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DESIGNING A NEW CASE OF TWO-STAGE DEA MODEL ABOUT THE INDIRECT RELATION OF INFORMATION TECHNOLOGY INVESTMENT ON FIRM PERFORMANCE IN INTUITIONISTIC FUZZY ENVIRONMENT

A. Hamzehee

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Abstract. Data Envelopment Analysis (DEA) is a theoretical framework for performance analysis and efficiency measurement. Traditional DEA models, which measure the efficiency of simple decision-making with multiple inputs and outputs, have several weaknesses, one of which is the inability to consider intermediate variables. Therefore, Network Data Envelopment Analysis (NDEA) has been developed to address this issue, which is especially important the analysis of two-stage processes. Also, since real-world data often are non-deterministic and imprecise, fuzzy sets theory and intuitionistic fuzzy sets theory, which are well-equipped to handle such information, can be used to improve the performance of two-stage DEA models. In this study, firstly NDEA models are discussed and then multiplicative method of NDEA is stated to obtain the individual efficiencies and the overall efficiency of the two stages. Also, it is explained how these models can be modified with intuitionistic fuzzy coefficients, and finally is described how arithmetic operators for intuitionistic fuzzy numbers can be used for a conversion into crisp two-stage structures. This paper presents a new two-stage DEA model to study the indirect impact of information technology investment on firm performance operating based on fuzzy intuitionistic numbers. Using this model, the efficiency of the first and second stages of a two-stage decision-making and ultimately its overall efficiency can be estimated with due to intermediate variables. The proposed method is used to solve a numerical example containing 12 DMUs with intuitionistic fuzzy triangular number coefficients.

Keywords: Two-stage DEA, Intuitionistic fuzzy set, Intuitionistic fuzzy triangular number, Information technology.
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✉ a_hamzehee@iauk.ac.ir, ORCID: 0000-0003-2610-998X
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1. **Introduction**

Data Envelopment Analysis (DEA) is a mathematical programming methodology for analyzing the efficiency of decision-making units (DMUs) with multiple inputs and outputs [2, 6, 20]. Many of the conducted studies on this methodology have shown that DEA can be used in a wide variety of fields. In many real-world systems or DMUs, the output is not produced in a single stage but rather through a chain of production stages with one or more intermediary products. Traditional DEA methods are not well-equipped to evaluate such multistage production systems, because their evaluations are based only on initial inputs and final outputs and fail to incorporate the internal structure of such units into efficiency calculations. The efficiency of multistage production systems can be studied by the use of the network data envelopment analysis (NDEA) paradigm, where intermediate variables play a key role in efficiency evaluations [27].

Uncertain data have effect studying and solving in most of studies especially decision making and optimization problems. These data contain statistical or random, fuzzy, interval, rough and even a combination of the aforementioned imprecise data. Fuzzy sets theory, proposed by Zadeh [44], is an effective tool which deals with imprecision and uncertainty and appears to be successful in different fields and it is one of the most significant data in uncertainty. Its concepts have received a lot of attention in many fields of science, industry, and management. Researchers in the field of linear programming and DEA have also sought to update the traditional DEA models with fuzzy concepts to make them produce more realistic results for real-world problems. In addition, for its general benefits, applying the concepts of fuzzy set theory to DEA reduces the sensitivity of its results toward the input and output data. Bellman and Zadeh [3] developed the concept of fuzziness to facilitate qualitative analyses based on imprecise data. Accordingly, incorporating the concepts of fuzzy sets theory into DEA models provides a way to estimate the efficiency of DMUs based on imprecise data [26]. Cooper et al. [11] were the first ones to propose a solution for handling ambiguous data such as bounded data, ordinal data, and ratio bounded data in DEA. Later, Kao and Liu [29] developed a method for obtaining membership functions for interval efficiency from pessimistic and optimistic perspectives. They also developed a two-level programming model for estimating the membership functions of fuzzy observations in a conventional process. Triantis and Girod [41] used membership functions to convert fuzzy inputs and outputs to crisp (non-fuzzy) data and proposed a mathematical programming approach, which involved calculating and then averaging the efficiency scores from different values of membership functions. Lertworasirikul et al. [30] proposed a probabilistic DEA model for fuzzy applications.

We know that one of the main characteristics of the fuzzy sets theory is which the sum of membership and non-membership degrees of an element is equal to one. However, in cases there are some ambiguities in the measurement of a fuzzy criterion and the obtained data are too vague or insufficient for decision making, so the usage of fuzzy set theory is ill-suited for problem representation. In such cases, it is better to use another fuzzy theory called the Intuitionistic Fuzzy Sets (IFS) theory, which IFSs are characterized by a membership function and a non-membership function. IFS theory has been proposed by Atanassov [1] and has been used with many studies in framework DEA, optimization and decision making. In these themes, it can be mentioned to literatures of the different authors as Boran et al. [4], Li et al. [31], Parvathi and Malathi [34], Parvathi et al. [34], Eslaminasab and Hamzehee [14, 15] and Daneshvar [12]. One of the other cases of fuzzy sets is Pythagorean Fuzzy Sets (PFS) that has introduced by Yager [43]. The structure of PFS is similar to IFS and more powerful tool to solve uncertain problems. In the recent years, the concept of PFS has developed in the different frameworks of sciences. For instance, Luqman et al. [32] defined triangular pythagorean fuzzy numbers and discussed digraph and matrix approach for risk evaluations under Pythagorean fuzzy information. But in the DEA context, in one study Hajiagha et al. [22] proposed an intuitionistic fuzzy DEA model with intuitionistic fuzzy outputs and used it to evaluate financial and credit institutions. In another study, Puri and Yadav [36] developed an intuitionistic fuzzy DEA model for the evaluation of the banking sector in India.

More of the used DEA models are criticized for treating units as black boxes and ignoring their internal processes, the efficiency of these processes and their relationships. This black box approach causes the analysis to miss a lot of valuable information about DMUs and limits their scope to the fundamental inputs and the ultimate outputs. In order to, Fare et al. [19] introduced NDEA and explained its importance for having a more accurate efficiency analysis of DMUs. In the following years, Sexton and Lewis [39] and Castelli et al. [5] used NDEA to examine the efficiency of DMUs with their internal structure taken into account. Kao [25, 27] investigated the use of NDEA in efficiency decomposition by considering serial and parallel structures for units. Tone and Tsutsui [40] introduced the slack-based measure (SBM) approach and also Cook et al. [10] introduced the additive efficiency decomposition method for NDEA. Kao and Huang [28] evaluated the efficiency of insurance companies in Taiwan by using independent and relational approach to efficiency calculation for two-stage DMUs and compared the results.

Chen and Zhu [7] developed an efficiency model that identifies the efficient frontier of a two-stage production process linked by intermediate measures. They designed a two-stage model for evaluating the efficiency of information technology (IT) units, which is the basis of the developed model in this paper. Also, they characterized the indirect impact of IT on firm performance and identified the efficient frontier of two principal value-added stages related to IT investment and profit generation. Wherever IT has become a key enabler of business process reengineering, such that if an organization is to survive and continue and prosper in a rapidly changing of business environment while facing competition in a global marketplace [7]. So, the motivation of this paper is to develop two-stage DEA models in intuitionistic fuzzy environment based on the Chen and Zhu [7] model and also to transform the proposed model with the expected value of intuitionistic fuzzy numbers to a linear programming. Such that transformation of the two-stage DEA models of intuitionistic fuzzy numbers to a linear programming problem, that really have computational complexity, overcomes the limitations in using of these models in intuitionistic fuzzy environments. In Section2, some basic knowledge and preliminaries of FIS, IFN, TIFN and NDEA are presented. In Section3, the desirable two-stage DEA model in intuitionistic fuzzy environment is discussed. In Section4, a numerical example is given to solve and evaluate the proposed method. Finally, concluding remarks are given in Section5.

