ 1)$. In addition from (21)-(25) we have

- If $E_1 = \frac{\lambda_1^k}{k} - \frac{\beta}{p^-}$, then for all $t \geq 0$ we have

$$\begin{aligned} \left(\frac{\mu p^-}{k} - 1\right) \lambda_\mu^k \frac{N_{a,b}^k u(t)}{\lambda_{sup}^k} \ - \ \mu p^- E_1 \ &\geq \ \mu \left(\frac{p^-}{k} - \frac{1}{\mu}\right) \lambda_\mu^k \ - \ \mu p^- E_1 \\ &\geq \ \mu \left(\frac{p^-}{k} - \beta \lambda_1^{-k}\right) \lambda_1^k - \mu p^- \left(\frac{\lambda_1^k}{k} - \frac{\beta}{p^-}\right) = 0. \end{aligned}$$

- If $E_1 = \left(\frac{1}{k} - \frac{1}{p^-}\right) \lambda_1^k$, then for all $t \geq 0$ we have

$$\begin{aligned} \left(\frac{\mu p^-}{k} - 1\right) \lambda_\mu^k \frac{N_{a,b}^k u(t)}{\lambda_{sup}^k} \ - \ \mu p^- E_1 \ &\geq \ \mu \left(\frac{p^-}{k} - \frac{1}{\mu}\right) \lambda_\mu^k \ - \ \mu p^- E_1, \\ &\geq \ \mu \left(\frac{p^-}{k} - 1\right) \lambda_1^k \ - \ \mu p^- \left(\frac{1}{k} - \frac{1}{p^-}\right) \lambda_1^k = 0. \end{aligned}$$

- If $E_1 = \left(\frac{1}{k} - \frac{1}{p^+}\right) \lambda_1^k$, then for all $t \geq 0$ we have

$$\begin{aligned} \left(\frac{\mu p^-}{k} - 1\right) \lambda_\mu^k \frac{N_{a,b}^k u(t)}{\lambda_{sup}^k} \ - \ \mu p^- E_1 \ &\geq \ \mu \left(\frac{p^-}{k} - \frac{1}{\mu}\right) \lambda_\mu^k \ - \ \mu p^- E_1, \\ &\geq \ \mu \left(\frac{p^-}{k} - \frac{p^-}{p^+}\right) \lambda_1^k \ - \ \mu p^- \left(\frac{1}{k} - \frac{1}{p^+}\right) \lambda_1^k = 0. \end{aligned}$$

Hence, (35) is established. $\square$

Theorem 3.7. *(Blow-up). Let the conditions of Theorem 2.7 be fulfilled. Assume that*

$$2 \leq \max\left\{k,\ q^+\right\} < p^- \leq p(x) \leq p^+ \begin{cases} \leq \frac{2(n-m)}{n-2m}, & \text{if} \quad n > 2m \\ < +\infty, & \text{if} \quad n \leq 2m \end{cases}. \tag{37}$$

If

$$E(0) \;<\; E_1 \qquad \textit{and} \qquad \lambda_{sup} \;>\; \overline{\lambda}_1, \tag{38}$$

then the solution of problem (1) *belonging to the class* (16) *blows up in finite time.*

Proof. Let us set

$$L(t) := \; H^{1-\sigma}(t) \;+\; \varepsilon \int_\Omega u u_t dx \tag{39}$$

for ε small to be chosen later and

$$0 \;<\; \sigma \;\leq\; \min\left\{ \frac{p^- - 2}{2p^-} \;,\; \frac{p^- - q^+}{p^-(q^+ - 1)} \right\}. \tag{40}$$

We notice that L is a small perturbation of the energy. By taking the time derivation of (39) and using a variational formulation, we obtain that

$$\begin{aligned} L'(t) \;&=\; (1-\sigma)H'(t)H^{-\sigma}(t) \;-\; \varepsilon\gamma \int_\Omega |u_t|^{q(x)-2} u_t . u \, dx \;+ \\ &+ \varepsilon\beta \int_\Omega |u|^{p(x)} dx \;+\; \varepsilon \|u_t\|_2^2 \;-\; \varepsilon N^k_{a,b} u(t). \end{aligned} \tag{41}$$

We add and subtract $\varepsilon\mu p^- H(t)$, for $\frac{k}{p^-} < \mu < 1$, from the right side of (41), to obtain

$$\begin{aligned} L'(t) \;\geq\; & (1-\sigma)H'(t)H^{-\sigma}(t) \;+\; \varepsilon\mu p^- H(t) \;+\; \varepsilon\left(\frac{\mu p^-}{2} + 1\right)\|u_t\|_2^2 \;+ \\ & + \varepsilon\left(\frac{\mu p^-}{k} - 1\right) N^k_{a,b} u(t) \;+\; \varepsilon\beta \int_\Omega \left(1 - \frac{\mu p^-}{p(x)}\right) |u|^{p(x)} dx \;+ \\ & - \varepsilon\gamma \int_\Omega |u_t|^{q(x)-2} u_t . u \, dx \;-\; \varepsilon\mu p^- E_1. \end{aligned}$$

