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CHAOS CONTROL AND HOPF BIFURCATION ANALYSIS OF A THREE-DIMENSIONAL CHAOTIC SYSTEM

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Abstract. In this paper, we study the effect of delayed feedback on the dynamics of a three-dimensional chaotic dynamical system and stabilize its chaotic behavior and control the respective unstable steady state. We derive an explicit formula in which a Hopf bifurcation occurs under some analytical conditions. Then the existence and stability of the Hopf bifurcation are analyzed by considering the time delay τ as a bifurcation parameter. Furthermore, by numerical calculation and appropriate ascertaining of both the feedback strength K and time delay τ, we find certain threshold values of time delay at which an unstable equilibrium of the considered system is successfully controlled. Finally, we use numerical simulations to examine the derived analytical results and reveal more dynamical behaviors of the system.

Keywords: Chaotic system, Chaos control, Time-delayed feedback, Stability, Hopf bifurcation.
2020 MSC: 34D05, 37G10, 37G15.

1. Introduction

Chaotic systems are nonlinear deterministic systems which can display complex and unpredictable behaviors. These systems are recognized by some special characteristics such as aperiodic solutions, positive Lyapunov exponents as well as high sensitivity to the variations of their initial conditions and system parameters [1, 3, 15, 19].

In recent years, the themes of analyzing chaotic systems and chaos control are growing with wide theoretical and practical applications in different scientific fields [10, 25]. For instance, some of their applications are found in secure communication, information processing, intelligent controls, power systems, liquid mixing, laser physics, nonlinear circuit, active wave propagation, biology, chemistry, mathematics, ecology and economy [3, 5, 15, 19].

In the discussion of chaos control, elimination of the chaotic behavior is its main target in order to stabilize the chaotic system towards either a periodic

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orbit or at one of the system's equilibrium points [9, 12, 23, 26]. There are several control techniques which have been developed, such as impulsive control, passive control, optimal control method, traditional linear and nonlinear control methods, fuzzy control methods and many others [9, 28]. These control methods are classified into the following two main categories.
The OGY (Ott–Grebogi–Yorke) method is the first category which is based on invariant manifold and it is a discrete technique. This method is introduced by Ott, Grebogi and Yorke [6]. The second one is the time-delayed feedback control scheme which was originally proposed by Pyragas [13] and this method is frequently used for the purpose of chaos control. The main purpose of the OGY control method is to make only small time dependent perturbations in the parameter of the chaotic systems in which the system attractors have embedded within an infinite number of unstable periodic orbits. While the idea and structure of Pyragas method is to inject an appropriate continuous controlling signal into the system which is proportional to the difference between the present state $Z(t)$ and the delayed state $Z(t-\tau)$ [9, 26, 28]. As it can be observed the Pyragas chaos control method is involved with time-delay which plays an important role in stabilization of unstable periodic orbit (UPO). Thus time-delay is unavoidable and can be existed everywhere. Because, it has an important influence in the system's dynamics and there are many utilization of one or several time-delays for different reasons in mathematical models particularly in biological and physical models [7, 18]. It can make complicated dynamics, such as the instability of an equilibrium point and fluctuation of the systems' solutions. Usually due to system process and information flow, the time-delay happens in a particular part of dynamical systems [4,7,18]. Hence to control chaos in a continuous nonlinear dynamical system, time-delayed feedback control is more convenient and it is a powerful tool to stabilize a system for which the time-delay is considered as a period of unstable periodic orbit (UPO) [8, 9, 14, 25]. Thus in the controlling process, the existing difference becomes zero when the system evolves close to the desired steady state or periodic orbit which means stabilization [8, 9, 14].

Compared to the other control methods, the main advantage of applying the Pyragas method is that it does not require the prior knowledge of the equations of the system and it can generate the control force from the information of the system itself [2, 9, 12]. Another advantage is that the time-delayed feedback control method has been successfully applied to various fields, such as biology, medicine, chemistry, engineering and physics. Particularly, it can be used to many practical chaotic systems including electronic oscillators, mechanical pendulums, lasers, gas discharge systems, high power ferromagnetic resonance, helicopter rotor blades, chemical systems and a cardiac system, see [9, 24] and the references therein.