2. Preliminaries

In this section, the definition and operations of IFSs, IFNs and TIFNs are briefly reviewed.

2.1. **Intuitionistic fuzzy sets.** The IFSs can be stated as a generalization of the classic fuzzy sets. Atanassov [1] concluded that IFSs are extended from fuzzy sets theory and so he characterized them by a membership function and a non-membership function.
Let X be a reference set, then Atanassov defined IFS as $\tilde{A}^I$ in X with form:
$\tilde{A}^I = \{\prec x, \mu_{\tilde{A}^I}(x), \nu_{\tilde{A}^I}(x) \succ\}$ wherever
$\mu_{\tilde{A}^I}(x) : X \longrightarrow [0,1]$ and $\nu_{\tilde{A}^I}(x) : X \longrightarrow [0,1]$
such that $0 \leq \mu_{\tilde{A}^I}(x) + \nu_{\tilde{A}^I}(x) \leq 1, \ \forall x \in X,$
where $\mu_{\tilde{A}^I}(x)$ and $\nu_{\tilde{A}^I}(x)$ are the degree of membership and the degree of non-membership of the element $x \in X$ of the set $\tilde{A}^I$, respectively.
In addition, for each $x \in X$ intuitionistic index or the hesitancy degree of x, denoted by π_x is defined as $\pi_x = 1 - \mu_{\tilde{A}^I}(x) - \nu_{\tilde{A}^I}(x)$.

2.1.1. *Intuitionistic Fuzzy Number.*

Definition 2.1. [33] $\tilde{A}^I$ is called an intuitionistic fuzzy number (IFN), if:
1) It is normal, i.e., $\forall x_0 \in X \ \mu_{\tilde{A}^I}(x_0) = 1$ and $\nu_{\tilde{A}^I}(x_0) = 0$.
2) $\mu_{\tilde{A}^I}(x)$ is convex, i.e.,
$\mu_{\tilde{A}^I}\big(\lambda x_1 + (1-\lambda)x_2\big) \geq \min\{\mu_{\tilde{A}^I}(x_1), \mu_{\tilde{A}^I}(x_2)\} \ \forall x_1, x_2 \in X \ , \ \forall \lambda \in [0,1]$.
3) $\nu_{\tilde{A}^I}(x)$ is concave, i.e.,
$\nu_{\tilde{A}^I}\big(\lambda x_1 + (1-\lambda)x_2\big) \leq \max\{\nu_{\tilde{A}^I}(x_1), \nu_{\tilde{A}^I}(x_2)\} \ \forall x_1, x_2 \in X \ , \ \forall \lambda \in [0,1]$.

Definition 2.2. An intuitionistic fuzzy number is shown with $\tilde{A}^I$ as follows:
$\tilde{A}^I = \prec x, \mu_{\tilde{A}^I}(x), \nu_{\tilde{A}^I}(x) \succ = \big(a^1, a^2, a^3, a^4; b^1, b^2, b^3, b^4\big),$

such that degrees of membership $\mu_{\tilde{A}^I}(x)$ and non-membership $\nu_{\tilde{A}^I}(x)$ are as follow:

(1)
$$\mu_{\tilde{A}^I}(x)=\begin{cases} f(x) & a^1\le x<a^2\\ 1 & a^2\le x\le a^3\\ g(x) & a^3<x\le a^4\\ 0 & \text{otherwise}\end{cases},\ \nu_{\tilde{A}^I}(x)=\begin{cases} h(x) & b^1\le x<b^2\\ 0 & b^2\le x\le b^3\\ k(x) & b^3<x\le b^4\\ 1 & \text{otherwise}\end{cases}$$

where f and k are monotonically increasing functions and g and h are monotonically decreasing functions.

2.1.2. *Triangular Intuitionistic Fuzzy Number.*

Definition 2.3. [33] An IFN is called a triangular intuitionistic fuzzy number (TIFN) if its membership function $\mu_{\tilde{A}^I}(x)$ and non-membership function $\nu_{\tilde{A}^I}(x)$ are as follows:

(2)
$$\mu_{\tilde{A}^I}(x)=\begin{cases} \frac{x-a^1}{a^2-a^1} & a^1\le x<a^2\\ 1 & x=a^2\\ \frac{x-a^3}{a^2-a^3} & a^2<x\le a^3\\ 0 & \text{otherwise}\end{cases},\ \nu_{\tilde{A}^I}(x)=\begin{cases} \frac{x-a^2}{a^{1\prime}-a^2} & a^{1\prime}\le x<a^2\\ 0 & x=a^2\\ \frac{x-a^2}{a^{3\prime}-a^3} & a^2<x\le a^{3\prime}\\ 1 & \text{otherwise}\end{cases}$$

such that $a^{1\prime}\le a^1\le a^2\le a^3\le a^{3\prime}$. This TIFN is represented as follows: $\left(a^1,a^2,a^3;a^{1\prime},a^2,a^{3\prime}\right)$.

2.1.3. *Arithmetic operations on intuitionistic fuzzy numbers.* Consider two values of the TIFNs like $\tilde{A}^I$ and $\tilde{B}^I$ as follows:
$\tilde{A}^I=\left(a^1,a^2,a^3;a^{1\prime},a^2,a^{3\prime}\right)$ and $\tilde{B}^I=\left(b^1,b^2,b^3;b^{1\prime},b^2,b^{3\prime}\right)$
Then the following relationships hold:

(3)
$$(i):\tilde{A}^I\oplus\tilde{B}^I=\left(a^1+b^1,a^2+b^2,a^3+b^3;a^{1\prime}+b^{1\prime},a^2+b^2,a^{3\prime}+b^{3\prime}\right).$$
$$(ii):\tilde{A}^I\otimes\tilde{B}^I=\left(a^1b^1,a^2b^2,a^3b^3;a^{1\prime}b^{1\prime},a^2b^2,a^{3\prime}b^{3\prime}\right)$$
$$(iii):\text{for all } k\in\mathbb{R}:k\tilde{A}^I=\begin{cases}\left(ka^1,ka^2,ka^3;ka^{1\prime},ka^2,ka^{3\prime}\right) & k>0\\ \left(ka^3,ka^2,ka^1;ka^{3\prime},ka^2,ka^{1\prime}\right) & k<0\end{cases}$$

2.1.4. *Expected values of intuitionistic fuzzy numbers and their characteristics.*

Definition 2.4. If $\left(a^1,a^2,a^3,a^4;b^1,b^2,b^3,b^4\right)$ is an IFN due to Definition 2.2, then the expected interval of this number is defined as follows:

(4)
$$EI(\tilde{A}^I)=\left[E_L(\tilde{A}^I),E_R(\tilde{A}^I)\right],$$
where $E_L(\tilde{A}^I)=\frac{b^1+a^2}{2}+\frac{1}{2}\int_{b^1}^{b^2}h(x)dx-\frac{1}{2}\int_{a^1}^{a^2}f(x)dx$,
and $E_R(\tilde{A}^I)=\frac{a^3+b^4}{2}+\frac{1}{2}\int_{a^3}^{a^4}g(x)dx-\frac{1}{2}\int_{b^3}^{b^4}k(x)dx$.