According to Lemma 3.6 and for $\mu \in (1-\delta_1,\; 1)$ with $\delta_1 \in \left(0, 1 - \frac{k}{p^-}\right)$ we have

$$\begin{aligned} \left(\frac{\mu p^-}{k} - 1\right) N^k_{a,b} u(t) - \mu p^- E_1 \;\;\geq\;& \left(\frac{\mu p^-}{k} - 1\right) \frac{\lambda^k_{sup} - \lambda^k_\mu}{\lambda^k_{sup}} N^k_{a,b} u(t) \;+ \\ &+\; \left(\frac{\mu p^-}{k} - 1\right) \lambda^k_\mu \frac{N^k_{a,b} u(t)}{\lambda^k_{sup}} \;-\; \mu p^- E_1 \\ \geq\;& C_1 N^k_{a,b} u(t), \end{aligned}$$

where $C_1 = \left(\frac{\mu p^-}{k} - 1\right) \frac{\lambda^k_{sup} - \lambda^k_\mu}{\lambda^k_{sup}} > 0$. Therefore

$$\begin{aligned} L'(t) \;\geq\; & (1-\sigma)H'(t)H^{-\sigma}(t) \;+\; \varepsilon\mu p^- H(t) \;+\; \varepsilon\left(\frac{\mu p^-}{2} + 1\right)\|u_t\|_2^2 \;+ \\ & + C_1 N^k_{a,b} u \;+\; \varepsilon\beta(1-\mu)\varrho_{p(\cdot)}(u) \;-\; \varepsilon\gamma \int_\Omega |u_t|^{q(x)-1}|u| \, dx, \end{aligned}$$

and so

$$\begin{aligned} L'(t) \;\geq\; & (1-\sigma)H'(t)H^{-\sigma}(t) \;-\; \varepsilon\gamma \int_\Omega |u_t|^{q(x)-1}|u|\,dx \;+ \\ & + \varepsilon\eta\Big[H(t) \;+\; \|u_t\|_2^2 \;+\; N_{a,b}^k u \;+\; \varrho_{p(\cdot)}(u)\Big], \end{aligned}$$

where

$$\eta \;=\; \min\left[\mu p^- \;,\; C_1 \;,\; \frac{\mu p^-}{k}+1 \;,\; \beta(1-\mu)\right] > 0. \tag{42}$$

Now by using Young's inequality, we get

$$\begin{aligned} \int_\Omega |u_t|^{q(x)-1}|u|dx \;\leq\; & \int_\Omega \frac{\delta^{q(x)}|u|^{q(x)}}{q(x)}\,dx \;+ \\ & + \int_\Omega \frac{q(x)-1}{q(x)}\delta^{-\frac{q(x)}{q(x)-1}}|u_t|^{q(x)}dx\,, \qquad \forall\, \delta>0 \\ \leq\; & \frac{1}{q^-}\int_\Omega \delta^{q(x)}|u|^{q(x)}dx \;+ \\ & + \frac{q^+-1}{q^+}\int_\Omega \delta^{-\frac{q(x)}{q(x)-1}}|u_t|^{q(x)}\,dx\,, \qquad \forall\, \delta>0. \end{aligned}$$

Taking in particular, $\delta^{-\frac{q(x)}{q(x)-1}} \;=\; \xi H^{-\sigma}(t)$, and for a large constant ξ to be specified later we obtain

$$\begin{aligned} \int_\Omega |u_t|^{q(x)-1}udx \;\leq\; & \frac{1}{q^-}\int_\Omega \xi^{1-q(x)}H^{\sigma(q(x)-1)}(t)|u|^{q(x)}dx \\ & + \frac{\xi(q^+-1)H^{-\sigma}(t)}{q^+}\int_\Omega |u_t|^{q(x)}dx, \end{aligned}$$

and since from (27), $H'(t) \;=\; \gamma\int_\Omega |u_t|^{q(x)}dt$, we have

$$\begin{aligned} \int_\Omega |u_t|^{q(x)-1}|u|dx \;\leq\; & \frac{1}{q^-}\int_\Omega \xi^{1-q(x)}H^{\sigma(q(x)-1)}(t)|u|^{q(x)}dx \;+ \\ & + \frac{\xi(q^+-1)}{\gamma q^+}H^{-\sigma}(t)H'(t). \end{aligned}$$

