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VORTEX SOLUTIONS FOR THERMOHALINE CIRCULATION EQUATIONS

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Abstract. The main objective of this article is to establish a new model and find some vortex axisymmetric solutions of finite core size for this model. We introduce the hydrodynamical equations governing the atmospheric circulation over the tropics, the Boussinesq equation with constant radial gravitational acceleration. Solutions are expanded into series of Hermite eigenfunctions. We find the coefficients of the series and show the convergence of them. These equations are critically important in mathematics. They are similar to the 3D Navier-Stokes and the Euler equations. The 2D Boussinesq equations preserve some important aspects of the 3D Euler and Navier-Stokes equations such as the vortex stretching mechanism. The inviscid 2D Boussinesq equations are known as the Euler equations for the 3D axisymmetric swirling flows.This model is the most frequently used for buoyancy-driven fluids, such as many largescale geophysical flows, atmospheric fronts, ocean circulation, clued dynamics. In addition, they play an important role in the Rayleigh-Benard convection

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In this paper, we represent some exact solutions of the vortex type for two dimensional Boussinesq equations defined over the entire plane. These systems describe the evolution of the velocity field u of an incompressible fluid under a centrist force which is proportional to some scalar field T (e.g., the temperature), the latter being transported by u. The standard 2D Boussinesq system with centrist force reads as:

$$
\begin{cases}
\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \varphi\mathbf{u} + 2\xi\times\mathbf{u} + \nabla p = \nu\triangle\mathbf{u} + \tau g\overrightarrow{\mathbf{e}_r}, \\
\partial_t T + \mathbf{u}\cdot\nabla T = K_T\triangle T, \\
\nabla\cdot\mathbf{u} = 0,
\end{cases} \tag{1}
$$

where $\mathbf{u}$ is the fluid speed, T stands for temperature, g is the gravitational acceleration constant, e_r is the monad vector in the r-direction, α is the thermal expansion coefficient, ξ is the earth's rotating angular velocity, Q is the heat

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sourse, K_T is diffusion coefficient of temperature, ν the kinematic viscosity and $\varphi = C_i h^2$ $(i = 0, 1)$ represents the turbulent friction. Here C_0 and C_1 are constants, and h is the vertical length scale. We can think about this added friction term as due to the turbulent averaging process, although the mathematical derivation of this scaling law is based on the analysis of the dynamic transitions of convection problems. Suppose that the field of vorticity $\xi = \nabla \times u$ is enough localized, and if we consider $\overrightarrow{\xi}$ constant, then we have:

$$(2) \quad \begin{cases} \partial_t \xi + u \cdot \nabla \xi + \varphi_1 \partial_{y_1} u_2 - \varphi_0 \partial_{y_2} u_1 = \nu \triangle \xi - \frac{\alpha \tau_0 g}{r} \mathbf{y}^{\perp} \nabla T, \\ \partial_t T + u \cdot \nabla T = k_T \triangle T, \\ \nabla \cdot \xi = 0. \end{cases}$$

We are able to restore the speed of the fluid through Biot-Savart legislation:

$$(3) \qquad u(y,t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(y-z)^{\perp}}{|y-z|^2} \xi(z,t) dz,$$

where $\mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$, $\mathbf{y}^{\perp} = (-y_2, y_1)$ and $|\mathbf{y}|^2 = y_1^2 + y_2^2$. For simplicity purposes, we focus on Equation (2), but our methods are applicable to the Thermohaline ocean circulation equations too. We employ Equations (2) and (3) to establish a vorticity representation of the two-dimensional viscous flow. Two-dimensional vortex motion studies go back to the work of Helmholtz [7], and later by Lord Kelvin [10], Sir Lamb [11], Prandtl [17], Milne-Thomson [1, 13, 16], Batchelor [4], and others. Bernoff and Lingevitch in [2,5] achieved that the motion of vortex is the integral of the background irrational current. For a comprehensive survey of the inviscid point vortex model and recent developments, see [3, 15]. Gallay and Wayne in [6] proved that the solutions of vorticity equation tend to Oseen vortex rapidly. Uminsky in [20] using Hermite eigenfunctions introduced a new multi-moment vortex method (MMVM). By using MMVM, Smith and Nagem in [19] studied vortex pairs and dipoles. Sharifi and Raesi in [18] presented the first solutions of vortex type for 2D Boussinesq equations under a vertical force.

In this paper, we extend the results of [18] to the Boussinesq equation under the central gravitational force on a rotating plane with turbulent friction terms. We express a moment expansion of the vorticity based on Hermite functions. Then, we establish a convergence criterion of the moment expansion. We show that, if this criterion meets for $t = 0$, then it meets, for all subsequent times $t > 0$. Our convergence criterion relies on the observation that for any value of t, the Hermite functions are the eigenfunctions of a self-adjoint linear operator in a weighted subspace of $L^2(\mathbb{R}^2)$. We prove that if the initial vorticity distribution lies in this space, then the solution of the vorticity equation with that initial condition lies in it. We rewrite the two-dimensional vorticity equation as a system of ODEs with simple, quadratic nonlinear terms whose coefficients

can be evaluated in terms of derivatives of a single explicit function. Furthermore, we establish a sufficient condition on the initial vorticity distribution to guarantee that the expansion of the vorticity generated by the solution of these ODEs converges for all time. Finally, we introduce the hydrodynamical equations governing the atmospheric circulation over the tropics, Boussinesq equation with constant radial gravitational acceleration. In the same way, we obtain the exact solutions of this model.
This article is organized as follows. In Section 2, we offer an expansion of solutions for the Boussinesq equations in the vorticity form. In Section 3, the convergence of the series of the solution is shown. In Section 4, we find the ODEs satisfied by the coefficient of the expansion in the Hermite base.