Accordingly, the expected value based on this IFN, is defined as:

(5)
$$EV(\tilde{A}^I)=\frac{E_L(\tilde{A}^I)+E_R(\tilde{A}^I)}{2}.$$

Proposition 2.5. *If* $\tilde{A}^I = \left(a^1, a^2, a^3; a^{1'}, a^2, a^{3'}\right)$ *is a TIFN, then* $EI(\tilde{A}^I)$ *is equal to:*

$$EI(\tilde{A}^I) = \left[\frac{a^{1'} + 2a^2 + a^1}{4}, \frac{a^3 + 2a^2 + a^{3'}}{4}\right] \tag{6}$$

So, the expected value for a TIFN is equal to:

$$EV(\tilde{A}^I) = \frac{a^{1'} + a^1 + 4a^2 + a^3 + a^{3'}}{8} \tag{7}$$

Proposition 2.6. *The expected value is a linear operator. In other words, the following relationship holds for any two IFNs like* $\tilde{A}^I$ *and* $\tilde{B}^I$:

$$EV(\tilde{A}^I + \tilde{B}^I) = EV(\tilde{A}^I) + EV(\tilde{B}^I) \tag{8}$$

2.2. **NDEA and two-stage structures.** DEA is one of the most effective methods to evaluate and compare the efficiency of a group of DMUs. With the expansion of this field, multiple variants of DEA have been developed for a more accurate evaluation of the efficiency of certain types of DMUs. NDEA is one of these variants. The two-stage structure was first introduced by Fare [16] and gradually was developed by Fare et al. [9]. Fare and Grosskopf [18] were the primary ones to formulate the relationships of different production processes in the framework of NDEA. In their formulation, the hierarchical structure of activities was replaced with a network structure. Initially, these structures were designed such that only the outputs of the first stage could be as the inputs of the second stage. But later, with the expansion of the two-stage network structure, the second stage was allowed to receive the other inputs than the outputs of the first stage. In other words, a two-stage system uses some resources as the inputs of stage 1 to produce some outputs. One part of the outputs is used as the input of the second stage, and another part is used as the final outputs. Stage 2 uses the intermediate products and additional inputs to generate the final outputs. Efficiency evaluation of two-stage systems is a major topic in DEA.

3. Structure of new model and the proposed method

As mentioned, this study aims to expand the DEA model of Chen and Zhu [7] for an intuitionistic fuzzy environment. Therefore, it is only appropriate to discuss about this model.

3.1. **Statement of two-stage DEA model in crisp environment.** Let there are n numbers of two-stage DMUs with form $DMU_j (j = 1, \ldots, n)$ that need to be evaluated. Each DMU_j takes m inputs denoted by $X_{ij} (i = 1, \ldots, m)$ to produce D outputs denoted by $Z_{dj} (d = 1, \ldots, D)$ in the first stage. These outputs are the inputs of the second stage and are called the intermediate products. The outputs of the second stage are denoted by $Y_{rj} (r = 1, \ldots, s)$.
As an applied example for two-stage DEA models, consider the tradeoff between information technology (IT) investment and number of employees in the

banking industry such that IT investment and employees are as inputs and also profit and loans as outputs. Then, an IT investment strategy can be defined as a DMU or firm performance.
Firstly, by using the next Property as "convexity" and also property "inefficiency" are introduced two models input-oriented and output-oriented of DEA [2,6].
Property 1: For all $i = 1, \dots, m$ and $r = 1, \dots, s$, the values $\sum_{j=1}^{n} \lambda_j X_{ij}$ and $\sum_{j=1}^{n} \lambda_j Y_{rj}$ are possible inputs and outputs achievable by DMUs respectively, where $\forall\ j = 1, \dots, n,\ \lambda_j \geq 0$ and those are scalars that $\sum_{j=1}^{n} \lambda_j = 1$.

The input-oriented DEA model is formulated as follows:

$$\begin{aligned} &\theta_1^* = \min \theta_1 \\ &s.t. \\ &\textstyle\sum_{j=1}^{n} \lambda_j X_{ij} \leq \theta_1 X_{ij_0},\ i = 1, \dots, m \\ &\textstyle\sum_{j=1}^{n} \lambda_j Y_{rj} \geq Y_{rj_0},\ r = 1, \dots, s \\ &\textstyle\sum_{j=1}^{n} \lambda_j = 1 \\ &\lambda_j \geq 0,\ j = 1, \dots, n. \end{aligned} \tag{9}$$

Where X_{ij_0} is the i *th* input and Y_{rj_0} is the r *th* output of the j_0 *th* observation (DMU) under evaluation. In Model (9), the goal is to minimize input usage while keeping the outputs at their current levels.
Also, the output-oriented DEA model is formulated as follows:

$$\begin{aligned} &\theta_2^* = \max \theta_2 \\ &s.t. \\ &\textstyle\sum_{j=1}^{n} \lambda_j X_{ij} \leq X_{ij_0},\ i = 1, \dots, m \\ &\textstyle\sum_{j=1}^{n} \lambda_j Y_{rj} \geq \theta_2 Y_{rj_0},\ r = 1, \dots, s \\ &\textstyle\sum_{j=1}^{n} \lambda_j = 1 \\ &\lambda_j \geq 0,\ j = 1, \dots, n. \end{aligned} \tag{10}$$

Similarly, in Model (10), the goal is to maximize the output production while keeping the inputs at their current levels.

Remark 3.1. If $\theta_1^* = 1$, then the DMU_{j_0} is efficient. Otherwise, if $\theta_1^* < 1$ then the DMU_{j_0} is inefficient. Note that both models input-oriented (9) and output-oriented (10) identify the same efficient frontier because $\theta_1^* = 1$ if and only if $\theta_2^* = 1$.

Now, remember the example of the banking industry again, then due to this fact that IT is indirectly linked with firm performance (IT investment strategy), consider an indirect impact of IT on firm performance. This indirect impact can be because of the presence of an intermediate measure. For instance, funds from the bank customers as deposits can be intermediate measures which in turn are transformed to realize firm performance, namely banks use the deposits as a source of funds to provide loan (an output). Therefore, according to above example, in a two-stage DEA Model, the first stage uses inputs to produce outputs which are the same intermediate measures, and then these measures are used as inputs in the second stage to produce final outputs. The following Theorem is stated for intermediate measures.

Theorem 3.2. *If a measure like Z_d is treated as both an input and both an output, then the optimal value for Model (9) must be equal to one.*

Proof. See [7]. □

Theorem 3.2 indicates that although Models (9) and (10) can measure the efficiency in each stage. Those models cannot deal with the two-stage efficiency with intermediate measures in a single implementation. In fact, Theorem 3.2 shows that a measure cannot be treated as an input and an output simultaneously in DEA Model (9) or Model (10). In other word, in the previous example of the banking industry, this discussion shows that the relationship between IT investments and performance cannot be simply characterized by Model (9) or (10). Consequently, Wang et al. [42] exclude the intermediate measures in one of the DEA implementations and obtain an overall efficiency with respect to the inputs of the first stage and the outputs of the second stage. Then Chen and Zhu [7] established the following liner programming problem to obtain the overall efficiency of the DMUs according to the two-stage DEA model for each DMU_{j_0}.

$$
\begin{aligned}
&\theta = \min \quad w_1\alpha - w_2\beta \\
&s.t. \\
&\textstyle\sum_{j=1}^{n} \lambda_j X_{ij} \le \alpha X_{ij_0},\ i = 1, \ldots, m \\
&\textstyle\sum_{j=1}^{n} \lambda_j Z_{dj} \ge Z_{dj_0},\ d = 1, \ldots, D \\
&\textstyle\sum_{j=1}^{n} \mu_j Z_{dj} \le Z_{dj_0},\ d = 1, \ldots, D \\
&\textstyle\sum_{j=1}^{n} \mu_j Y_{rj} \ge \beta Y_{rj_0},\ r = 1, \ldots, s \\
&\lambda_j,\ \mu_j \ge 0, \qquad j = 1, \ldots, n \\
&\alpha, \beta \ge 0.
\end{aligned} \tag{11}
$$

In Model (11), all changes in the inputs and outputs of DMUs are denoted by α and β, respectively. These parameters measure the efficiency of DMU with a two-stage process. The weights w_1 and w_2 represent the importance of the first and second stages for the overall efficiency, respectively. In other words, this Model defines the efficient frontier of a two-stage process and of viewpoint applied it can characterize the indirect impact of IT on firm performance in a single linear programming problem.