Therefore

$$\begin{aligned} L'(t) \;\geq\; & \left[(1-\sigma) \;-\; \frac{\xi(q^+-1)}{q^+}\right]H'(t)H^{-\sigma}(t) \;+ \\ & - \varepsilon\frac{\gamma\xi^{1-q^-}}{q^-}H^{\sigma(q^+-1)}(t)\int_\Omega |u|^{q(x)}dx + \\ & + \varepsilon\eta\Big[H(t) \;+\; \|u_t\|_2^2 \;+\; N_{a,b}^k u \;+\; \varrho_{p(\cdot)}(u)\Big]. \end{aligned} \tag{43}$$

From (32) and (33)

$$
\begin{aligned}
H^{\sigma(q^+-1)}(t)\int_\Omega |u|^{q(x)}dx \;&\leq\; C\left[\frac{\beta}{p^-}\varrho_{p(\cdot)}(u)\right]^{\sigma(q^+-1)} \times \\
&\quad\times\left[\left(\varrho_{p(\cdot)}(u)\right)^{\frac{q^-}{p^-}} \;+\; \left(\varrho_{p(\cdot)(u)}\right)^{\frac{q^+}{p^-}}\right] \\
&\leq\; C\left[\left(\varrho_{p(\cdot)}(u)\right)^{\frac{q^-}{p^-}+\sigma(q^+-1)} + \right. \\
&\quad\left. + \;\left(\varrho_{p(\cdot)}(u)\right)^{\frac{q^+}{p^-}+\sigma(q^+-1)}\right]
\end{aligned}
$$

and from (40)

$$
s_1 \;=\; \frac{q^-}{p^-} \;+\; \sigma(q^+-1) \;\leq\; 1 \tag{44}
$$

and

$$
s_2 \;=\; \frac{q^+}{p^-} \;+\; \sigma(q^+-1) \;\leq\; 1. \tag{45}
$$

Using the following algebraic inequality: for all $z \geq 0,\ 0 < \pi \;\leq\; 1,\ d \;>\; 0,$

$$
z^\pi \;\leq\; z+1 \;\leq\; \left(1+\frac{1}{d}\right)(z+d), \tag{46}
$$

and assuming that $z = \varrho_{p(\cdot)}(u)$ and $d = N^k_{a,b}u$ we obtain

$$
\begin{aligned}
H^{\sigma(q^+-1)}(t)\int_\Omega |u|^{q(x)}dx \;&\leq\; C\left(1 \;+\; \lambda^{-1}_{sup}\right)\left(\varrho_{p(\cdot)}(u) \;+\; N^k_{a,b}u\right) \\
&\leq\; C\big(\varrho_{p(\cdot)}(u) \;+\; N^k_{a,b}u\big).
\end{aligned} \tag{47}
$$

At this point, we choose ξ large enough so that

$$
\eta - \frac{\gamma\xi^{1-q^-}}{q^-}C \;>\; 0. \tag{48}
$$

Using Lemma 3.4 and (47), we obtain

$$
\begin{aligned}
L'(t) \;\geq\; &\left[(1-\sigma) \;-\; \varepsilon\frac{\xi(q^+-1)}{q^+}\right]H'(t)H^{-\sigma}(t) \;+ \\
&+\varepsilon\left(\eta - \frac{\gamma\xi^{1-q^-}}{q^-}C\right)\left[H(t) \;+\; \|u_t\|_2^2 \;+\; c_0\,\|u\|_{p^-}^{p^-} \;+\; N^k_{a,b}u\right].
\end{aligned}
$$

Once ξ is fixed, we choose ε small enough so that

$$
\begin{cases}
(1-\sigma) \;-\; \varepsilon\frac{\xi(q^+-1)}{q^+} \geq 0 \\
L(0) := H^{1-\sigma}(0) \;+\; \varepsilon\displaystyle\int_\Omega u_0u_1\,dx > 0.
\end{cases} \tag{49}
$$

Since $H'(t) \geq 0$, $H(t) > 0$,

$$L'(t) \quad \geq \quad \varepsilon C_0 \Big[H(t) \ + \ \|u_t\|_2^2 \ + \ \|u\|_{p^-}^{p^-} \ + \ N_{a,b}^k u \Big].$$

Next, using the algebraic inequality

$$(z \ + \ d)^i \ \leq \ 2^{i-1}(z^i \ + \ d^i), \quad z, \quad d \ \geq \ 0, \qquad i \ > \ 1,$$