Since the last decades, numerous results are dedicated in the context of chaos control and many scholars have been discussed the control of chaos problems in various fields of science and engineering. For example, G. M. Mahmoud et al. in [9], investigated the control of chaotic Burke-Shaw system using Pyragas method. In [12], J. H. Yang et al. discussed the effect of delayed feedback on a finance system. Their results show that, when the delay passes through a certain critical value, chaos vanishes, i.e., the chaotic oscillation is converted into a stable equilibrium or a periodic orbit. The dynamics of a three-dimensional Jerk chaotic system with only one stable equilibrium is studied in [28] by applying a delayed feedback control scheme. While in [11], H. Zhao et al. have been focused on control of Hopf bifurcation and chaos in a delayed Lotka-Volterra predator-prey system by means of time-delayed feedbacks control method.

Motivated by [10, 16, 27, 28] and following the idea of Pyragas, this work focuses on controlling of a three-dimensional chaotic system given by

$$\begin{aligned} \frac{dx(t)}{dt} &= \beta x(t) - y^2(t), \\ \frac{dy(t)}{dt} &= \mu\big(z(t) - y(t)\big), \\ \frac{dz(t)}{dt} &= x(t)y(t) + (\alpha - \mu)y(t) + \alpha z(t). \end{aligned} \tag{1}$$

System (1) is proposed by P. P. Singh et al. [17] based on Bhalekar and Gejji (BG) [20] chaotic system which for $\beta = -10, \mu = 55, \alpha = 37$, displays chaotic behavior.

Then for controlling the chaos, we add a time delayed feedback control $K(z(t) - z(t-\tau))$ to the third equation of (1) as follows:

$$\begin{aligned} \frac{dx(t)}{dt} &= \beta x(t) - y^2(t), \\ \frac{dy(t)}{dt} &= \mu\big(z(t) - y(t)\big), \\ \frac{dz(t)}{dt} &= x(t)y(t) + (\alpha - \mu)y(t) + \alpha z(t) + K\big(z(t) - z(t-\tau)\big), \end{aligned} \tag{2}$$

where $K \in \mathbb{R}$ is the feedback strength which represents the intensity of control per unit of time.

The organization of this paper is as follows. In Section 2, we first analyze the system's stability and determine the range of the control parameters τ and K for which one of the unstable equilibrium point is controlled to a stable state. Then, we analytically derive the conditions for the occurrence of a Hopf bifurcation. In Section 3, to illustrate the obtained analytical results, numerical simulations are performed for a set of parameters as given in [17]. The brief conclusions are finally given in Section 4.

2. **Stability and Hopf bifurcation analysis of system** (2)

In this section, we study the dynamical behaviors of system (2), when the delay τ is considered as a free parameter. We first determine stability of the system at the equilibrium $E^*(x^*, y^*, z^*)$ for which

$$x^* = -(2\alpha - \mu) = \mu - 2\alpha, \qquad y^* = \sqrt{\beta(\mu - 2\alpha)}, \quad \text{and} \quad z^* = \sqrt{\beta(\mu - 2\alpha)}.$$

This equilibrium point is feasible if the condition (H_1) holds.

(H_1) $\quad \beta(\mu - 2\alpha) > 0.$

Under the transformation $X = x(t) - x^*, Y = y(t) - y^*, Z = z(t) - z^*$ and hypothesis (H_1), we linearize the system as follows:

$$(3) \quad \begin{cases} \frac{dX(t)}{dt} = \beta X(t) - 2y^* Y(t), \\ \frac{dY(t)}{dt} = -\mu Y(t) + \mu Z(t), \\ \frac{dZ(t)}{dt} = y^* X(t) + \big(x^* + (\alpha - \mu)\big)Y(t) + (\alpha + K)Z(t) - KZ(t - \tau), \end{cases}$$

which can also be written as

$$(4) \qquad \begin{bmatrix} \dot{X}(t) \\ \dot{Y}(t) \\ \dot{Z}(t) \end{bmatrix} = A_1 \begin{bmatrix} X(t) \\ Y(t) \\ Z(t) \end{bmatrix} + A_2 \begin{bmatrix} X(t-\tau) \\ Y(t-\tau) \\ Z(t-\tau) \end{bmatrix},$$

where

$$A_1 = \begin{bmatrix} \beta & -2y^* & 0 \\ 0 & -\mu & \mu \\ y^* & x^* + \alpha - \mu & \alpha + K \end{bmatrix}, \qquad \text{and} \qquad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -K \end{bmatrix}.$$