Theorem 3.3. *([7]) In Model (11) if $\alpha^* = \beta^* = 1$, then there must exist an optimal solution such that $\lambda^*_{j_0} = \mu^*_{j_0} = 1$.*

Theorem 3.4. *([7]) In Mode (11) if $\alpha^* = \beta^* = 1$, then $\theta^*_1 = 1$ and $\theta^*_2 = 1$, where θ^*_1 and θ^*_2 are the optimal values for Models (9) and (10), respectively.*

Remark 3.5. Assume that α^* and β^* are the optimal efficiencies of Model (11), i.e. $\theta_1 = \alpha^*$ is the first stage efficiency based on a model with minimization type and $\theta^*_2 = \frac{1}{\beta^*}$ is the second stage efficiency based on a model with maximization type and the overall efficiency is obtained by combining these two models. With this assumption, the model of Chen and Zhu [7] can be used to find the inverse of the second-stage efficiency (because the goal is to minimize the input of the first stage without reducing its output). Therefore, according to the relation between the overall efficiency and the efficiency of individual stages in multiplicative method the overall efficiency of Model (11) can be obtained with the equation: $\theta^* = \theta^*_1 \times \theta^*_2 = \frac{\alpha^*}{\beta^*}$.

3.2. **Development of two-stage DEA model in Intuitionistic fuzzy environment.** Indisputable, in order to study the real-world problems, the use of fuzzy data or each type of the inexact data does not provide a benefit rather than crisp ones. But, since most of these real-world problems are not crisp, the methods of classical mathematics are not usually suitable for dealing with them. In fact, almost all concepts which we are using in natural language are vague and in modern times scholars are often faced with complex problems concerning uncertainty. In other words, the inputs and outputs of these real-world problems are not always deterministic and precise and some data and/or variables can only be expressed in vague verbal and subjective terms and thus have a fuzzy or intuitionistic fuzzy nature. The use of fuzzy sets in mathematical modeling is imperative for overcoming with the challenges of dealing with such data. Hence the IFNs can be used as a replacement of the data, parameters, variables and/or coefficients in the multiple problems.

According to the stated discussions, assume all of the input and output data and only the variables $\lambda_j, (j = 1, \ldots, n)$ and $\mu_j, (j = 1, \ldots, n)$ of the Model (11) are in the form of IFNs, then this Model can be rewritten to Model (12) as follows:

$$
(12)\qquad
\begin{aligned}
&\theta = \min \quad w_1\alpha - w_2\beta\\
&s.t.\\
&\textstyle\sum_{j=1}^{n} \tilde{\lambda}_j^I \tilde{X}_{ij}^I \lessapprox \alpha \tilde{X}_{ij_0}^I,\ i = 1,\dots,m\\
&\textstyle\sum_{j=1}^{n} \tilde{\lambda}_j^I \tilde{Z}_{dj}^I \gtrapprox \tilde{Z}_{dj_0}^I,\ d = 1,\dots,D\\
&\textstyle\sum_{j=1}^{n} \tilde{\mu}_j^I \tilde{Z}_{dj}^I \lessapprox \tilde{Z}_{dj_0}^I,\ d = 1,\dots,D\\
&\textstyle\sum_{j=1}^{n} \tilde{\mu}_j^I \tilde{Y}_{rj}^I \gtrapprox \beta \tilde{Y}_{rj_0}^I,\ r = 1,\dots,s\\
&\tilde{\lambda}_j^I,\ \tilde{\mu}_j^I \gtrapprox \tilde{0}^I, \qquad j = 1,\dots,n\\
&\alpha, \beta \geq 0,
\end{aligned}
$$

where the symbol $\sim^I$ represents unknown data and decision variables of type intuitionistic fuzzy and also $\tilde{0}^I$ is the intuitionistic fuzzy number equal to zero, namely $\tilde{0}^I = (0,0,0,0;0,0,0,0)$.

Remark 3.6. In the Models (11) and (12) α and β are two scalars which they represent rations of inputs contraction and outputs expansion respectively, according to $\alpha, \beta > 1$ or $\alpha, \beta < 1$. So it is better which they be crisp in Model (12). Also, it is possible that in a model all of the variables and coefficients in the objective function or constraints are not imprecise and only some of them are imprecise or IFS. Therefore, in the other cases of fuzziness in Model(12) like the variables α and β in the objective function, further research is needed to analyze and find solutions of the two-stage DEA problems in intuitionistic fuzzy environment.

Due to the proposed method for efficiency calculation, the two-stage DEA model can be expanded for TIFNs. Here, this is done by rewriting Model (12) based on TIFNs and the method of Puri and Yadav [36] to reach Model (13) .

$$\theta = \min \quad w_1\alpha - w_2\beta$$
$$s.t.$$
$$\sum_{j=1}^{n} \left(\lambda^1{}_j, \lambda^2{}_j, \lambda^3{}_j; \lambda^{1\prime}{}_j, \lambda^2{}_j, \lambda^{3\prime}{}_j\right) \otimes \left(X^1{}_{ij}, X^2{}_{ij}, X^3{}_{ij}; X^{1\prime}{}_{ij}, X^2{}_{ij}, X^{3\prime}{}_{ij}\right)$$
$$\lessapprox \alpha\left(X^1{}_{ij_0}, X^2{}_{ij_0}, X^3{}_{ij_0}; X^{1\prime}{}_{ij_0}, X^2{}_{ij_0}, X^{3\prime}{}_{ij_0}\right), \qquad i = 1, \ldots, m$$

$$\sum_{j=1}^{n} \left(\lambda^1{}_j, \lambda^2{}_j, \lambda^3{}_j; \lambda^{1\prime}{}_j, \lambda^2{}_j, \lambda^{3\prime}{}_j\right) \otimes \left(Z^1{}_{dj}, Z^2{}_{dj}, Z^3{}_{dj}; Z^{1\prime}{}_{dj}, Z^2{}_{dj}, Z^{3\prime}{}_{dj}\right)$$
$$\gtrapprox \left(Z^1{}_{dj_0}, Z^2{}_{dj_0}, Z^3{}_{dj_0}; Z^{1\prime}{}_{dj_0}, Z^2{}_{dj_0}, Z^{3\prime}{}_{dj_0}\right), \qquad d = 1, \ldots, D$$

$$(13) \quad \sum_{j=1}^{n} \left(\mu^1{}_j, \mu^2{}_j, \mu^3{}_j; \mu^{1\prime}{}_j, \mu^2{}_j, \mu^{3\prime}{}_j\right) \otimes \left(Z^1{}_{dj}, Z^2{}_{dj}, Z^3{}_{dj}; Z^{1\prime}{}_{dj}, Z^2{}_{dj}, Z^{3\prime}{}_{dj}\right)$$
$$\lessapprox \left(Z^1{}_{dj_0}, Z^2{}_{dj_0}, Z^3{}_{dj_0}; Z^{1\prime}{}_{dj_0}, Z^2{}_{dj_0}, Z^{3\prime}{}_{dj_0}\right), \qquad d = 1, \ldots, D$$