and Hölder and Young's inequality we obtain successively

$$\begin{aligned}
L^{\frac{1}{1-\sigma}}(t) \quad &= \quad \left[H^{1-\sigma}(t) \ + \ \varepsilon \int_\Omega u u_t \, dx \right]^{\frac{1}{1-\sigma}} \\
&\leq \quad 2^{\frac{\sigma}{1-\sigma}} \left[H(t) \ + \ \varepsilon^{\frac{1}{1-\sigma}} \left| \int_\Omega u u_t \, dx \right|^{\frac{1}{1-\sigma}} \right] \\
&\leq \quad 2^{\frac{\sigma}{1-\sigma}} \left[H(t) \ + \ \varepsilon^{\frac{1}{1-\sigma}} \left[\mathrm{mes}(\Omega)\right]^{\frac{p^- - 2}{2p^-(1-\sigma)}} \|u\|_{p^-}^{\frac{1}{1-\sigma}} \ \|u_t\|_2^{\frac{1}{1-\sigma}} \right] \\
&\leq \quad C \Big[H(t) \ + \ \|u\|_{p^-}^{\frac{2}{1-2\sigma}} \ + \ \|u_t\|_2^2 \Big]. \qquad (50)
\end{aligned}$$

From (40), $0 \ < \ \frac{2}{p^-(1-2\sigma)} \ \leq 1$ and using again (46) we have

$$\|u\|_{p^-}^{\frac{2}{1-2\sigma}} \ = \ \|u\|_{p^-}^{p^-\left[\frac{2}{p^-(1-2\sigma)}\right]} \ \leq \ \left(1 \ + \ \lambda_{sup}^{-1}\right) \left(\|u\|_{p^-}^{p^-} \ + \ N_{a,b}^k u \right).$$

It follows that

$$L^{\frac{1}{1-\sigma}}(t) \quad \leq \quad C \Big[H(t) \ + \ \|u\|_{p^-}^{p^-} + N_{a,b}^k u \ + \ \|u_t\|_2^2 \Big]. \qquad (51)$$

Therefore

$$L'(t) \geq \Gamma L^{\frac{1}{1-\sigma}}(t), \qquad \Gamma > 0, \text{ for all } t \geq 0. \qquad (52)$$

By integrating (52)

$$L^{\sigma/(1-\sigma)}(t) \quad \geq \quad \frac{1}{L^{-\sigma/(1-\sigma)}(0) \ - \ \Gamma t \sigma/(1-\sigma)}. \qquad (53)$$

Then (53) shows that $L(t)$ blows up in finite time

$$T^* \quad \leq \quad \frac{(1-\sigma) L^{\sigma/(1-\sigma)}(0)}{\Gamma \sigma}.$$

□

Corollary 3.8. *Assume that the conditions of the theorem 2.7 be fulfilled and that (37) holds. Then any solution of problem* (1) *with initial data satisfying*

$$E(0) \ < \ Q\left(\overline{\lambda}_1\right) \qquad \text{and} \qquad N_{a,b} u_0 \ > \ \lambda_1, \qquad (54)$$

blows up in finite time.

Proof. From (25) $\overline{\lambda}_1 \geq \lambda_1$ and since Q decreases in $[\lambda_1, +\infty)$, we have $Q(\overline{\lambda}_1) \leq Q(\lambda_1) = E_1$. Now suppose by contradiction that there exists $t_2 \geq 0$ such that $N_{a,b}u(t_2) \leq \overline{\lambda}_1$. By the continuity of $N_{a,b}u(\cdot)$ we can choose $t_2 \geq 0$ such that $\lambda_1 \leq N_{a,b}u(t_2) \leq \overline{\lambda}_1$. Then we have

$$Q\left(N_{a,b}u(t_2)\right) \leq E(t_2) \leq E(0) < Q(\overline{\lambda}_1).$$

This is a contradiction. It follows that $N_{a,b}u(t) > \overline{\lambda}_1$ for all $t \geq 0$ and so $\lambda_{sup} > \overline{\lambda}_1$. This completes the proof according to Theorem 3.7. □

3.2. **Second blow-up result.** Now we establish the blow-up for solutions with the positive energy E_1^* following:

$$\begin{aligned} E_1^* &= P(\lambda_1^*) \\ &= \left(\frac{1}{k} - \frac{1}{p^-}\right)\lambda_1^{*k} + \frac{\beta B^{p^+}}{p^-}\left(\frac{p^+}{p^-} - 1\right)\lambda_1^{*p^+} \\ &= \left(\frac{1}{k} - \frac{1}{p^+}\right)\lambda_1^{*k} - \beta B^{p^-}\left(\frac{1}{p^-} - \frac{1}{p^+}\right)\lambda_1^{*p^-} \\ &= \frac{\beta}{p^-}\left[B^{p^-}\left(\frac{p^-}{k} - 1\right)\lambda_1^{*p^-} + B^{p^+}\left(\frac{p^+}{k} - 1\right)\lambda_1^{*p^+}\right], \end{aligned} \tag{55}$$

where

$$P(y) = \frac{1}{k}y^k - \frac{\beta}{p^-}\left[B^{p^-}y^{p^-} + B^{p^+}y^{p^+}\right], \quad y \in \mathbb{R}_+. \tag{56}$$