Then the characteristic equation can be described by

$$\triangle(\lambda, \tau) = \det \begin{pmatrix} \lambda - \beta & 2y^* & 0 \\ 0 & \lambda + \mu & -\mu \\ -y^* & -x^* - \alpha + \mu & \lambda - \alpha - K + Ke^{-\lambda\tau} \end{pmatrix} = 0,$$

which implies that

$$(5) \qquad \triangle(\lambda, \tau) = \lambda^3 + L_2\lambda^2 + L_1\lambda + L_0 + (S_2\lambda^2 + S_1\lambda + S_0)e^{-\lambda\tau} = 0,$$

in which

$$L_2 = -K - \alpha + \mu - \beta, \quad L_1 = \mu^2 + (-K - x^* - 2\alpha - \beta)\mu + (\alpha + K)\beta,$$
$$L_0 = \big(-\beta\mu + (K + x^* + 2\alpha)\beta + 2y^*\big)\mu, \quad S_2 = K, \quad S_1 = -(\beta K - K\mu),$$
$$S_0 = -K\mu\beta.$$

When $\tau = 0$, Eq. (5) becomes

$$(6) \qquad \triangle(\lambda) = \lambda^3 + (L_2 + S_2)\lambda^2 + (L_1 + S_1)\lambda + L_0 + S_0 = 0.$$

Hence, E^* becomes asymptotically stable if the following conditions hold.

(H_2) $\quad L_2 + S_2 > 0, \qquad L_0 + S_0 > 0, \qquad (L_2 + S_2)(L_1 + S_1) > L_0 + S_0.$

For the Hopf bifurcation analysis, let $\lambda = \mathrm{i}\omega$ $(\omega > 0)$ be a root of Eq. (5), then we obtain

(7)
$$-\mathrm{i}\omega^3 - L_2\omega^2 + \mathrm{i}L_1\omega + L_0 + (-S_2\omega^2 + S_0 + \mathrm{i}S_1\omega)(\cos(\omega\tau) - \mathrm{i}\sin(\omega\tau)) = 0.$$

The corresponding real and imaginary parts can be acquired as

$$\begin{aligned} &(S_0 - S_2\omega^2)\cos(\omega\tau) + S_1\omega\sin(\omega\tau) = L_2\omega^2 - L_0,\\ &S_1\omega\cos(\omega\tau) - (S_0 - S_2\omega^2)\sin(\omega\tau) = \omega^3 - L_1\omega, \end{aligned} \tag{8}$$

which leads to

$$\omega^6 + (L_2^2 - S_2^2 - 2L_1)\omega^4 + (L_1^2 - S_1^2 - 2L_2L_0 + 2S_2S_0)\omega^2 + L_0^2 - S_0^2 = 0. \tag{9}$$

Let $\omega^2 = \eta$. Then from (9) we get

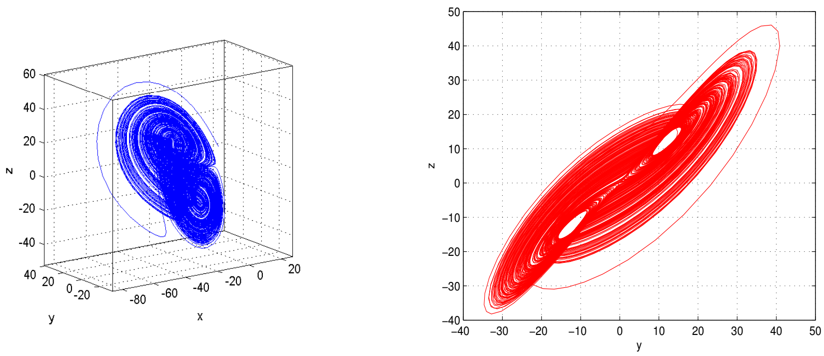

Figure 1. System (2) for $\tau = 0$ or $K = 0$ is chaotic. The initial value is $(0.1, 0.1, 10.38)$.

$$\eta^3 + p\eta^2 + q\eta + r = 0, \tag{10}$$

where

$$p = L_2^2 - S_2^2 - 2L_1, \qquad q = L_1^2 - S_1^2 - 2L_2L_0 + 2S_2S_0, \qquad r = L_0^2 - S_0^2.$$

Suppose $f(\eta) = \eta^3 + p\eta^2 + q\eta + r$ and $f'(\eta) = 3\eta^2 + 2p\eta + q$. If $\lim_{\eta\longrightarrow+\infty} f(\eta) = +\infty$ and $f(0) = r = L_0^2 - S_0^2 < 0$, then Eq. (10) has at least one positive real root, hence we can derive the following results based on [21, 22].