1. Review of the single center vortex method

In this section, we express the solutions of Equation (2), based on Hermite functions. Let

$$\Theta_{00}(\mathbf{y},t;\eta) = \frac{1}{\pi\eta^2} e^{-|\mathbf{y}|^2/\eta^2}, \tag{4}$$

$$\Upsilon_{00}(\mathbf{y},t;\Omega) = \frac{1}{\pi\Omega^2} e^{-|\mathbf{y}|^2/\Omega^2}, \tag{5}$$

where $\eta^2 = \eta_0^2 + 4\nu t$ and $\Omega^2 = \Omega_0^2 + 4k_T t$ and η_0 and Ω_0 represent the initial core size of our localized vortex structure. Note that for any value of η_0 and Ω_0, Θ_{00} and Υ_{00} are exact solutions of the two-dimensional vorticity equation recognized as the Lamb-Oseen vortex. As a consequence, we can choose any value of η_0 and Ω_0 in the definition of our Hermite spectral method; η_0 and Ω_0 are chosen to portray a typical length scale in the initial vorticity distribution. The Hermite functions of degree (κ_1, κ_2) are defined as follows:

$$\Theta_{\kappa_1,\kappa_2}(\mathbf{y},t;\eta) = D_{y_1}^{\kappa_1} D_{y_2}^{\kappa_2} \Theta_{00}(\mathbf{y},t;\eta), \tag{6}$$

$$\eta_{\kappa_1,\kappa_2}(\mathbf{y},t;\Omega) = D_{y_1}^{\kappa_1} D_{y_2}^{\kappa_2} \Upsilon_{00}(\mathbf{y},t;\Omega).$$

An expansion of the solution of the vorticity equation based on Hermite functions, which are called the moment expansion, are defined as follows:

$$\begin{aligned} \xi(\mathbf{y},t) &= \sum_{\kappa_1,\kappa_2=1}^{\infty} N[\kappa_1,\kappa_2;t]\Theta_{\kappa_1,\kappa_2}(\mathbf{y},t;\eta), \\ T(\mathbf{y},t) &= \sum_{\kappa_1,\kappa_2=1}^{\infty} J[\kappa_1,\kappa_2;t]\eta_{\kappa_1,\kappa_2}(\mathbf{y},t;\Omega). \end{aligned} \tag{7}$$

The vorticity function $\xi(\mathbf{y},t) = \varsigma\Theta_{00}(\mathbf{y},t)$ is an exact solution (the Oseen, or Lamb, vortex) of the two dimensional vorticity equation for all values of ς. Let $(\xi, T)(\mathbf{y},t)$ be the resolvent of Equation (2), then Biot-Savrat law implies that the velocity field is as below:

$$\mathbf{V}(\mathbf{y},t)=\sum_{\kappa_1,\kappa_2=1}^{\infty}N[\kappa_1,\kappa_2;t]V_{\kappa_1,\kappa_2}(\mathbf{y},t;\eta),$$

where $V_{\kappa_1,\kappa_2}(\mathbf{y},t;\eta)=D_{y_1}^{\kappa_1}D_{y_2}^{\kappa_2}\mathbf{V}_{00}(\mathbf{y},t;\eta)$ and $\mathbf{V}_{00}(\mathbf{y},t;\eta)$ is the induced velocity from $\Theta_{00}(\mathbf{y},t;\eta)$ which is specifed as follows:

$$\mathbf{V}_{00}(\mathbf{y},t;\eta)=\frac{1}{2\pi}\frac{(-y_2,y_1)}{|\mathbf{y}|^2}(1-e^{-|\mathbf{y}|^2/\eta^2}).$$

It can be easily seen, for any value of t, the Hermite functions $\Theta_{\kappa_1,\kappa_2}(\mathbf{y},t;\eta)$ are the eigenfunctions of the self-adjoint linear operator:

$$L^\eta\Theta=\frac{1}{4}\eta^2\triangle\Theta+\frac{1}{2}\nabla\cdot(\mathbf{y}\Theta).$$

Note that, L^η can be transformed into the Hamiltonian quantum mechanical harmonic oscillator. The eigenfunctions of L^η construct an orthogonal set in the $X^\eta=\{f\in L^2(\mathbb{R}^2)\mid\Theta_\eta^{-1/2}f\in L^2(\mathbb{R}^2)\}$, which is a Hilbert space. Let

$$\Theta_\eta(\mathbf{y},t)=\Theta_{00}(\mathbf{y},t;\eta),\qquad\qquad \eta_\Omega(\mathbf{y},t)=\Upsilon_{00}(\mathbf{y},t;\Omega).$$

Nagem et al. [14] showed the convergence of expansions (7), under the following conditions:

$$\int_{\mathbb{R}^2}\Theta_\eta^{-1}(\mathbf{y})(\xi(\mathbf{y},t))^2d\mathbf{y}<\infty, \tag{8}$$

$$\int_{\mathbb{R}^2}\eta_\Omega^{-1}(\mathbf{y})(T(\mathbf{y},t))^2d\mathbf{y}<\infty. \tag{9}$$

In the next section, we prove Theorem (2.3). Under the initial vorticity distribution satisfies Equation (8) for some $\eta=\eta_0$ and $\Omega=\Omega_0$, Theorem (2.3) shows that the solution of the vorticity equation with that initial condition will satisfy Equation (8) for all time t with
$\eta=\sqrt{4\nu t+\eta_0^2}$, $\Omega=\sqrt{4K_Tt+\Omega_0^2}$. Hence, if the initial vorticity distribution satisfies Equation (8), then our moment expansion converges for all times t.

2. Existence of solution for vorticity equation

In this section, we are ready to prove a theorem on the existence of solution for the vorticity equation. In the following, we first prove two lemmas to prepare the ground for this theorem.

Lemma 2.1.

$$\frac{d\gamma(t)}{dt}\le(\frac{4c(\xi_0,T_0)}{K_T}+\frac{4K_T}{\Omega^2})\gamma(t).$$

Proof: According to Lemma 2.1 in [6], we have: $||u||_\infty\le c||\xi||_p^\alpha||\xi||_q^{1-\alpha}$, where $1\le p<2<q\le\infty$ and $\frac{\alpha}{p}+\frac{1-\alpha}{q}=\frac{1}{2}$. Similar to the proof of Theorem 3.4 in [14] it could be proved this lemma. This means that $\gamma(t)$ is limited for

each $t > 0$ if $\gamma(0)$ is finite. Now to prove that $\epsilon(t) < \infty$, differentiate $\epsilon(t)$, we have:

$$\begin{aligned}
\frac{d\epsilon(t)}{dt} &= \frac{4\nu}{\eta^2}\epsilon(t) - \frac{4\nu}{\eta^4}\int_{\mathbb{R}^2}|y|^2\Theta_\eta^{-1}(\xi(y,t))^2dy \\
&+ 2\int_{\mathbb{R}^2}|y|^2\Theta_\eta^{-1}\xi(y,t)\partial_t\xi(y,t)dy \\
&= \frac{4\nu}{\eta^2}\epsilon(t) - \frac{4\nu}{\eta^4}\int_{\mathbb{R}^2}|y|^2\Theta_\eta^{-1}(\xi(y,t))^2dy \\
&+ 2\int_{\mathbb{R}^2}\Theta_\eta^{-1}\xi\,(\nu\triangle\xi - u\cdot\nabla\xi + \Omega_1\partial_{y_1}u_2 - \Omega_0\partial_{y_2}u_1 \\
&+ \frac{\alpha\tau_0 g}{r}(y_2\partial_{y_1}T - y_1\partial_{y_2}T)\Big)\,dy.
\end{aligned} \tag{10}$$