$$\sum_{j=1}^{n} \left(\mu^1{}_j, \mu^2{}_j, \mu^3{}_j; \mu^{1\prime}{}_j, \mu^2{}_j, \mu^{3\prime}{}_j\right) \otimes \left(Y^1{}_{rj}, Y^2{}_{rj}, Y^3{}_{rj}; Y^{1\prime}{}_{rj}, Y^2{}_{rj}, Y^{3\prime}{}_{rj}\right)$$
$$\gtrapprox \beta\left(Y^1{}_{rj_0}, Y^2{}_{rj_0}, Y^3{}_{rj_0}; Y^{1\prime}{}_{rj_0}, Y^2{}_{rj_0}, Y^{3\prime}{}_{rj_0}\right), \qquad r = 1, \ldots, s$$

$$\lambda^1{}_j, \lambda^2{}_j, \lambda^3{}_j, \lambda^{1\prime}{}_j, \lambda^2{}_j, \lambda^{3\prime}{}_j > 0, \qquad j = 1, \ldots, n$$

$$\mu^1{}_j, \mu^2{}_j, \mu^3{}_j, \mu^{1\prime}{}_j, \mu^2{}_j, \mu^{3\prime}{}_j > 0, \qquad j = 1, \ldots, n$$

$$\alpha, \beta \geq 0.$$

Using the multiplication operator of TIFNs in (3), Model (13) is rewritten into Model (14).

(14)

$$\theta = \min \quad w_1\alpha - w_2\beta$$

$s.t.$

$$\Big(\sum_{j=1}^{n} \lambda^1{}_j X^1{}_{ij}, \sum_{j=1}^{n} \lambda^2{}_j X^2{}_{ij}, \sum_{j=1}^{n} \lambda^3{}_j X^3{}_{ij}; \sum_{j=1}^{n} \lambda^{1\prime}{}_j X^{1\prime}{}_{ij}, \sum_{j=2}^{n} \lambda^2{}_j X^2{}_{ij}, \sum_{j=1}^{n} \lambda^{3\prime}{}_j X^{3\prime}{}_{ij} \Big)$$
$$\lessapprox \Big(\alpha X^1{}_{ij_0}, \alpha X^2{}_{ij_0}, \alpha X^3{}_{ij_0}; \alpha X^{1\prime}{}_{ij_0}, \alpha X^2{}_{ij_0}, \alpha X^{3\prime}{}_{ij_0} \Big), \qquad i = 1, \ldots, m$$

$$\Big(\sum_{j=1}^{n} \lambda^1{}_j Z^1{}_{dj}, \sum_{j=1}^{n} \lambda^2{}_j Z^2{}_{dj}, \sum_{j=1}^{n} \lambda^3{}_j Z^3{}_{dj}; \sum_{j=1}^{n} \lambda^{1\prime}{}_j Z^{1\prime}{}_{dj}, \sum_{j=2}^{n} \lambda^2{}_j Z^1{}_{dj}, \sum_{j=1}^{n} \lambda^{3\prime}{}_j Z^{3\prime}{}_{dj} \Big)$$
$$\gtrapprox \Big(Z^1{}_{dj_0}, Z^2{}_{dj_0}, Z^3{}_{dj_0}; Z^{1\prime}{}_{dj_0}, Z^2{}_{dj_0}, Z^{3\prime}{}_{dj_0} \Big), \qquad d = 1, \ldots, D$$

$$\Big(\sum_{j=1}^{n} \mu^1{}_j Z^1{}_{dj}, \sum_{j=1}^{n} \mu^2{}_j Z^2{}_{dj}, \sum_{j=1}^{n} \mu^3{}_j Z^3{}_{dj}; \sum_{j=1}^{n} \mu^{1\prime}{}_j Z^{1\prime}{}_{dj}, \sum_{j=2}^{n} \mu^2{}_j Z^1{}_{dj}, \sum_{j=1}^{n} \mu^{3\prime}{}_j Z^{3\prime}{}_{dj} \Big)$$
$$\lessapprox \Big(Z^1{}_{dj_0}, Z^2{}_{dj_0}, Z^3{}_{dj_0}; Z^{1\prime}{}_{dj_0}, Z^2{}_{dj_0}, Z^{3\prime}{}_{dj_0} \Big), \qquad d = 1, \ldots, D$$

$$\Big(\sum_{j=1}^{n} \mu^1{}_j Y^1{}_{rj}, \sum_{j=1}^{n} \mu^2{}_j Y^2{}_{rj}, \sum_{j=1}^{n} \mu^3{}_j Y^3{}_{rj}; \sum_{j=1}^{n} \mu^{1\prime}{}_j Y^{1\prime}{}_{rj}, \sum_{j=2}^{n} \mu^2{}_j Y^2{}_{rj}, \sum_{j=1}^{n} \mu^{3\prime}{}_j Y^{3\prime}{}_{rj} \Big)$$
$$\gtrapprox \Big(\beta Y^1{}_{rj_0}, \beta Y^2{}_{rj_0}, \beta Y^3{}_{rj_0}; \beta Y^{1\prime}{}_{rj_0}, \beta Y^2{}_{rj_0}, \beta Y^{3\prime}{}_{rj_0} \Big), \qquad r = 1, \ldots, s$$

$$\lambda^1{}_j, \lambda^2{}_j, \lambda^3{}_j, \lambda^{1\prime}{}_j, \lambda^{3\prime}{}_j > 0, \qquad j = 1, \ldots, n$$

$$\mu^1{}_j, \mu^2{}_j, \mu^3{}_j, \mu^{1\prime}{}_j, \mu^{3\prime}{}_j > 0, \qquad j = 1, \ldots, n$$

$$\alpha, \beta \geq 0.$$

As the coefficients of Model (14) indicate, this model works with IFNs (with 6 components). Therefore, the expected value of IFNs is used to convert this model to a crisp linear programming model. After obtaining the expected value from the objective function and constraints of Model (15), this model is turned into Model (15).

(15)

$$\theta = \min \quad w_1\alpha - w_2\beta$$

$s.t.$

$$EV\Big(\textstyle\sum_{j=1}^{n}\lambda^1{}_jX^1{}_{ij}, \sum_{j=1}^{n}\lambda^2{}_jX^2{}_{ij}, \sum_{j=1}^{n}\lambda^3{}_jX^3{}_{ij}; \sum_{j=1}^{n}\lambda^{1\prime}{}_jX^{1\prime}{}_{ij}, \sum_{j=2}^{n}\lambda^2{}_jX^2{}_{ij}, \sum_{j=1}^{n}\lambda^{3\prime}{}_jX^{3\prime}{}_{ij}\Big)$$
$$\leq EV\Big(\alpha X^1{}_{ij_0}, \alpha X^2{}_{ij_0}, \alpha X^3{}_{ij_0}; \alpha X^{1\prime}{}_{ij_0}, \alpha X^2{}_{ij_0}, \alpha X^{3\prime}{}_{ij_0}\Big), \qquad i = 1, \ldots, m$$

$$EV\Big(\textstyle\sum_{j=1}^{n}\lambda^1{}_jZ^1{}_{dj}, \sum_{j=1}^{n}\lambda^2{}_jZ^2{}_{dj}, \sum_{j=1}^{n}\lambda^3{}_jZ^3{}_{dj}; \sum_{j=1}^{n}\lambda^{1\prime}{}_jZ^{1\prime}{}_{dj}, \sum_{j=2}^{n}\lambda^2{}_jZ^1{}_{dj}, \sum_{j=1}^{n}\lambda^{3\prime}{}_jZ^{3\prime}{}_{dj}\Big)$$
$$\geq EV\Big(Z^1{}_{dj_0}, Z^2{}_{dj_0}, Z^3{}_{dj_0}; Z^{1\prime}{}_{dj_0}, Z^2{}_{dj_0}, Z^{3\prime}{}_{dj_0}\Big), \qquad d = 1, \ldots, D$$