Lemma 2.1. *For the polynomial Eq.* (10), *the following statements hold.*

- **(i):** *If* $(\mathrm{H}_3):\ r > 0 \quad \& \quad \Delta = p^2 - 3q < 0$, *then Eq.* (10) *has no positive roots, i.e., if* $\Delta \geq 0$, *then this equation have positive real roots.*

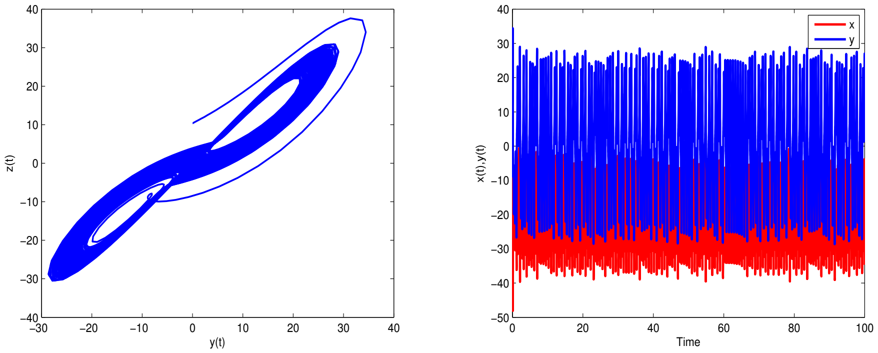

FIGURE 2. Phase portrait and time series solution of system (2) when $K = -7.75$, $\tau = 0.034$, E^* become unstable and chaos still exists. Here, the initial value is $(0.1, 0.1, 10.38)$.

(ii): *The Eq.* (10) *have positive real roots if and only if* $\Delta > 0$, $\eta_1^* = \frac{-p+\sqrt{\Delta}}{4} > 0$ *and* $f(\eta_1^*) \leq 0$. *More precisely, if the condition* $(\mathrm{H}_4): r > 0,\ \eta_1^* > 0,\ f(\eta_1^*) < 0$ *holds, then* (10) *has two positive roots,* η_1 *and* η_2.

Suppose $\Delta \geq 0, \eta_1^* > 0,\ f(\eta_1^*) \leq 0$, without loss of generality, we assume that Eq. (10) has three positive roots $\eta_k,\ k = 1, 2, 3$; consequently, Eq. (9) also has three positive roots $\omega_k = \sqrt{\eta_k}$ $(k = 1, 2, 3)$. Then from (8), we can determine the corresponding critical values as

$$\tau_k^{(j)} = \begin{cases} \frac{1}{\omega_k}[\arccos(P) + 2j\pi], & Q \geq 0 \\ \frac{1}{\omega_k}[2\pi - \arccos(P) + 2j\pi], & Q < 0, \end{cases} \tag{11}$$

where

$$Q = \sin(\omega_k \tau_k) = \frac{(S_2\omega_k^4 + (L_2S_1 - L_1S_2 - S_0)\omega_k^2 - L_0S_1 + S_0L_1)\omega_k}{S_2^2\omega_k^4 + (S_1^2 - 2S_0S_2)\omega_k^2 + S_0^2},$$

and

$$P = \cos(\omega_k \tau_k) = \frac{(S_1 - L_2S_2)\omega_k^4 + (L_0S_2 - L_1S_1 + L_2S_0)\omega_k^2 - L_0S_0}{S_2^2\omega_k^4 + (S_1^2 - 2S_2S_0)\omega_k^2 + S_0^2}.$$

Now based on the above analysis, the following result can be presented.

Lemma 2.2. *When* $\tau = \tau_k^{(j)}$ $(k = 1, 2, 3; j = 0, 1, 2, \cdots)$, *where* $\tau_k^{(j)}$ *is defined by* (11), *and if* (H_4) *in Lemma 2.1 holds, then Eq. (5) has a pair of complex conjugate pure imaginary roots* $\pm \mathrm{i}\omega_0$, *and all other roots have nonzero real parts.*

Additionally, let $\lambda(\tau) = \varphi(\tau) + \mathrm{i}\omega(\tau)$ be a root of (5) near $\tau = \tau_k^{(j)}$ such that $\varphi(\tau_k^{(j)}) = 0$ and $\omega(\tau_k^{(j)}) = \omega_k$, then the following transversality condition holds.