and λ_1^* is unique positive real number such that $P'(\lambda_1^*) = 0$. We set

$$\begin{aligned} \lambda_\mu^* &= \lambda_1^*\left[\frac{\frac{p^-}{k} - 1}{\frac{p^-}{k} - \frac{1}{\mu}} + \beta B^{p^+}\left(\frac{\frac{p^+}{p^-} - 1}{\frac{p^-}{k} - \frac{1}{\mu}}\right)\lambda_1^{*p^+ - k}\right]^{\frac{1}{k}} \\ &= \lambda_1^*\left[\frac{\frac{p^-}{k} - \frac{p^-}{p^+}}{\frac{p^-}{k} - \frac{1}{\mu}} - \beta B^{p^-}p^-\left(\frac{\frac{1}{p^-} - \frac{1}{p^+}}{\frac{p^-}{k} - \frac{1}{\mu}}\right)\lambda_1^{*p^- - k}\right]^{\frac{1}{k}} \end{aligned} \tag{57}$$

and

$$\overline{\lambda}_1^* = \lim_{\mu \to 1^-} \lambda_\mu^*. \tag{58}$$

Theorem 3.9. *(Blow-up). Let the conditions of Theorem 2.7 be fulfilled. Assume further (37) holds. If* $E(0) < E_1^*$ *and* $\lambda_{sup}^* > \overline{\lambda}_1^*$*, then the solution of problem* (1) *belonging to the class* (16) *blows up in finite time.*

The proof of this theorem is done in a similar way to that of Theorem 3.7 using the following lemmas.

Lemma 3.10. *Let u be a solution of* (1) *with initial data satisfying*

$$E(0) < E_1^* \quad \text{and} \quad N_{a,b}u_0 > \lambda_1^*. \tag{59}$$

Then there exists a constant $\lambda_2^* > \lambda_1^* > 0$ *such that:*

$$N_{a,b}u(t) \;\geq\; \lambda_2^* \;>\; \lambda_1^*, \quad \forall\, t \;\geq\; 0. \tag{60}$$

Proof. From (19) we have

$$E(t) \;\geq\; \frac{1}{k} N_{a,b}^k u \;-\; \frac{\beta}{p^-}\int_\Omega |u|^{p(x)}\,dx, \tag{61}$$

and since

$$\int_\Omega |u|^{p(x)}\,dx \quad\leq\quad B^{p^-} N_{a,b}^{p^-} u \;+ B^{p^+} N_{a,b}^{p^+} u, \tag{62}$$

we have

$$E(t) \;\geq\; \frac{1}{k} N_{a,b}^k u \;-\; \frac{\beta}{p^-}\left[B^{p^-} N_{a,b}^{p^-} u \;+\; B^{p^+} N_{a,b}^{p^+} u\right] \;=\; P\left(N_{a,b}u\right). \tag{63}$$

In addition, we have

$$\begin{aligned}
P'(y) \;&=\; y^{k-1} - \beta B^{p^-} y^{p^- -1} \;-\; \frac{p^+\beta B^{p^+}}{p^-} y^{p^+ -1}\;, \qquad \forall\, y \in \mathbb{R}_+ && (64)\\
&=\; y^{k-1}\left[1 - \beta B^{p^-} y^{p^- -k} \;-\; \frac{p^+\beta B^{p^+}}{p^-} y^{p^+ -k}\right] && (65)\\
&=\; yR(y),
\end{aligned}$$

where R is defined by

$$R(y) \;=\; 1 - \beta B^{p^-} y^{p^- -k} \;-\; \frac{p^+\beta B^{p^+}}{p^-} y^{p^+ -k}, \quad \forall\, y \in \mathbb{R}_+ \tag{66}$$

and its derivative R' on $\mathbb{R}_+^*$ is defined by

$$R'(y) \;=\; -\beta(p^- - k) B^{p^-} y^{p^- -k-1} - \;\frac{p^+\beta(p^+ - k) B^{p^+}}{p^-} y^{p^+ -k-1}. \tag{67}$$

We have the following properties:

- $R'(y) < 0$ for all $y \in \mathbb{R}_+^*$.
- R is continuous on $\mathbb{R}_+$, $R(y) \longrightarrow -\infty$ as $y \to +\infty$, and $R(0) = 1$, then there is a unique $\lambda_1^* > 0$ such that $R(\lambda_1^*) = 0$. Therefore

$$\begin{array}{lll}
P'(y) = 0 & \Longleftrightarrow & y \in \;\; \{0,\ \lambda_1^*\},\\
P'(y) > 0 & \Longleftrightarrow & y \in \;\; (0,\ \lambda_1^*),\\
P'(y) < 0 & \Longleftrightarrow & y \in \;\; (\lambda_1^*,\ +\infty).
\end{array} \tag{68}$$

- P is increasing on $[0,\ \lambda_1^*]$.
- P is decreasing on $[\lambda_1^*,\ +\infty)$.