$$EV\Big(\textstyle\sum_{j=1}^{n}\mu^1{}_jZ^1{}_{dj}, \sum_{j=1}^{n}\mu^2{}_jZ^2{}_{dj}, \sum_{j=1}^{n}\mu^3{}_jZ^3{}_{dj}; \sum_{j=1}^{n}\mu^{1\prime}{}_jZ^{1\prime}{}_{dj}, \sum_{j=2}^{n}\mu^2{}_jZ^1{}_{dj}, \sum_{j=1}^{n}\mu^{3\prime}{}_jZ^{3\prime}{}_{dj}\Big)$$
$$\leq EV\Big(Z^1{}_{dj_0}, Z^2{}_{dj_0}, Z^3{}_{dj_0}; Z^{1\prime}{}_{dj_0}, Z^2{}_{dj_0}, Z^{3\prime}{}_{dj_0}\Big), \qquad d = 1, \ldots, D$$

$$EV\Big(\textstyle\sum_{j=1}^{n}\mu^1{}_jY^1{}_{rj}, \sum_{j=1}^{n}\mu^2{}_jY^2{}_{rj}, \sum_{j=1}^{n}\mu^3{}_jY^3{}_{rj}; \sum_{j=1}^{n}\mu^{1\prime}{}_jY^{1\prime}{}_{rj}, \sum_{j=2}^{n}\mu^2{}_jY^2{}_{rj}, \sum_{j=1}^{n}\mu^{3\prime}{}_jY^{3\prime}{}_{rj}\Big)$$
$$\geq EV\Big(\beta Y^1{}_{rj_0}, \beta Y^2{}_{rj_0}, \beta Y^3{}_{rj_0}; \beta Y^{1\prime}{}_{rj_0}, \beta Y^2{}_{rj_0}, \beta Y^{3\prime}{}_{rj_0}\Big), \qquad r = 1, \ldots, s$$

$$\lambda^{3\prime}{}_j \geq \lambda^3{}_j \geq \lambda^2{}_j \geq \lambda^1{}_j \geq \lambda^{1\prime}{}_j > 0, \qquad j = 1, \ldots, n$$

$$\mu^{3\prime}{}_j \geq \mu^3{}_j \geq \mu^2{}_j \geq \mu^1{}_j \geq \mu^{1\prime}{}_j > 0, \qquad j = 1, \ldots, n$$

$$\alpha, \beta \geq 0.$$

Next, the Equation (7) is used to calculate the expected value of TIFNs. Using this Equation and Equation (8) in Proposition 2.6, Model (15) is turned into the following crisp linear programming model (16):

$$
\begin{aligned}
&\theta = \min \quad w_1\alpha - w_2\beta \\
&s.t. \\
&\tfrac{1}{8}\Big(\textstyle\sum_{j=1}^{n} \Big({\lambda^1}'_j {X^1}'_{ij} + \lambda^1{}_j X^1{}_{ij} + 4\lambda^2{}_j X^2{}_{ij} + \lambda^3{}_j X^3{}_{ij} + {\lambda^3}'_j {X^3}'_{ij} \Big) \Big) \\
&\le \tfrac{1}{8}\Big(\alpha {X^1}'_{ij_0} + \alpha X^1{}_{ij_0} + 4\alpha X^2{}_{ij_0} + \alpha X^3{}_{ij_0} + \alpha {X^3}'_{ij_0} \Big), \qquad i = 1, \ldots, m \\
&\tfrac{1}{8}\Big(\textstyle\sum_{j=1}^{n} \Big({\lambda^1}'_j {Z^1}'_{dj} + \lambda^1{}_j Z^1{}_{dj} + 4\lambda^2{}_j Z^2{}_{dj} + \lambda^3{}_j Z^3{}_{dj} + {\lambda^3}'_j {Z^3}'_{dj} \Big) \Big) \\
&\ge \tfrac{1}{8}\Big({Z^1}'_{dj_0} + Z^1{}_{dj_0} + 4Z^2{}_{dj_0} + Z^3{}_{dj_0} + {Z^3}'_{dj_0} \Big), \qquad d = 1, \ldots, D \\
(16) \quad &\tfrac{1}{8}\Big(\textstyle\sum_{j=1}^{n} \Big({\mu^1}'_j {Z^1}'_{dj} + \mu^1{}_j Z^1{}_{dj} + 4\mu^2{}_j Z^2{}_{dj} + \mu^3{}_j Z^3{}_{dj} + {\mu^3}'_j {Z^3}'_{dj} \Big) \Big) \\
&\le \tfrac{1}{8}\Big({Z^1}'_{dj_0} + Z^1{}_{dj_0} + 4Z^2{}_{dj_0} + Z^3{}_{dj_0} + {Z^3}'_{dj_0} \Big), \qquad d = 1, \ldots, D \\
&\tfrac{1}{8}\Big(\textstyle\sum_{j=1}^{n} \Big({\mu^1}'_j {Y^1}'_{rj} + \mu^1{}_j Y^1{}_{rj} + 4\mu^2{}_j Y^2{}_{rj} + \mu^3{}_j Y^3{}_{rj} + {\mu^3}'_j {Y^3}'_{rj} \Big) \Big) \\
&\ge \tfrac{1}{8}\Big(\beta {Y^1}'_{rj_0} + \beta Y^1{}_{rj_0} + 4\beta Y^2{}_{rj_0} + \beta Y^3{}_{rj_0} + \beta {Y^3}'_{rj_0} \Big), \qquad r = 1, \ldots, s \\
&{\lambda^3}'_j \ge \lambda^3{}_j \ge \lambda^2{}_j \ge \lambda^1{}_j \ge {\lambda^1}'_j > 0, \qquad j = 1, \ldots, n \\
&{\mu^3}'_j \ge \mu^3{}_j \ge \mu^2{}_j \ge \mu^1{}_j \ge {\mu^1}'_j > 0, \qquad j = 1, \ldots, n \\
&\alpha, \beta \ge 0.
\end{aligned}
$$

Finally, solving Model (16) gives the overall efficiency value (θ^*) of each DMU based on TIFN inputs.
Until hither, we discussed NDEA models and then stated multiplicative method of NDEA to obtain the individual efficiencies and the overall efficiency of the two stages. In the next Section, a numerical example is given to solve and evaluate the proposed method for an NDEA model with TIFNs and finally based on this method the overall efficiency is achieved.

4. Numerical Example

To illustrate the computational process of the proposed model, here, it is implemented on a data set of intuitionistic fuzzy data for 12 DMUs, such that each DMU is contained with 3 inputs in the first stage, 2 intermediate products, and 3 outputs in the second stage. The general model of these DMUs is illustrated in Figure 1.