Lemma 2.3. *If* $f'(\eta_k) \neq 0$ *and* (H_3) *in Lemma 2.1 holds, then* $\frac{d\left[\mathrm{Re}\lambda\left(\tau_k^{(j)}\right)\right]}{d\tau} \neq 0$ *and* $f'(\eta_k)$ *have the same sign.*

Proof. Substituting $\lambda(\tau)$ into characteristic Eq. (5) and differentiating its both sides with respect to τ, we obtain

$$\left[3\lambda^2 + 2L_2\lambda + L_1 + \Big(2\lambda S_2 + S_1\Big)e^{-\lambda\tau} - \Big(S_2\lambda^2 + S_1\lambda + S_0\Big)\tau \mathrm{e}^{-\lambda\tau}\right]\frac{d\lambda}{d\tau} = \Big(S_2\lambda^2 + S_1\lambda + S_0\Big)\lambda \mathrm{e}^{-\lambda\tau}.$$

This gives

$$\begin{aligned}\left(\frac{d\lambda}{d\tau}\right)^{-1} &= \frac{\Big(3\lambda^2 + 2L_2\lambda + L_1\Big) + \Big(S_1 + 2S_2\lambda\Big)\mathrm{e}^{-\lambda\tau}}{\Big(S_2\lambda^2 + S_1\lambda + S_0\Big)\lambda \mathrm{e}^{-\lambda\tau}} \\ &\quad - \frac{\Big(S_2\lambda^2 + S_1\lambda + S_0\Big)\tau \mathrm{e}^{-\lambda\tau}}{\Big(S_2\lambda^2 + S_1\lambda + S_0\Big)\lambda \mathrm{e}^{-\lambda\tau}} \\ &= \frac{\Big(3\lambda^2 + 2L_2\lambda + L_1\Big)\mathrm{e}^{\lambda\tau}}{\Big(S_2\lambda^2 + S_1\lambda + S_0\Big)\lambda} + \frac{\Big(S_1 + 2S_2\lambda\Big)}{\Big(S_2\lambda^2 + S_1\lambda + S_0\Big)\lambda} - \frac{\tau}{\lambda}.\end{aligned} \tag{12}$$

From (8), we have

$$\begin{aligned}\Big[(3\lambda^2 + 2L_2\lambda + L_1)\mathrm{e}^{\lambda\tau}\Big]_{\tau=\tau_k^{(j)}} &= \big[(L_1 - 3\omega_k^2)\cos(\omega_k\tau_k^{(j)}) - 2L_2\omega_k\sin(\omega_k\tau_k^{(j)})\big] \\ &\quad + \big[2L_2\omega_k\cos(\omega_k\tau_k^{(j)}) + (L_1 - 3\omega_k^2)\sin(\omega_k\tau_k^{(j)})\big]\mathrm{i},\end{aligned} \tag{13}$$

$$\Big[(2S_2\lambda + S_1)\Big]_{\tau=\tau_k^{(j)}} = S_1 + 2S_2\omega_k\mathrm{i}, \tag{14}$$

and

$$\Big[(S_2\lambda^2 + S_1\lambda + S_0)\lambda\Big]_{\tau=\tau_k^{(j)}} = -S_1\omega_k^2 + (S_0\omega_k - S_2\omega_k^3)\mathrm{i}. \tag{15}$$