Furthermore $E(0) \;<\; E_1^*$, then there exists $\lambda_2^* \;>\; \lambda_1^*$ such that $P(\lambda_2^*) \;=\; E(0)$ and

$$P\left(N_{a,b}u_0\right) \;\leq\; E(0) \;=\; P(\lambda_2^*).$$

Therefore $N_{a,b}u_0 \geq \lambda_2^* > \lambda_1^*$. Assume that there exists $t_2 > 0$ such that $\lambda_1^* < N_{a,b}u(t_2) < \lambda_2^*$. It follows that

$$E(t_2) \geq P\left(N_{a,b}u(t_2)\right) > P(\lambda_2^*) = E(0). \tag{69}$$

This is a contradiction, because E is decreasing. Therefore (60) is established.

□

Lemma 3.11. *Let the conditions of Lemma 3.10 be fulfilled. We set*

$$\lambda_{sup}^* = \inf_{t\geq 0} N_{a,b}u(t). \tag{70}$$

If $\lambda_{sup}^* > \overline{\lambda}_1^*$, *then there exists* $\delta_1^* \in \left(0,\ 1-\frac{k}{p^-}\right)$ *such that: for all* $\mu \in (1-\delta_1^*,\ 1)$,

$$\begin{cases} \lambda_{sup}^* > \lambda_\mu^* > \lambda_1^* \\ \left(\frac{\mu p^-}{k}-1\right)\lambda_\mu^{*k}\frac{N_{a,b}^k u(t)}{\lambda_{sup}^{*k}} - \mu p^- E_1^* \geq 0,\ \forall\ t \geq 0. \end{cases} \tag{71}$$

Proof. According to Lemma 3.10,

$$\lambda_1^* < \lambda_{sup}^* = \inf_{t\geq 0} N_{a,b}u(t) \leq N_{a,b}u_0 < +\infty, \tag{72}$$

and so $\lambda_1^* < \lambda_{sup}^* < +\infty$. Let $\epsilon = \lambda_{sup}^* - \overline{\lambda}_1^*$. There exists $\delta_1^* \in \left(0,\ 1-\frac{k}{p^-}\right)$ such that

$$\begin{aligned} 1-\delta_1^* < \mu < 1 \quad &\Longrightarrow \quad \left|\lambda_\mu^* - \bar{\lambda}_1^*\right| < \epsilon \\ &\Longrightarrow \quad \lambda_1^* < \lambda_\mu^* < \lambda_{sup}^*, \end{aligned}$$

and from (55) and (57), we have for all $t \geq 0$

$$\begin{aligned} \left(\frac{\mu p^-}{k}-1\right)\lambda_\mu^{*k}\frac{N_{a,b}^k u(t)}{\lambda_{sup}^{*k}} - \mu p^- E_1^* \quad &\geq \quad \mu\left(\frac{p^-}{k}-\frac{1}{\mu}\right)\lambda_\mu^{*k} - \mu p^- E_1^*, \\ &\geq \quad \mu\left(\frac{p^-}{k}-1\right)\lambda_1^{*k} \;+ \\ &\quad + \beta B^{p^+}\left(\frac{p^+}{p^-}-1\right)\lambda_1^{*p^+} - \mu p^- E_1^* = 0. \end{aligned}$$

□

3.3. **Global existence.** We consider the functionals:

$$I_\rho(u) = N_{a,b}^k u - \rho\int_\Omega \frac{\beta}{p(x)}|u|^{p(x)}\,dx, \quad \forall\ \rho > k \tag{73}$$

$$J(u) = \frac{1}{k}N_{a,b}^k u - \int_\Omega \frac{\beta}{p(x)}|u|^{p(x)}\,dx \tag{74}$$

for $u \in H_0^m(\Omega)$ and the stable set $\mathbb{H}$ defined by

$$\mathbb{H} = \left\{u \in H_0^m(\Omega) :\ \exists\rho > k,\ \ I_\rho(u) > 0\right\} \cup \{0\}. \tag{75}$$

We set:

$$\theta_{p,\rho} = \beta B^{p^-}\left[\frac{k\rho}{\rho-k}E(0)\right]^{\frac{p^--k}{k}} \times \tag{76}$$
$$\times \max\left\{1,\ \left[\frac{k\rho}{\rho-k}E(0)B^k\right]^{\frac{p^+-p^-}{k}}\right\}, \quad \forall\ \rho > k.$$

Lemma 3.12. *Let u be a solution to the problem (1). If there exists $\rho_0 > k$ such that*

$$I_{\rho_0}(u_0) > 0 \quad \text{and} \quad \theta_{p,\rho_0} < p^-\rho_0^{-1}. \tag{77}$$

Then $u(t) \in \mathbb{H}$ for each $t \in [0, T)$.