The TIFN data of this example are presented in Tables 1, 2 and 3. As mentioned, the TIFN data in this Example and these Tables are arbitrary and expository to state of the proposed method. The proposed models of Puri and

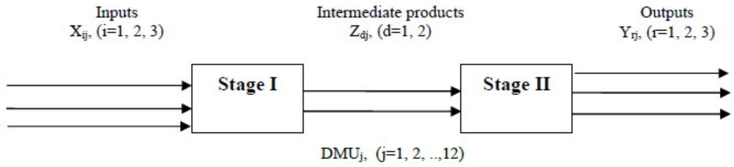

Figure 1. Two-stage model of the numerical example

Table 1. The assumed TIFN inputs in stage I for 12 DMUs

Input3	Input2	Input1	DMUs
(5.6, 6.1, 6.6; 5.1, 6.1, 7.6)	(4.87, 5.37, 5.87; 4.37, 5.37, 6.87)	(6, 6.5, 7; 5.5, 6.5, 8)	1
(3.94, 4.44, 4.94; 3.44, 4.44, 5.94)	(4.88, 5.38, 5.88; 4.38, 5.38, 6.88)	(5.33, 5.83, 6.33; 4.83, 5.83, 7.33)	2
(5.37, 5.87, 6.37; 4.87, 5.87, 7.37)	(4.79, 5.29, 5.79; 4.29, 5.29, 6.79)	(5.31, 5.81, 6.31; 4.81, 5.81, 7.31)	3
(5.38, 5.88, 6.38; 4.88, 5.88, 7.38)	(4.81, 5.31, 5.81; 4.31, 5.31, 6.81)	(5.5, 6, 6.5; 5, 6, 7.5)	4
(4.45, 4.95, 5.45; 3.95, 4.95, 6.45)	(4.23, 4.73, 5.23; 3.73, 4.73, 6.23)	(4.65, 5.15, 5.65; 4.15, 5.15, 6.65)	5
(4.36, 4.86, 5.36; 3.86, 4.86, 6.36)	(4.51, 5.01, 5.51; 4.01, 5.01, 6.51)	(4.45, 4.95, 5.45; 3.95, 4.95, 6.45)	6
(4.78, 5.28, 5.78; 4.28, 5.28, 6.78)	(3.71, 4.21, 4.71; 3.21, 4.21, 5.71)	(4.35, 4.85, 5.35; 3.85, 4.85, 6.35)	7
(4.64, 5.14, 5.64; 4.14, 5.14, 6.64)	(4.14, 4.64, 5.14; 3.64, 4.64, 6.14)	(4.92, 5.42, 5.92; 4.42, 5.42, 6.92)	8
(4.64, 5.14, 5.64; 4.14, 5.14, 6.64)	(4.14, 4.64, 5.14; 3.64, 4.64, 6.14)	(4.85, 5.35, 5.85; 4.35, 5.35, 6.85)	9
(4.63, 5.13, 5.63; 4.13, 5.13, 6.63)	(4.05, 4.55, 5.05; 3.55, 4.55, 6.05)	(4.6, 5.1, 5.6; 4.1, 5.1, 6.6)	10
(3.71, 4.21, 4.71; 3.21, 4.21, 5.71)	(3.38, 3.88, 4.38; 2.88, 3.88, 5.38)	(3.48, 3.98, 4.48; 2.98, 3.98, 5.48)	11
(5.43, 5.93, 6.43; 4.93, 5.93, 7.43)	(4.41, 4.91, 5.41; 3.91, 4.91, 6.41)	(5.31, 5.81, 6.31; 4.81, 5.81, 7.31)	12

Table 2. The assumed TIFN outputs in stage II for 12 DMUs

Output3	Output2	Output1	DMUs
(3.3, 3.8, 4.3; 2.8, 3.8, 5.3)	(4.18, 4.68, 5.18; 3.68, 4.68, 6.18)	(4, 4.5, 5; 3.5, 4.5, 6)	1
(3.36, 3.86, 4.36; 2.86, 3.86, 5.36)	(3.88, 4.38, 4.88; 3.38, 4.38, 5.88)	(3.53, 4.03, 4.53; 3.03, 4.03, 5.53)	2
(3.49, 3.99, 4.49; 2.99, 3.99, 5.49)	(3.88, 4.38, 4.88; 3.38, 4.38, 5.88)	(3.57, 4.07, 4.57; 3.07, 4.07, 5.57)	3
(4.62, 5.12, 5.62; 4.12, 5.12, 6.62)	(4.87, 5.37, 5.87; 4.37, 5.37, 6.87)	(5, 5.5, 6; 4.5, 5.5, 7)	4
(3.1, 3.6, 4.1; 2.6, 3.6, 5.1)	(3.03, 3.53, 4.03; 2.53, 3.53, 5.03)	(3.32, 3.82, 4.32; 2.82, 3.82, 5.32)	5
(3.39, 3.89, 4.39; 2.89, 3.89, 5.39)	(3.14, 3.64, 4.14; 2.64, 3.64, 5.14)	(3.38, 3.88, 4.38; 2.88, 3.88, 5.38)	6
(3.65, 4.15, 4.65; 3.15, 4.15, 5.65)	(3.47, 3.97, 4.47; 2.97, 3.97, 5.47)	(3.27, 3.77, 4.27; 2.77, 3.77, 5.27)	7
(5.19, 5.69, 6.19; 4.69, 5.69, 7.19)	(5.1, 5.6, 6.1; 4.6, 5.6, 7.1)	(4.78, 5.28, 5.78; 4.28, 5.28, 6.78)	8
(3.7, 4.2, 4.7; 3.2, 4.2, 5.7)	(3.89, 4.39, 4.89; 3.39, 4.39, 5.89)	(3.91, 4.41, 4.91; 3.41, 4.41, 5.91)	9
(3.77, 4.27, 4.77; 3.27, 4.27, 5.77)	(4.04, 4.54, 5.04; 3.54, 4.54, 6.04)	(3.78, 4.28, 4.78; 3.28, 4.28, 5.78)	10
(2.98, 3.48, 3.98; 2.48, 3.48, 4.98)	(3.45, 3.95, 4.45; 2.95, 3.95, 5.45)	(3.41, 3.91, 4.41; 2.91, 3.91, 5.41)	11
(3.9, 4.4, 4.9; 3.4, 4.4, 5.9)	(3.89, 4.39, 4.89; 3.39, 4.39, 5.89)	(3.6, 4.1, 4.6; 3.1, 4.1, 5.6)	12

Yadav [36] have been survived with three numerical examples by using the data as crisp, fuzzy number and intuitionistic fuzzy number, such that all of those data have been used arbitrary and expository. In order to handle intuitionistic fuzzy numbers, readers can recourse to many of the different literatures and papers. For instance, in the newest of the papers, recently Erdebilli et al. [13] proposed a decision-making method for dental supplier selection with TOPSIS method by using linear programming methodology. Wherever, the dental supplier selection is a multi-criteria group decision-making problem that contains many different criteria about the decision-makers generally ambiguous information. In their study, that some of the criteria are intuitionistic fuzzy numbers, decision-making for selection the most appropriate orthodontic brackets supplier was aimed, i.e., the purpose of it is to analyze the orthodontic brackets suppliers according to the specified measurements and to select appropriate the best.

Table 3. The assumed TIFN intermediate products for 12 DMUs

Intermediate2	Intermediate1	DMUs
(5.2, 5.7, 6.2; 4.7, 5.7, 7.2)	(2.65, 3.05, 3.45; 2.35, 3.05, 4.15)	1
(4.68, 5.18, 5.68; 4.18, 5.18, 6.68)	(2.66, 3.06, 3.46; 2.36, 3.06, 4.16)	2
(4.68, 5.18, 5.68; 4.18, 5.18, 6.68)	(4, 4.4, 4.8; 3.7, 4.4, 5.5)	3
(6.05, 6.55, 7.05; 5.55, 6.55, 8.05)	(4.44, 4.84, 5.24; 4.14, 4.84, 5.94)	4
(4.6, 5.1, 5.6; 4.1, 5.1, 6.6)	(2.05, 2.45, 2.85; 1.75, 2.45, 3.55)	5
(4.3, 4.8, 5.3; 3.8, 4.8, 6.3)	(2.68, 3.08, 3.48; 2.38, 3.08, 4.18)	6
(4.36, 4.86, 5.36; 3.86, 4.86, 6.36)	(2.57, 2.97, 3.37; 2.27, 2.97, 4.07)	7
(4.78, 5.28, 5.78; 4.28, 5.28, 6.78)	(2.92, 3.32, 3.72; 2.62, 3.32, 4.42)	8
(5.14, 5.64, 6.14; 4.64, 5.64, 7.14)	(2.92, 3.32, 3.72; 2.62, 3.32, 4.42)	9
(3.96, 4.46, 4.96; 3.46, 4.46, 5.96)	(2.66, 3.06, 3.46; 2.36, 3.06, 4.16)	10
(3.57, 4.07, 4.57; 3.07, 4.07, 5.57)	(1.42, 1.82, 2.22; 1.12, 1.82, 2.92)	11
(4.75, 5.25, 5.75; 4.25, 5.25, 6.75)	(2, 2.4, 2.8; 1.7, 2.4, 3.5)	12

Table 4 shows the efficiency of the first and second stages and the overall efficiency of DMUs according to linear programming Model (16), which has been solved by *Lingo software* with considering $w_1 = w_2 = 0.5$.