From (12)–(15) and (8), we obtain

$$\begin{aligned}
\left[\frac{d(Re(\lambda))}{d\tau}\right]^{-1}_{\lambda=\mathrm{i}\omega_k} &= Re\left[\frac{(3\lambda^2+2L_2\lambda+L_1)\mathrm{e}^{\lambda\tau}}{\big(S_2\lambda^2+S_1\lambda+S_0\big)\lambda}\right]_{\lambda=\mathrm{i}\omega_k} \\
&\quad + Re\left[\frac{\big(2S_2\lambda+S_1\big)}{\big(S_2\lambda^2+S_1\lambda+S_0\big)\lambda}\right]_{\lambda=\mathrm{i}\omega_k} \\
&= \frac{1}{\Lambda}\Bigg[-S_1\omega_k^2\Big\{(L_1-3\omega_k^2)\cos(\omega_k\tau_k^{(j)})-2L_2\omega_k\sin(\omega_k\tau_k^{(j)})\Big\} \\
&\quad - S_1^2\omega_k^2+\omega_k(S_0-S_2\omega_k^2)\Big\{2L_2\omega_k\cos(\omega_k\tau_k^{(j)}) \\
&\quad + (L_1-3\omega_k^2)\sin(\omega_k\tau_k^{(j)})+2S_2\omega_k\Big\}\Bigg] \\
&= \frac{1}{\Lambda}\Bigg[(L_1-3\omega_k^2)\omega_k\Big\{(S_0-S_2\omega_k^2)\sin(\omega_k\tau_k^{(j)}) \\
&\quad - S_1\omega_k\cos(\omega_k\tau_k^{(j)})\Big\}+2L_2\omega_k^2\Big\{(S_0-S_2\omega_k^2)\cos(\omega_k\tau_k^{(j)}) \\
&\quad + S_1\omega_k\sin(\omega_k\tau_k^{(j)})\Big\}-S_1^2\omega_k^2+2S_2\omega_k^2(S_0-S_2\omega_k^2)\Bigg].
\end{aligned}$$

Therefore

$$\begin{aligned}
\left[\frac{d(Re(\lambda))}{d\tau}\right]^{-1}_{\lambda=\mathrm{i}\omega_k} &= \frac{1}{\Lambda}\Bigg[3\omega_k^6+2(L_2^2-S_2^2-2L_1)\omega_k^4 \\
&\quad + \big\{L_1^2-2L_0L_2+2S_2S_0-S_1^2\big\}\omega_k^2\Bigg] \\
&= \frac{1}{\Lambda}\Big(3\omega_k^6+2p\omega_k^4+q\omega_k^2\Big) \\
&= \frac{1}{\Lambda}\Big[\eta_k(3\eta_k^2+2p\eta_k+q)\Big] \\
&= \frac{\eta_k}{\Lambda}f'(\eta_k),
\end{aligned}$$

where $\Lambda=\omega_k^4S_1^2+(S_0\omega_k-S_2\omega_k^3)^2$.

Thus, we have

$$sign\left[\frac{dRe(\lambda)}{d\tau}\right]_{\tau=\tau_k^{(j)},\lambda=\mathrm{i}\omega_k} = sign\left[\frac{dRe(\lambda)}{d\tau}\right]^{-1}_{\tau=\tau_k^{(j)},\lambda=\mathrm{i}\omega_k} = sign\left[\frac{\eta_k}{\Lambda}f'(\eta_k)\right].$$

Furthermore, since $\eta_k > 0$ and $\Lambda > 0$, we conclude that $\left[\frac{dRe(\lambda)}{d\tau}\right]_{\tau=\tau_k^{(j)},\lambda=i\omega_k}$ and $f'(\eta_k)$ have the same sign. Also, if we assume that $\left[\frac{dRe(\lambda)}{d\tau}\right]_{\tau=\tau_k^{(j)}} < 0$, then the characteristic equation has roots with positive real parts when $\tau < \tau_k$. It contradictions the local stability of the positive equilibrium point. Hence, $\left[\frac{dRe(\lambda)}{d\tau}\right]_{\tau=\tau_k^{(j)}} > 0$ and the proof of Lemma 2.3 is complete. □