Integrating by parts in the last term in Equation (10) implies that:

$$\begin{aligned}
& 2\int_{\mathbb{R}^2}\Theta_\eta^{-1}\xi(\nu\triangle\xi)dy \\
&= -2\nu\int_{\mathbb{R}^2}\Theta_\eta^{-1}(y)(|\nabla\xi|^2 + \frac{2}{\eta^2}\xi y\cdot\nabla\xi)dy,
\end{aligned} \tag{11}$$

and the second item in the right side of Equation (11) satisfies the following relation:

$$\begin{aligned}
2\nu\int_{\mathbb{R}^2}\Theta_\eta^{-1}(y)(\frac{2}{\eta^2}\xi y\cdot\nabla\xi)dy &\le \nu\int_{\mathbb{R}^2}\Theta_\eta^{-1}(y)|\nabla\xi|^2dy \\
&+ \frac{4\nu}{\eta^4}\int_{\mathbb{R}^2}\Theta_\eta^{-1}(y)(y^2\xi^2)dy.
\end{aligned} \tag{12}$$

Now using $||u||_\infty \le c(\xi_0, T_0)$ and Cauchy's inequality we have :

$$\begin{aligned}
2\int_{\mathbb{R}^2}\Theta_\eta^{-1}\xi(u\cdot\nabla\xi)dy &\le 2c(\xi_0,T_0)\int_{\mathbb{R}^2}\Theta_\eta^{-1}|\xi(y,t)||\nabla\xi|dy \\
&\le \frac{c^2(\xi_0,T_0)}{\nu}\int_{\mathbb{R}^2}\Theta_\eta^{-1}(\xi(y,t))^2dy + \nu\int_{\mathbb{R}^2}\Theta_\eta^{-1}(y)|\nabla\xi|^2dy,
\end{aligned}$$

also

$$\begin{aligned}
& 2\alpha\tau_0 g\int_{\mathbb{R}^2}\Theta_\eta^{-1}\xi\left(\frac{y_2\partial_{y_1}T - y_1\partial_{y_2}T}{r}\right)d\mathbf{y} \\
&\le 2\alpha\tau_0 g\int_{\mathbb{R}^2}\Theta_\eta^{-1}\xi\left(\partial_{y_1}T + \partial_{y_2}T\right)d\mathbf{y} \\
&\le \frac{\alpha\tau_0 g\mu(t)}{\epsilon^2} + \frac{2\alpha\tau_0 g}{\epsilon^2}\delta(t) \\
&\Rightarrow \frac{d\gamma(t)}{dt} \le (\frac{4c(\xi_0,T_0)}{K_T} + \frac{4K_T}{\Omega^2})\gamma(t).
\end{aligned} \tag{13}$$

Lemma 2.2. *Define:*

$$\delta(t) = \int_{\mathbb{R}^2} \Theta_\eta^{-1}(\nabla T(y,t))^2 dy.$$

The term $||\nabla T||_\eta^2$ *is bounded.*

Proof: According to [18], if $\delta(0)$ is limited, then $\delta(t)$ will be limited for all $t > 0$.
Also:

$$\begin{aligned}
&2\int_{\mathbb{R}^2} \Theta_\eta^{-1}\xi(\Omega_1\partial_{y_1}u_2 - \Omega_0\partial_{y_2}u_1)dy \\
&\le \frac{\Omega_1}{4}\int_{\mathbb{R}^2} \Theta_\eta^{-1}\xi^2 dy + \Omega_1\int_{\mathbb{R}^2} \Theta_\eta^{-1}(\partial_{y_1}u_2)^2 dy \\
&+\frac{\Omega_0}{4}\int_{\mathbb{R}^2} \Theta_\eta^{-1}\xi^2 dy + \Omega_0\int_{\mathbb{R}^2} \Theta_\eta^{-1}(\partial_{y_2}u_1)^2 dy \\
&\le \frac{\max(\Omega_1,\Omega_0)}{4}\epsilon(t) + \max(\Omega_1,\Omega_0)\int_{\mathbb{R}^2} \Theta_\eta^{-1}(\partial_{y_1}u_2)^2 dy \\
&\le \frac{\max(\Omega_1,\Omega_0)}{4}\epsilon(t) + \max(\Omega_1,\Omega_0)\int_{\mathbb{R}^2} \Theta_\eta^{-1}||\nabla u||^2 d\mathbf{y}.
\end{aligned}$$

Now we bound the term $||\nabla u||^2$, let $f(y,t) = \nabla u(y,t)$ and define:

$$\zeta(t) = \int_{\mathbb{R}^2} \Theta_\eta^{-1}(\nabla u(y,t))^2 dy.$$

Differentiate $\zeta(t)$ obtain the following equation:

$$\begin{aligned}
\frac{d\zeta(t)}{dt} &= \frac{4\nu}{\eta^2}\zeta(t) - \frac{4\nu}{\eta^4}\int_{\mathbb{R}^2} |y|^2\Theta_\eta^{-1}f^2(y,t)dy \\
&+ 2\int_{\mathbb{R}^2} \Theta_\eta^{-1}f(y,t)\partial_t f(y,t)dy \\
&= \frac{4\nu}{\eta^2}\zeta(t) - \frac{4\nu}{\eta^4}\int_{\mathbb{R}^2} |y|^2\Theta_\eta^{-1}(f(y,t))^2 dy \\
&+ 2\int_{\mathbb{R}^2} \Theta_\eta^{-1}f\nabla(\nu\triangle u - u\cdot\nabla u - \Omega u \\
&- 2\overrightarrow{\xi}\times u - \nabla p + \tau g\overrightarrow{e_r})dy.
\end{aligned} \tag{14}$$