Proof. Since $I_{\rho_0}(u_0) > 0$ then there exists $t_m \leq T$ such that $I_{\rho_0}(u(t)) \geq 0$ for all $t \in [0, t_m)$. From (73) and (74)

$$J(u) = \frac{\rho_0-k}{k\rho_0}N^k_{a,b}u + \frac{1}{\rho_0}I_{\rho_0}(u) \geq \frac{\rho_0-k}{k\rho_0}N^k_{a,b}u, \quad \forall t \in [0,\ t_m). \tag{78}$$

Then

$$N^k_{a,b}u(t) \leq \frac{k\rho_0}{\rho_0-k}J(u) \leq \frac{k\rho_0}{\rho_0-k}E(0), \quad \forall t \in [0,\ t_m). \tag{79}$$

Therefore

$$\begin{aligned}
\int_\Omega \frac{\beta}{p(x)}|u|^{p(x)}dx &\leq \frac{\beta}{p^-}\max\left\{B^{p^-}N^{p^-}_{a,b}u\ ,\ B^{p^+}N^{p^+}_{a,b}u\right\} \\
&\leq \frac{\beta B^{p^-}}{p^-}\left[\frac{k\rho_0}{\rho_0-k}E(0)\right]^{\frac{p^--k}{k}} \times \\
&\quad \times \max\left\{1,\ \left[\frac{k\rho_0}{\rho_0-k}E(0)B^k\right]^{\frac{p^+-p^-}{k}}\right\}N^k_{a,b}u \\
&\leq \frac{\theta_{p,\rho_0}}{p^-}N^k_{a,b}u \qquad (80)\\
&< \rho_0^{-1}N^k_{a,b}u\ ,\ \forall t \in [0,\ t_m), \qquad (81)
\end{aligned}$$

which implies that $u(t) \in \mathbb{H}$ for all $t \in [0,\ t_m)$. And since E is decreasing, we have

$$\beta B^{p^-}\left[\frac{k\rho_0}{\rho_0-k}E(t_m)\right]^{\frac{p^--k}{k}} \times \tag{82}$$
$$\times \max\left\{1,\ \left[\frac{k\rho_0}{\rho_0-k}E(t_m)B^k\right]^{\frac{p^+-p^-}{k}}\right\} \leq \theta_{p,\rho_0} < p^-\rho_0^{-1}$$

and so $I_{\rho_0}(u(t_m)) > 0$. We repeat the steps (78)-(81) to extend t_m to $2t_m$. By continuing the procedure, the assertion of Lemma 3.12 is proved. $\square$

Theorem 3.13. *(Global existence theorem). Suppose that the conditions of Theorem 2.7 hold. If either $p^+ < q^-$ or (77) is satisfied, then the solution of problem (1) exists globally.*

Proof. First, assume that there exist $\rho_0 > k$ such that

$$I_{\rho_0}(u_0) \;>\; 0 \text{ and } \theta_{p,\rho_0} \;<\; p^-\rho_0^{-1}. \tag{83}$$

Then $I_{\rho_0}(u) > 0$ and so from (79)

$$0 \;\leq\; \frac{\rho_0 - k}{\rho_0 k} N^k_{a,b} u \;<\; \frac{1}{k} N^k_{a,b} u \;-\; \int_\Omega \frac{\beta}{p(x)}\,|u|^{p(x)}\,dx. \tag{84}$$

It follows that

$$\frac{1}{2}\,\|u_t\|_2^2 \;+\; \frac{\rho_0 - k}{\rho_0 k} N^k_{a,b} u \;\leq\; E(t) \leq E(0) \;<\; +\infty. \tag{85}$$

The above inequality allows us to conclude that the solution exists when $T \to +\infty$. Thus, the solution u of the problem (1) is global.

Assume now that $p^+ < q^-$. Let

$$K(t) \;=\; E(t) \;+\; \frac{2\beta}{p^+}\Big(\int_\Omega |u|^{p^-}\,dx \;+\; \int_\Omega |u|^{p^+}\,dx\Big). \tag{86}$$

We have

$$\begin{aligned} K'(t) \;&=\; -\,\gamma\int_\Omega |u_t|^{q(x)}dx \;+\; \frac{2\beta}{p^+}\Big(p^-\int_\Omega uu_t|u|^{p^- -2}dx \;+ \\ &\qquad + p^+\int_\Omega uu_t|u|^{p^+ -2}dx\Big) \\ &\leq\; -\,\gamma\int_\Omega |u_t|^{q(x)}dx \;+\; 2\beta\Big(\int_\Omega |u_t||u|^{p^- -1}dx \;+\; \int_\Omega |u_t||u|^{p^+ -1}dx\Big). \end{aligned}$$