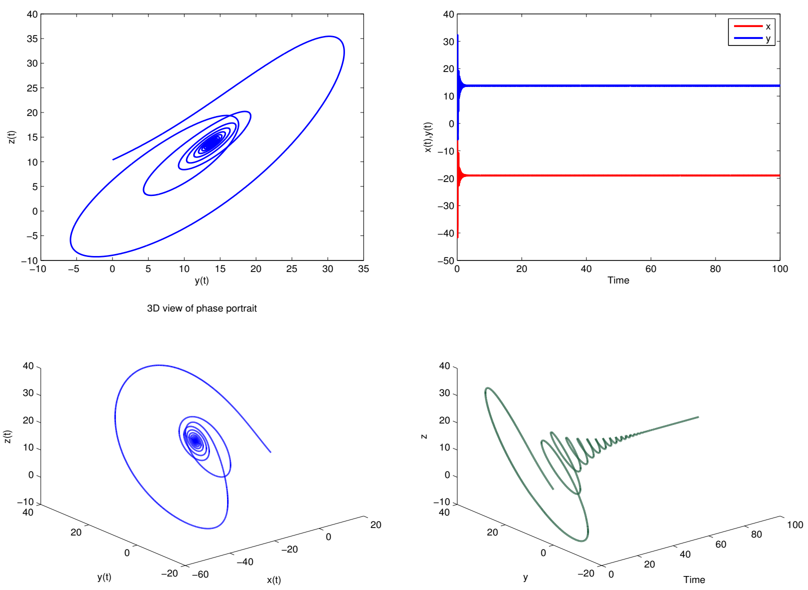

FIGURE 3. Phase portraits and time series diagrams of system (2) when $K = -7.75$ and $\tau = 0.12$. Chaos vanishes and equilibrium point E^* becomes locally asymptotically stable. The initial value is $(0.1, 0.1, 10.38)$.

Define $\tau_0 = \tau_{k_0} = \min_{1\le k\le 3}\{\tau_k\}, \omega_0 = \omega_{k_0}, \eta_0 = \omega_0^2$, according to the derived Lemmas 2.1, 2.2, 2.3, a Hopf bifurcation occurs.

Theorem 2.4. *For system (2), we have.*

- **(i):** *If the condition* (H_3) *in Lemma (2.1) holds, then* E^* *is asymptotically stable for all* $\tau > 0$.
- **(ii):** *If* $\Delta = p^2 - 3q > 0$, $f'(\eta_0) \neq 0$ *and if there exists only one positive real root, then there exists a positive number* τ_0 *such that the equilibrium* E^* *is locally asymptotically stable when* $\tau \in [0, \tau_0)$ *and unstable when* $\tau > \tau_0$. *Moreover, system (2) undergoes a Hopf bifurcation at* E^* *when* $\tau = \tau_0$.

(iii): *If* $\tau_1^{(0)} < \tau_2^{(0)}$ *then there exists* $m \in \mathbb{N}$ *such that* $\tau_1^{(0)} < \tau_2^{(0)} < \tau_1^{(1)} < \tau_2^{(1)} < \cdots < \tau_1^{(m)} < \tau_2^{(m)} < \tau_2^{(m+1)} < \tau_1^{(m+1)}$*, and* E^* *is asymptotically stable for* $\tau \in [0, \tau_1^{(0)}) \cup \bigcup_{n=1}^{m}(\tau_2^{(n-1)}, \tau_1^{(n)}) \cup (\tau_2^{(m)}, \tau_1^{(m+1)})$ *and unstable for* $\tau \in \bigcup_{n=0}^{m}(\tau_1^{(n)}, \tau_2^{(n)})$*. Moreover, if* (H_4) *in Lemma (2.1) and* $f'(\eta_k) \neq 0$ *hold, then the system (2) undergoes a Hopf bifurcation at* E^* *when* $\tau = \tau_k^{(j)}$ *for* $k = 1, 2;\ j = 0, 1, 2, \cdots$.

3. Numerical simulations

In this section, we use MATLAB 2013a and Maple 2017 as the calculation tools to carry out some numerical simulations for verifying the analytical results obtained in the previous section. For the parameters $\beta = -10, \alpha = 37, \mu = 55$, we get $E^*(-19, 13.784, 13.784)$. Also, for chaos control, we suppose $K < -1.72$, especially $K \in (-13.75, -1.72)$. Thus when $\tau = 0$ or $K = 0$, system (2) becomes chaotic (see Fig. 1).

Let $K = -7.75 \in (-13.75, -1.72)$. By equations (10), (11) and lemma 2.1, we obtain

(16)

$$\eta_1 = 391, \quad \omega_1 = 19.77, \quad \tau_1^{(j)} = 0.213 + \frac{2j\pi}{\omega_1}, \quad f'(\eta_1) = 208372.84,$$
$$\eta_2 = 110.59, \quad \omega_2 = 10.52, \quad \tau_2^{(j)} = 0.325 + \frac{2j\pi}{\omega_2}, \quad f'(\eta_2) = -129737.125.$$

From lemma 2.3, we have

$$\left[\frac{\mathrm{d}\big(Re\lambda(\tau_1^{(j)})\big)}{\mathrm{d}\tau_1}\right] \approx 0.0020 > 0, \qquad \left[\frac{\mathrm{d}\big(Re\lambda(\tau_2^{(j)})\big)}{\mathrm{d}\tau_2}\right] \approx -0.0032 < 0.$$