Now by considering that the last term in Equation (14) we have:

$$\begin{aligned}
2\int_{\mathbb{R}^2} \Theta_\eta^{-1}f\nabla(\nu\triangle u)dy &= 2\nu\int_{\mathbb{R}^2} \Theta_\eta^{-1}f(\triangle f)dy \\
&= -2\nu\int_{\mathbb{R}^2} \Theta_\eta^{-1}(y)(|\nabla f|^2 + \frac{2}{\eta^2}f\cdot y\cdot\nabla f)dy.
\end{aligned} \tag{15}$$

The second term in the last part of Equation (15) satisfies the following inequality:

$$2\nu \int_{\mathbb{R}^2} \Theta_\eta^{-1}(y)(\frac{2}{\eta^2} f \cdot y \cdot \nabla f)dy \\ \leq \frac{4\nu}{\eta^4} \int_{\mathbb{R}^2} \Theta_\eta^{-1}(y^2 f^2)dy + \frac{\nu^2}{\nu} \int_{\mathbb{R}^2} \Theta_\eta^{-1} |\nabla f|^2 dy. \tag{16}$$

On the other hand inequalitis $||u||_\infty \leq c(\xi_0, T_0, t)$ and $||\nabla u||_\infty \leq c(\xi_0, T_0, t)$ in [8] impliy that:

$$-2 \int_{\mathbb{R}^2} \Theta_\eta^{-1} f \nabla(u \cdot \nabla u)dy = \\ -2 \int_{\mathbb{R}^2} (\Theta_\eta^{-1} f) \nabla u \cdot \nabla u dy - 2 \int_{\mathbb{R}^2} (\Theta_\eta^{-1} f) u \cdot \nabla(\nabla u)dy \\ \leq 2c(\xi_0, T_0, t) \int_{\mathbb{R}^2} \Theta_\eta^{-1} f^2 dy + 2c(\xi_0, T_0, t) \int_{\mathbb{R}^2} \Theta_\eta^{-1} |f||\nabla f|dy. \\ \leq 2c(\xi_0, T_0, t)\zeta(t) + 2c(\xi_0, T_0, t) \int_{\mathbb{R}^2} \Theta_\eta^{-1} |f||\nabla f|dy.$$

Now we have:

$$2c(\xi_0, T_0, t) \int_{\mathbb{R}^2} \Theta_\eta^{-1} |f||\nabla f|dy \\ \leq \frac{c^2(\xi_0, T_0, t)}{\nu} \int_{\mathbb{R}^2} \Theta_\eta^{-1}(f^2(y,t))dy + \nu \int_{\mathbb{R}^2} \Theta_\eta^{-1} |\nabla f|^2 dy. \tag{17}$$

Theorem 2.3. *Define*

$$\epsilon(t) = \int_{\mathbb{R}^2} \Theta_\eta^{-1}(\xi(y,t))^2 dy, \tag{18}$$

$$\gamma(t) = \int_{\mathbb{R}^2} \eta_\Omega^{-1}(T(y,t))^2 dy. \tag{19}$$

If $k_T < 2\nu$ and the primary vorticity and temperature, i.e. ξ_0 and T_0, guarantee that $\epsilon(0) < \infty$ and $\gamma(0) < \infty$ for some η_0 and Ω_0, respectively, and ξ_0 and T_0 are in the L^∞, then $\epsilon(t)$ and $\gamma(t)$ will be finite for all times of $t > 0$.

Proof:

$$\frac{d\zeta(t)}{dt} \leq (2c(\xi_0, T_0, t) + \frac{c^2(\xi_0, T_0, t)}{\nu} + \frac{4\nu}{\eta^2})\zeta(t), \tag{20}$$

and this means that if $\delta(0)$ is limited then $\delta(t)$ will be limited for all $t > 0$. Also:

$$2 \int_{\mathbb{R}^2} \Theta_\eta^{-1} f \nabla(-2\overrightarrow{\xi} \times u)dy = -4 \int_{\mathbb{R}^2} \Theta_\eta^{-1} f(f^\perp)dy = 0,$$

and

$$2 \int_{\mathbb{R}^2} \Theta_\eta^{-1} f \nabla(-\Omega u)dy = -2\Omega \int_{\mathbb{R}^2} \Theta_\eta^{-1} f^2 = -2\Omega\zeta(t).$$

So according to Equations (10)-(13) we can write:

$$\frac{d\epsilon(t)}{dt} \leq (\frac{4\nu}{\eta^2} + \frac{4c(\xi_0, T_0)}{\nu} + 2\alpha\tau_0 g)\epsilon(t) + 4\alpha\tau_0 g c_1(\xi_0, t_0, t),$$

where $||\nabla T||_\eta^2 \leq c_1(\xi_0, t_0, t)$. Using the Gronwall lemma if $\epsilon(0)$ is limited, then $\epsilon(t)$ remains limited for all $t > 0$.

3. ODEs of the coefficients of the expansion

In this section, we rewrite the two-dimensional vorticity equation as a system of ODEs with simple, quadratic nonlinear terms whose coefficients can be evaluated in terms of derivatives of a single explicit function. In other words we show that the coefficients in this expansion satisfy a system of ordinary differential equations whose coefficients can be explicitly represented in terms of a fixed, computable kernel function. In the following, we look for differential equations generating the coefficient $N[\kappa_1, \kappa_2; t]$, $J[\kappa_1, \kappa_2; t]$. Assuming that the $(\xi, T)(y, t)$ is a solution of Equation (2) and define

$$\xi^m(y,t) = \sum_{\kappa_1,\kappa_2}^{m} N[\kappa_1, \kappa_2; t]\Theta_{\kappa_1,\kappa_2}(y, t; \eta), \tag{21}$$

$$u^m(y,t) = \sum_{\kappa_1,\kappa_2}^{m} N[\kappa_1, \kappa_2; t]V_{\kappa_1,\kappa_2}(y, t; \eta), \tag{22}$$

$$T^m(y,t) = \sum_{\kappa_1,\kappa_2}^{m} J[\kappa_1, \kappa_2; t]\eta_{\kappa_1,\kappa_2}(y, t; \Omega), \tag{23}$$

where ξ^m, u^m, and T^m are Hermit approximations of order m (Glerkin approximation by Hermit functions). Then by the use of Glerkin standard approximation for Equation (2) we have:

$$\begin{aligned}
\partial_t \xi^m &= \sum_{\kappa_1,\kappa_2}^{m} \frac{dN[\kappa_1, \kappa_2; t]}{dt}\Theta_{\kappa_1,\kappa_2}(y, t; \eta) + \sum_{\kappa_1,\kappa_2}^{m} N[\kappa_1, \kappa_2; t]\partial_t\Theta_{\kappa_1,\kappa_2} \\
&= \sum_{\kappa_1,\kappa_2}^{m} N[\kappa_1, \kappa_2; t](\nu\triangle\Theta_{\kappa_1,\kappa_2}(y, t; \eta))
\end{aligned}$$

$$P^m\left[(\sum_{l_1,l_2}^{m} M[l_1, l_2; t]V_{l_1,l_2}(y, t; \eta)) \cdot \nabla(\sum_{\kappa_1,\kappa_2}^{m} N[\kappa_1, \kappa_2; t]\Theta_{\kappa_1,\kappa_2}(y, t; \eta))\right]$$

$$+\alpha\tau_0 g P^m\left[y_2\partial_{y_1}(\sum_{\kappa_1,\kappa_2}^{m} J[\kappa_1, \kappa_2; t]\eta_{\kappa_1,\kappa_2}(y, t; \Omega)) - y_1\partial_{y_2}(\sum_{\kappa_1,\kappa_2}^{m} J[\kappa_1, \kappa_2; t]\eta_{\kappa_1,\kappa_2}(y, t; \Omega))\right]$$

$$2P^m\left[\varphi_0\partial_{y_2}((\sum_{l_1,l_2}^{m} M[l_1, l_2; t]V^2_{l_1,l_2}(y, t; \eta))) - \varphi_1\partial_{y_1}(\sum_{l_1,l_2}^{m} M[l_1, l_2; t]V^1_{l_1,l_2}(y, t; \eta)))\right],$$

where $P^m[.]$ is a projector on the subspace produced by Hermit functions of degree m or less. Noting that:

$$\partial_t \Theta_{\kappa_1,\kappa_2} = \nu \triangle \Theta_{\kappa_1,\kappa_2}.$$

Then we have:

$$\begin{aligned}
\frac{dN[\kappa_1,\kappa_2;t]}{dt} &= \\
-P_{\kappa_1,\kappa_2} &\left[(\sum_{l_1,l_2}^{m} M[l_1,l_2;t] V_{l_1,l_2}(y,t;\eta)) \cdot \nabla (\sum_{m_1,m_2}^{m} M[m_1,m_2;t] \Theta_{m_1,m_2}(y,t;\eta)) \right] \\
+P_{\kappa_1,\kappa_2} &\left[\frac{\alpha \tau_0 g y_2}{\sqrt{{y_1}^2+{y_2}^2}} \partial_{y_1} (\sum_{m_1,m_2}^{m} I[m_1,m_2;t] \eta_{m_1,m_2}(y,t;\Omega)) \right] \\
-P_{\kappa_1,\kappa_2} &\left[\frac{\alpha \tau_0 g y_1}{\sqrt{{y_1}^2+{y_2}^2}} \partial_{y_2} (\sum_{m_1,m_2}^{m} I[m_1,m_2;t] \eta_{m_1,m_2}(y,t;\Omega)) \right] \\
+2P_{\kappa_1,\kappa_2} &\left[\varphi_0 \partial_{y_2} (\sum_{l_1,l_2}^{m} M[l_1,l_2;t] V^2_{l_1,l_2}(y,t;\eta))) - \varphi_1 \partial_{y_1} (\sum_{l_1,l_2}^{m} M[l_1,l_2;t] V^1_{l_1,l_2}(y,t;\eta))) \right], \\
\frac{dJ[\kappa_1,\kappa_2;t]}{dt} &= \\
-Q_{\kappa_1,\kappa_2} &\left[(\sum_{l_1,l_2}^{m} M[l_1,l_2;t] V_{l_1,l_2}(y,t;\eta)) \cdot \nabla (\sum_{m_1,m_2}^{m} I[m_1,m_2;t] \eta_{m_1,m_2}(y,t;\Omega)) \right].
\end{aligned}$$

Note that $\kappa_1 + \kappa_2 \le m$, then

$$\begin{aligned}
\Theta_{m_1,m_2}(y,t;\eta) &= (D^{m_1}_{a_1} D^{m_2}_{a_2} \Theta_{00}(y+a,\eta))|_{a=0}, \\
V_{l_1,l_2}(y,t;\eta) &= (D^{l_1}_{b_1} D^{l_2}_{b_2} V_{00}(y+b,\eta))|_{b=0}, \\
\eta_{m_1,m_2}(y,t;\Omega) &= (D^{m_1}_{c_1} D^{m_2}_{c_2} \eta_{00}(y+c,\Omega))|_{c=0}.
\end{aligned}$$

The system of ordinary differential Equations (24) and (24) become as follows:

$$
\begin{aligned}
&\frac{dN[\kappa_1,\kappa_2;t]}{dt} = \\
&-\tau(\kappa_1,\kappa_2\eta)\sum_{l_1,l_2=1}^{m}\sum_{m_1,m_2=1}^{m} M[l_1,l_2;t]M[m_1,m_2;t] \\
&\times\int_{\mathbb{R}^2} H_{\kappa_1,\kappa_2}(y)(D_{y_1}^{l_1}D_{y_2}^{l_2}V_{00}(y,\eta))\cdot\nabla_y(D_{y_1}^{m_1}D_{y_2}^{m_2}\Theta_{00}(y,\eta))dy \\
&+\tau(\kappa_1,\kappa_2,\eta)\alpha\tau_0 g\sum_{m_1,m_2=1}^{m} I[m_1,m_2;t]\int_{\mathbb{R}^2} H_{\kappa_1,\kappa_2}(y)\frac{y_2}{r}(D_{y_1}^{m_1+1}D_{y_2}^{m_2}\Upsilon_{00}(y,\Omega))dy \\
&-\tau(\kappa_1,\kappa_2,\eta)\alpha\tau_0 g\sum_{m_1,m_2=1}^{m} I[m_1,m_2;t]\int_{\mathbb{R}^2} H_{\kappa_1,\kappa_2}(y)\frac{y_1}{r}(D_{y_1}^{m_1}D_{y_2}^{m_2+1}\Upsilon_{00}(y,\Omega))dy \\
&+2\varphi_0\tau(\kappa_1,\kappa_2,\eta)\sum_{l_1,l_2=1}^{m} M[l_1,l_2;t]\int_{\mathbb{R}^2} H_{\kappa_1,\kappa_2}(y)(D_{y_1}^{l_1+1}D_{y_2}^{l_2}V_{00}^2(y,\eta))dy \\
&-2\varphi_1\tau(\kappa_1,\kappa_2,\eta)\sum_{l_1,l_2=1}^{m} M[l_1,l_2;t]\int_{\mathbb{R}^2} H_{\kappa_1,\kappa_2}(y)(D_{y_1}^{l_1}D_{y_2}^{l_2+1}V_{00}^1(y,\eta))dy, \\
&\frac{dJ[\kappa_1,\kappa_2;t]}{dt} = \\
&-\tau(\kappa_1,\kappa_2,\Omega)\sum_{l_1,l_2=1}^{m}\sum_{m_1,m_2=1}^{m} M[l_1,l_2;t]I[m_1,m_2;t] \\
&\times\int_{\mathbb{R}^2} F_{\kappa_1,\kappa_2}(y)(D_{y_1}^{m_1}D_{y_2}^{m_2}V_{00}(y,\eta))\cdot\nabla_y(D_{y_1}^{l_1}D_{y_2}^{l_2}\Upsilon_{00}(y,\Omega))dy.
\end{aligned}
$$

The first integral in (24) is calculated in [20] and the last integral in Equation (24) is calculated in [18] the two remaining integrals in Equation (24) are calculated in appendix.

Finally, using appendix and Equations (24)-(24) we have corrected the differential equations for $M[\kappa_1, \kappa_2, t]$ and $I[\kappa_1, \kappa_2, t]$ to:

(24)
$$\begin{aligned}
&\frac{dN[\kappa_1,\kappa_2;t]}{dt} = \\
&\tau(\kappa_1,\kappa_2\eta)\sum_{l_1,l_2=1}^{m}\sum_{m_1,m_2=1}^{m} M[l_1,l_2;t]M[m_1,m_2;t] \\
\times\ & \tilde{\Gamma}[\kappa_1,\kappa_2,l_1,l_2,m_1,m_2;\eta] + \tau(\kappa_1,\kappa_2,\eta)\alpha\tau_0 g \\
\times\ & \sum_{m_1,m_2}^{m} I[m_1,m_2;t]B[\kappa_1,\kappa_2,m_1,m_2;\eta,\Omega] \\
+\ & 2\varphi_0\tau(\kappa_1,\kappa_2,\eta)\sum_{l_1,l_2=1}^{m} M[l_1,l_2;t]A[\kappa_1,\kappa_2,m_1,m_2;\eta,\Omega] \\
-\ & 2\varphi_1\tau(\kappa_1,\kappa_2,\eta)\sum_{l_1,l_2=1}^{m} M[l_1,l_2;t]\tilde{A}[\kappa_1,\kappa_2,m_1,m_2;\eta,\Omega],
\end{aligned}$$

(25)
$$\begin{aligned}
&\frac{dJ[\kappa_1,\kappa_2;t]}{dt} = \\
&\tau(\kappa_1,\kappa_2\Omega)\sum_{l_1,l_2=1}^{m}\sum_{m_1,m_2=1}^{m} M[l_1,l_2;t]I[m_1,m_2;t]
\end{aligned}$$

(26)
$$\times\ \tilde{\theta}[\kappa_1,\kappa_2,l_1,l_2,m_1,m_2;\eta,\Omega],$$

where $\tilde{\theta}$ is calculated in [18], $\tilde{\Gamma}$ is calcualted in [14], $\tilde{A}$ is calcualted in [9], A calculated in appendix and B introduced as follows:

$$B = \begin{cases}
\xi_1\xi_2\kappa_1!\kappa_2!(m_1+1)!m_2!\Gamma(\frac{1}{2}(p+q+1))\Omega^{p+q+1} & \text{if } p \text{ and } q \text{ be even.} \\
\times \sum_{r=0}^{n}\sum_{\ell=0}^{n-r}\sum_{r_2=0}^{n_2}\sum_{\ell_2=0}^{n_2-r_2}\frac{(-1)^{\kappa_1+\kappa_2+n+n_2}2^{r+r_2}}{n!n_2!\eta^{2n}\Omega^{2n_2+2}} & \\
\times\binom{n}{r}\binom{\kappa_2-2\ell}{r}\binom{n-r}{\ell}\binom{n_2}{r_2}\binom{r_2}{m_2-2\ell_2}\binom{n_2-r_2}{\ell_2} & \\
\times\left(1-\frac{(r_2-m_2+2\ell_2)(m_2+1)}{(m_2-2\ell_2+1)(m_1+1)}\right) & \\
0 & \text{otherwise,}
\end{cases}$$

where $\ell = \frac{2n-2r-\kappa_1}{2}$ and $\ell_2 = \frac{2n_2-2r_2-(m_1+1)}{2}$ such that

$$\xi_1 := \prod_{i=0}^{\frac{q}{2}-1}\frac{q-(2i+1)}{q+p-2i}, \qquad \xi_2 := \prod_{i=0}^{\frac{p}{2}-1}\frac{p-(2i+1)}{p-2i},$$

$q := m_2 - 2\ell_2 + \kappa_2 - 2\ell + 1$, $p := r_2 - m_2 + 2\ell_2 + r - \kappa_2 + 2\ell$.

4. Conclusions

In this paper, we have derived a system of ordinary differential equations whose solutions give a representation of solutions of the two-dimensional vorticity equation in terms of a system of interacting vortices. We have also derived a sufficient condition on the initial vorticity distribution which guarantees that this representation in terms of interacting vortices is equivalent to the original solution of the two-dimensional vorticity equation. Considering different value for ν, the effect of viscosity coefficients on the vortex and also the vortex symmetry rate can be investigated. The coefficient K_T is also effective in determining the rate of vortex symmetry so that time of destroying or symmetry of the vortex will be a different value of this coefficient. One of the important fields of research for the future could be to find vortex solutions for 3D equations, to expand its solutions into a series of Hermite eigenfunctions, and to confirm the convergence of series of the solutions.