Corollary 4.1. *When the optimal solutions of Model (16) are* $\alpha^* = \beta^* = 1$, *the efficiency of the two stages can be considered equal to overall efficiency. Therefore, Model (16) correctly defines the efficient boundary of the two-stage production process. In the first stage for DMU2 and DMU4, the inputs can be reduced with the same outputs. In the second stage for DMU8 and DMU11, the outputs can be increased with the same inputs. Therefore, for the other DMUs, should be used to produce less inputs in the first stage and should be used to produce more outputs in the second stage, so can calculated the efficiency of the overall system.*

TABLE 4. The final values of the overall efficiency and the efficiencies of the first and second stages

$\theta^* = \frac{\alpha^*}{\beta^*}$	$\frac{1}{\beta^*}$	β^*	α^*	DMUs
0.699772	0.81196	1.231588	0.8618307	1
0.813901	0.813901	1.228650	1	2
0.751113	0.799513	1.250762	0.9394634	3
0.84121	0.84121	1.188764	1	4
0.730278	0.791503	1.263419	0.9226472	5
0.721113	0.810797	1.233354	0.8893881	6
0.758971	0.810794	1.233359	0.9360831	7
0.923053	1	1	0.9230526	8
0.779479	0.791147	1.263987	0.9852518	9
0.771198	0.960199	1.041451	0.8031646	10
0.935616	1	1	0.9356158	11
0.840126	0.967854	1.033214	0.8680303	12

5. Conclusion

DEA is a powerful tool for assessing the efficiency of production and service DMUs. However, conventional DEA models can only evaluate the efficiency of those units whose inputs and outputs are given in the form of deterministic numbers. Since many real-world data are imprecise, using the conventional DEA methods for real-world efficiency evaluations can lead to errors in decision-making. Therefore, in order to make sensible decisions based on realistic DEA-based efficiency assessments, it is essential to combine DEA with fuzzy logic as a means to handle imprecise information. Due to the introduction of the concept of decision-making in fuzzy environments by Bellman and Zadeh [3], many methods have been developed to incorporate fuzzy data into DEA. Sengupta [38] was the first one to investigate the application of fuzzy sets theory in DEA and used the principles of this theory to introduce fuzzy concepts into the objective function and constraint of conventional DEA models. In a study by Rostami Mal-Khalife and Mollaeian [37], they expanded the method of Chiang and Shiang [9] in order to produce fuzzy efficiency scales for the DMUs with fuzzy observations. By using DEA, this paper presented a model for assessing the efficiency of two-stage production units with intuitionistic fuzzy data based on intermediate variables, which helps managers to identify the inefficient DMUs and evaluate the overall efficiency of DMUs according to the efficiency of their first and second stages. Although the review of the literature suggests that a few studies have already used the IFS theory in DEA [12, 33, 36], this paper is the first study to investigate the application of IFS in two-stage DEA. Since in the recent years, many of NDEA models have been exposed by different authors in the multiple articles but only a few of these literatures and the studies of two-stage DEA models have been stated framework IFS, so this research has been done.

To compare the proposed method and model of this paper with the previous works, it can be mentioned to the various cases like (i) almost all of the proposed models in intuitionistic fuzzy environment of those works were only of type traditional DEA, (ii) further those models have not been used for all intuitionistic fuzzy data i.e., either only the coefficients of model were intuitionistic fuzzy or only the variables, (iii) the structure or type of the models in the previous works were different and etc. For example, Puri and Yadav [36] developed models to measure optimistic and pessimistic efficiencies of each DMU in intuitionistic fuzzy environment. To show the overall efficiency using optimistic and pessimistic situations together in intuitionistic fuzzy environments, they proposed a hybrid intuitionistic fuzzy DEA performance decision model. Now, to compare the proposed model of this study with the proposed models of Puri and Yadav is in the intuitionistic in fuzzy environment, it work has paid attention to the structure more than one stage, in which the source of inefficiency can be well identified, i.e., the basic crisp models were of type traditional DEA. While, in the studying of the structures of the two-stage models in intuitionistic fuzzy environment, the relationship between overall efficiency and the efficiency of the lower stages is less exposed to errors, and the optimal value of the intermediate variables is well determined.

In a paper of Javaherian et al. [23], the DEA model based on the network two-stage and slack variables and triangular intuitionistic fuzzy data was used to identify the efficiency of units. The importance of stated model of them was to measure the values of slack variables, which based on the Tone and Tsutsui model [40] and optimized the intermediate values for inefficient units and ultimately showed better inefficiency. Also, the optimized intermediate values were considered in their proposed model and thus were improved the overall efficiency of the system. The motivation of the other study from Javaherian et al. [24], was to develop two-stage DEA models in intuitionistic fuzzy environment with the assumption variable returns to scale based on the Chen et al. [8] model. In their work, by using expected value, the two-stage DEA models of all intuitionistic fuzzy data became the crisp linear programming problem and discussed with the evaluation of the performance of the units and their internal structures. In other word, the basic crisp model in [23] was a two-stage DEA model proposed by Tone and Tsutsui [40], such that their model was stated based on the slacks variables and they discussed with intermediate products formally and evaluated divisional efficiencies along with the overall efficiency

of DMUs. Also, the basic crisp model in [24] was a two-stage DEA model proposed by Chen et al. [8], that computed the overall efficiency under the assumption variable returns to scale. While, the basic crisp model in this study was a two-stage DEA model proposed by Chen and Zhu [7], that compounded two traditional DEA model by using two variables to represent rations of inputs contraction and outputs expansion. Briefly, the proposed method and procedure in this paper is similar to the used methods in the studies of Javaherian et al. [23, 24], but every which of the stated NDEA models in these works are different, because of the presence of the extension and expanse of NDEA models. Consequently, this study aimed to measure the efficiency of two-stage DMUs with intuitionistic fuzzy data to determine the minimum and maximum input and output levels of these units.
The proposed model has been solved for a numerical example with 12 DMUs, with 3 inputs in the first stage, 2 intermediate products, and 3 outputs in the second stage using Lingo software. This is partly certain due to the intuitionistic fuzzy nature of data in the proposed model, which it can be expected to produce more accurate results than the crisp model. For further use, this research can be investigated by other types of uncertain environments and also future studies are suggested to design multiplicative versions of two-stage DEA models with IFNs or TIFNs.

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Ali Hamzehee
Orcid number: 0000-0003-2610-998X
Department of Mathematics
Kerman Branch, Islamic Azad University
Kerman, Iran
Email address: hamzehee_ali@yahoo.com (a_hamzehee@iauk.ac.ir)