In addition, notice that

$$\tau_1^{(0)} = 0.213 < \tau_2^{(0)} = 0.325 < \tau_1^{(1)} = 0.5308 < \tau_2^{(1)} = 0.9222 < \cdots.$$

Thus all the conditions in Lemmas 2.1, 2.3 and Theorem 2.4 are satisfied. The graphical results with initial value $(0.1, 0.1, 10.38)$ show that when $\tau < 0.048$, the equilibrium E^* still displays chaotic behavior (see Fig. 2). When $\tau \in (0.048, 0.13]$, E^* becomes asymptotically stable. Fig. 3 indicates the phase trajectories of system (2) for $\tau = 0.12$. For the critical value $\tau = \tau_1^{(0)} = \tau_0 = 0.213$, the system undergoes a Hopf bifurcation and a periodic orbit emerges around E^*. Thus for $\tau = 0.231$, a limit cycle appears which is depicted in Fig. 3. By further increasing of τ, stability of E^* is changed and the system regains its complex dynamical behavior, i.e., it becomes chaotic again when $\tau > 0.36$.

4. Conclusion

In this study, a state feedback control method with single time delay is used to stabilize UPOs and unstable equilibrium point of a chaotic system that was

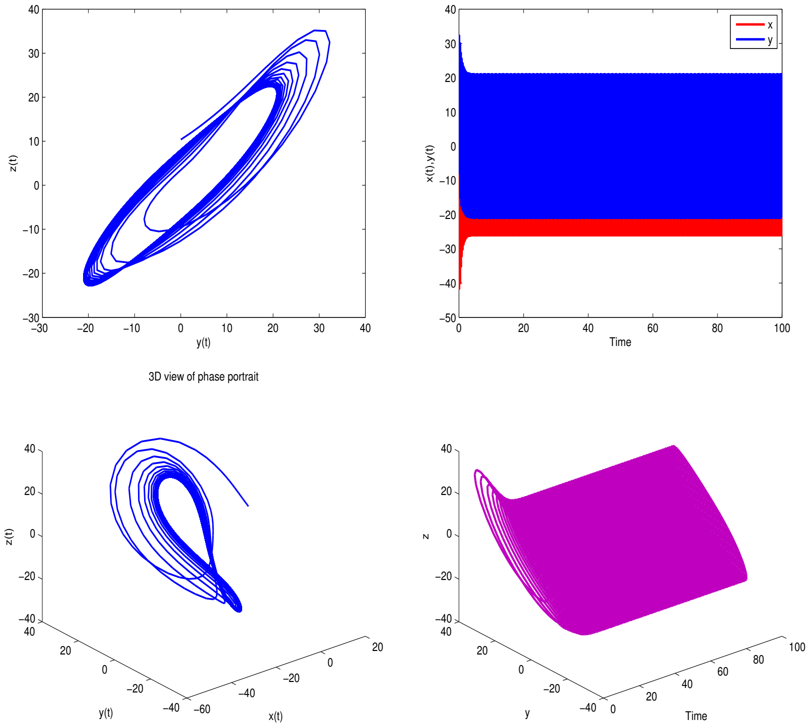

FIGURE 4. Phase portraits and time series solutions of system (2) when $K = -7.75$ and $\tau = 0.231$. Chaos vanishes and a stable periodic solution bifurcates from E^*. The initial value is $(0.1, 0.1, 10.38)$.

studied in [17]. We investigated the existence and stability of a Hopf bifurcation both analytically and numerically by analyzing the distribution of the roots of the corresponding characteristic equation. An explicit formula which determines the critical values for occurrence of a Hopf bifurcation is derived. Then a necessary condition is proposed and proved under which this bifurcation occurs. Thus to eliminate the chaotic behaviors, the feedback strength K and time delay τ are adequately designed and applied to stabilize one of the unstable equilibrium point of the system is stabilized. It is shown that the appropriate choice of two important parameters K and τ has a prominent influence on the problem of chaos control. Furthermore, numerical calculations can be easily implemented in this scheme and the chaotic behaviors of the system can be controlled successfully by proper selections of the feedback gain and the corresponding critical value of the time delay. According to the numerical results, we find out that the time-delayed feedback control is an efficient method for control of the chaos phenomenon.

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