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Appendix A.

$$A[\kappa_1,\kappa_2,m_1,m_2;\eta;\Omega] = \int_{\mathbb{R}^2} \Theta_{00}^{-1}(y)\Theta_{00}(y+b;t)V_{00}(y+a;t)dy =$$

$$\int_{\mathbb{R}^2} \pi\eta^2 e^{\frac{y_1^2+y_2^2}{\eta^2}} \cdot \frac{1}{\pi\eta^2} e^{\frac{-y_1^2-2b_1y_1-b_1^2-y_2^2-2b_2y_2-b_2^2}{\eta^2}}$$

$$\cdot \frac{1}{\pi\Omega^2} e^{\frac{-y_1^2-2a_1y_1-a_1^2-y_2^2-2a_2y_2-a_2^2}{\Omega^2}} dy|_{a=0,b=0} =$$

$$\frac{1}{\pi\Omega^2} \cdot e^{\frac{-b_1^2-b_2^2}{\eta^2}} \cdot e^{\frac{-a_1^2-a_2^2}{\Omega^2}} \times \int_{\mathbb{R}^2} e^{\frac{-2b_1y_1-2b_2y_2}{\eta^2}} \cdot e^{\frac{-y_1^2-2a_1y_1-y_2^2-2a_2y_2}{\Omega^2}} dy$$

$$= \beta_1\beta_2 \int_{\mathbb{R}^2} e^{\frac{-(y_1+\Omega^2b_1+\eta^2a_1)^2}{\eta^2\Omega^2}} dy_1 \cdot \int_{\mathbb{R}^2} e^{\frac{-(y_2+\Omega^2b_2+\eta^2a_2)^2}{\eta^2\Omega^2}} dy_2 = \beta_1\beta_2 \cdot \pi\Omega^4,$$

where

$$\beta_1 = \frac{1}{\pi\Omega^2} e^{\frac{-b_1^2-b_2^2}{\eta^2}} \cdot e^{\frac{-a_1^2-a_2^2}{\Omega^2}},\ \beta_2 = e^{\frac{(\Omega^2b_1+\eta^2a_1)^2}{\eta^4\Omega^2}} \cdot e^{\frac{(\Omega^2b_2+\eta^2a_2)^2}{\eta^4\Omega^2}},$$

and this implies that:

$$\int_{\mathbb{R}^2} H_{\kappa_1,\kappa_2}(y)(D_{y_1}^{m_1}D_{y_2}^{m_2}\Upsilon_{00}(y,\Omega))dy = (-1)^{\kappa_1+\kappa_2} D_{b_1}^{\kappa_1} D_{b_2}^{\kappa_2} D_{a_1}^{m_1} D_{a_2}^{m_2}[\beta_1\beta_2.\pi\Omega^4]. \tag{27}$$

Note that:

(28)

$$\begin{aligned}\beta_1\beta_2 &= \frac{1}{\pi\Omega^2}\cdot e^{-\frac{|b|^2}{\eta^2}}\cdot e^{-\frac{|a|^2}{\Omega^2}}\cdot e^{-\frac{\Omega^2|b|^2}{\eta^4}}\cdot e^{-\frac{|a|^2}{\Omega^2}}\cdot e^{-\frac{2\eta^2\Omega^2 a_1 b_1}{\Omega^2\eta^4}}.e^{-\frac{2\eta^2\Omega^2 a_2 b_2}{\Omega^2\eta^4}}\\ &= \frac{1}{\pi\Omega^2}e^{\frac{1}{\eta^2}[(-1+\frac{\Omega^2}{\eta^2})|b|^2+\frac{2(a_1b_1+a_2b_2)}{\eta^2}]},\end{aligned}$$

so

(29)

$$\begin{aligned}&\int_{\mathbb{R}^2} H_{\kappa_1,\kappa_2}(y)(D_{y_1}^{m_1}D_{y_2}^{m_2}\Upsilon_{00}(y,\Omega))dy =\\ &(-1)^{\kappa_1+\kappa_2}(\frac{1}{\pi\Omega})^2\pi\Omega^4 D_{b_1}^{\kappa_1}D_{b_2}^{\kappa_2}D_{a_1}^{m_1}D_{a_2}^{m_2}[e^{\frac{1}{\eta^2}[(-1+\frac{\Omega^2}{\eta^2})|b|^2+\frac{2(a_1b_1+a_2b_2)}{\eta^2}]}]|_{a=0,b=0}\\ &= (-1)^{\kappa_1+\kappa_2}\frac{\Omega^2}{\pi}D_{b_1}^{\kappa_1}D_{b_2}^{\kappa_2}D_{a_1}^{m_1}D_{a_2}^{m_2}[e^{\frac{1}{\eta^2}[(-1+\frac{\Omega^2}{\eta^2})|b|^2+\frac{2(a_1b_1+a_2b_2)}{\eta^2}]}]|_{a=0,b=0},\end{aligned}$$

but

$$\begin{aligned}&e^{\frac{1}{\eta^2}[(-1+\frac{\Omega^2}{\eta^2})|b|^2+\frac{2(a_1b_1+a_2b_2)}{\eta^2}]} =\\ &\sum_{n=0}^{\infty}\frac{1}{\eta^{2n}}\cdot\frac{1}{n!}\sum_{r=0}^{n}\binom{n}{r}2^r\cdot(-1+\frac{\Omega^2}{\eta^2})^{n-r}(a_1b_1+a_2b_2)^r|b|^{2(n-r)}\\ &\sum_{n=0}^{\infty}\frac{1}{\eta^{2n}}\cdot\frac{1}{n!}\sum_{r=0}^{n}\binom{n}{r}2^r\cdot(-1+\frac{\Omega^2}{\eta^2})^{n-r}\left(\sum_{h_1=0}^{r}\binom{r}{h_1}(a_1b_1)^{h_1}(a_2b_2)^{r-h_1}\right)\\ &\times\left(\sum_{h_2=0}^{n-r}\binom{n-r}{h_2}(b_1)^{2h_2}(b_2)^{2(n-r-h_2)}\right) = \sum_{n=0}^{\infty}\frac{1}{\eta^{2n}}\cdot\frac{1}{n!}\sum_{r=0}^{n}\binom{n}{r}2^r\cdot(-1+\frac{\Omega^2}{\eta^2})^{n-r}\\ &\sum_{h_1=0}^{r}\sum_{h_2=0}^{n-r}\binom{r}{h_1}\binom{n-r}{h_2}(a_1)^{h_1}(a_2)^{r-h_1}(b_1)^{h_1+2h_2}(b_2)^{2(n-r-h_2)+r-h_1},\end{aligned}$$