The Hölder's inequality yields

$$\begin{aligned} \int_\Omega |u_t||u|^{p^- -1}dx \;+\; \int_\Omega |u_t||u|^{p^+ -1}dx \;&\leq\; \|u_t\|_{p^-}\,\|u\|_{p^-}^{p^- -1} \;+\; \|u_t\|_{p^+}\,\|u\|_{p^+}^{p^+ -1} \\ &\leq\; \delta_1\,\|u_t\|_{p^-}^{p^-} \;+\; C_{\delta_1}^{p^- -1}\,\|u\|_{p^-}^{p^-} \;+ \\ &\quad + \delta_2\,\|u_t\|_{p^+}^{p^+} \;+\; C_{\delta_2}^{p^+ -1}\,\|u\|_{p^+}^{p^+} \\ &\leq\; \delta\left(\|u_t\|_{q^-}^{p^-} \;+\; \|u_t\|_{q^-}^{p^+}\right) \;+ \\ &\quad + C_\delta\left(\|u\|_{p^-}^{p^-} \;+\; \|u\|_{p^+}^{p^+}\right), \end{aligned}$$

where

$$\delta \;=\; \max\left\{\delta_1\,[\mathrm{mes}(\Omega)]^{1-\frac{p^-}{q^-}}\;,\;\delta_2\,[\mathrm{mes}(\Omega)]^{1-\frac{p^+}{q^-}}\right\}$$

and

$$C_\delta \;=\; \max\left\{C_{\delta_1}^{p^- -1},\; C_{\delta_2}^{p^+ -1}\right\}.$$

It follows that

$$\begin{aligned} K'(t) \quad &\leq \quad -\gamma_1 \min \left\{ \|u_t\|_{q^-}^{q^-},\ \|u_t\|_{q^-}^{q^+} \right\} + 2\beta\delta \left(\|u_t\|_{q^-}^{p^-} + \|u_t\|_{q^-}^{p^+} \right) \ + \\ &\quad + 2\beta C_\delta \left(\|u\|_{p^-}^{p^-} + \|u\|_{p^+}^{p^+} \right). \end{aligned}$$

At this point we distinguish two cases: if $\|u_t\|_{q^-} > 1$, we choose δ small enough so that

$$-\gamma_1 \|u_t\|_{q^-}^{q^-} \ + \ 2\beta\delta \left(\|u_t\|_{q^-}^{p^-} + \|u_t\|_{q^-}^{p^+} \right) \ \leq \ 0 \tag{87}$$

hence, $K'(t) \ \leq \ 2\beta C_\delta \left(\|u\|_{p^-}^{p^-} \ + \ \|u\|_{p^+}^{p^+} \right)$. If $\|u_t\|_{q^-} \ \leq \ 1$, then $K'(t) \ = \ 4\beta\delta + \beta C_\delta \left(\|u\|_{p^-}^{p^-} + \|u\|_{p^+}^{p^+} \right)$. Therefore, in either case, we have

$$K'(t) \ \leq \ c_3 \ + \ c_4 K(t), \ c_3 \ \geq \ 0, \ c_4 \ > \ 0. \tag{88}$$

Then the Gronwall inequality yields

$$K(t) \ \leq \ \left(K(0) \ + \ \frac{c_3}{c_4} \right) e^{c_4 t}. \tag{89}$$

This last estimate and the principle of continuation [26], complete the proof. □

4. Conclusion

This work deals essentially with the phenomena of blow-up and global existence of solutions for Higher-Order Kirchhoff-Type problems with variable exponents. Using the concavity-type method, we establish under appropriate conditions, the finite-time blow-up results that improve and generalize several interesting well-known results, notably [2,6,8,20,21]. These results yield a partial classification of blow-up and global existence of solutions with respect to the type of energy, and would be undoubtedly useful for qualitative analysis of Partial Differential Equations. We have also provided a global existence result based on some suitable properties of variable Lebesgue spaces.

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F.M.D. Dohemeto
Orcid number: 0000-0003-1439-8710
Department of Mathematics
Institut de Mathématiques et de Sciences Physiques (IMSP), Université d'Abomey Calavi (UAC)
Bénin, Dangbo
Email address: fortune.dohometo@imsp-uac.org

G.A. Degla
Orcid number: 0000-0003-1162-6140
Department of Mathematics
Institut de Mathématiques et de Sciences Physiques (IMSP), Université d'Abomey Calavi (UAC)
Bénin, Dangbo
Email address: gdegla@imsp-uac.org

S.C. Dansou
Orcid number: 0000-0001-9428-4216
Department of Mathematics
Institut de Mathématiques et de Sciences Physiques (IMSP), Université d'Abomey Calavi (UAC)
Bénin, Dangbo
Email address: cyrille.dansou@imsp-uac.org