so

(30)

$$\begin{aligned}&D_{b_1}^{\kappa_1}D_{b_2}^{\kappa_2}D_{a_1}^{m_1}D_{a_2}^{m_2}(e^{\frac{1}{\eta^2}[(-1+\frac{\Omega^2}{\eta^2})|b|^2+\frac{2(a_1b_1+a_2b_2)}{\eta^2}]})|_{a=0,b=0} =\\ &\Bigg[\sum_{n=0}^{\infty}\frac{1}{\eta^{2n}}.\frac{1}{n!}\sum_{r=0}^{n}\binom{n}{r}2^r.(-1+\frac{\Omega^2}{\eta^2})^{n-r}\sum_{h_1=0}^{r}\sum_{h_2=0}^{n-r}\binom{r}{h_1}\binom{n-r}{h_2}\\ &\times\frac{h_1!}{(h_1-m_1)!}(a_1)^{h_1-m_1}.\frac{(r-h_1)!}{(r-h_1-m_2)!}.(a_1)^{(r-h_1-m_2)}.\frac{(h_1+2h_2)!}{(h_1+2h_2-\kappa_1)!}\\ &\times(b_1)^{h_1+2h_2-\kappa_1}\frac{(2(n-r-h_2)+r-h_1)!}{(2(n-r-h_2)+r-h_1-\kappa_2)!}.(b_2)^{2(n-r-h_2)+r-h_1-\kappa_2}\Bigg]\Bigg|_{a=0,b=0}.\end{aligned}$$

Assume $h_1 = m_1$, $r = m_1 + m_2$, $h_2 = \frac{\kappa_1 - m_1}{2}$, $n = \frac{m_1+\kappa_1+m_1+\kappa_2}{2}$, and define:

$$A[\kappa_1,\kappa_2,m_1,m_2;\eta,\Omega] = \begin{cases} \frac{\Omega^2 g\alpha}{\pi} \cdot \frac{2^{m_1+m_2-\kappa_1-\kappa_2}}{\eta^{(m_1+m_2-\kappa_1-\kappa_2+2)}} & \text{if } \kappa_1 - m_1 \\ \times(-1+\frac{\Omega^2}{\eta^2})^{\frac{\kappa_1+\kappa_2-m_1-m_2}{2}} & \text{and } \kappa_2 - m_2 \\ \times\frac{1}{(\frac{\kappa_1\kappa_1-m_1}{2})!(\frac{\kappa_2-m_2}{2})!} & \text{is positive and even} \\ 0 & \text{otherwise} \end{cases}$$

References

[1] M.A Abolfathi, "On the stability of 2-dimensional Pexider quadratic functional equation in non-Archimedean spaces", *Journal of Mahani Mathematical Research*, (2022), 11(2), 31-34.

[2] HM. Srivastava, and M. Izadi. "The Rothe-Newton approach to simulate the variable coefficient convection-diffusion equations." *Journal of Mahani Mathematical Research Center*, (2022), 11(2), 141-157.

[3] M. Vanday Baseri, and H. Fatemidokht. "Survey of different techniques for detecting selfish nodes in MANETs." *Journal of Mahani Mathematical Research Center*, (2022), 11(2), 45-59.

[4] Md. Nur Alam, C. Tunç, New solitary wave structures to the (2+1)-dimensional KD and KP equations with spatio-temporal dispersion. *Journal of King Saud University-Science*. **32** (2020), 3400-3409.

[5] Md. Nur Alam, C. Tunç, The new solitary wave structures for the (2+1)-dimensional time-fractional Schrodinger equation and the space-time nonlinear conformable fractional Bogoyavlenskii equations. *Alexandria Engineering Journal*, **59** (2020), 2221-2232.

[6] T. Gallay, C. E. Wayne, *Invariant manifolds and the long-time asymptotics of the Navier-Stokes and vorticity equations on* $\mathbb{R}^2$, Arch. Ration. Mech. Anal., **163**, (2012), 209–258.

[7] H. V. Helmholtz, *Über Integrale der hydrodynamischen Gleichungen*, Journal fur die reine und angewandte Mathematik, **55**, (1858), 25–55.

[8] T. Hou, C. Li, *Global well-posedness of the viscous Boussinesq equations*, Discrete and Cont. Dyn. Syst, **12**, (2005), 1–12.

[9] M. kamandar, B. Raesi, *Point Vortex Dynamics for the 2D Boussinesq Equations Over the Tropics*, Iranian Journal of Science and Technology, Transactions A: Science (2022), 1-10.

[10] L. Kelvin, *On vortex motion*, Trans. R. Soc. Edimb **25**, (2000), 217-60.

[11] H. Lamb, *Hydrodynamics*, Cambridge Univ Press, Cambridge, 2002.

[12] T. Ma, Sh. wang, *Phase Transition Dynamics*, Springer, 2014.

[13] L. M. Milne-Thomson, *Theoretical hydrodynamics*, Dover Publications, 1998.

[14] R. J. Nagem, G. Sandri, D. Uminsky, C. E. Wayne, *Generalized Helmholtz–Kirchhoff model for two-dimensional distributed vortex motion*, SIAM Journal on Applied Dynamical Systems, **8**, (2019), 160–179.

[15] P. K. Newton, *The N-Vortex Problem: Analytical Techniques*, Springer-Verlag, 2019.

[16] S. Özer, T. Şengül, *Transitions of Spherical Thermohaline Circulation to Multiple Equilibria*, Journal of Mathematical Fluid Mechanics, **20**, (2018), 499-515.

[17] L. Prandtl, *Essentials of fluid dynamics*, Hafner Pub. Co., 2004.

[18] M. Sharifi, B. Raesi, *Vortex Theory for Two Dimensional Boussinesq Equations*, Applied Mathematics and Nonlinear Sciences 5, **2**, (2020), 67-84.

[19] S. G. L. Smith, R. J. Nagem, *Vortex pairs and dipoles*, Regular and Chaotic Dynamics, **18**, (2013), 194-201.

[20] D. Uminsky, C. E. Wayne, A. Barbaro, *A multi-moment vortex method for 2D viscous fluids*, Journal of Computational Physics, **231**, (2012), 1705–1727.

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