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# THE SMALL INTERSECTION GRAPH OF FILTERS OF A BOUNDED DISTRIBUTIVE LATTICE

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Abstract. Let $L$ be a lattice with 1 and 0. The small intersection graph of filters of $L$, denoted by $\Gamma(L)$, is defined to be a graph whose vertices are in one to one correspondence with all non-trivial filters of $L$ and two distinct vertices are adjacent if and only if the intersection of corresponding filters of $L$ is a small filter of $L$. In this paper, the basic properties and possible structures of the graph $\Gamma(L)$ are investigated. Moreover, the complemented property, the domination number and the planar property of $\Gamma(L)$ are considered.

*Keywords*: Lattice, small filter, small intersection graph.
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## 1. Introduction

The study of algebraic structures, using the properties of graph theory, tends to an exciting research topic in the last decade. Associating a graph with an algebraic structure allows us to obtain characterizations and representations of special classes of algebraic structures in terms of the graphs and vice versa. The purpose of the paper is to investigate the interplay between lattice properties of a lattice $L$ and properties of its small intersection graph. This will result in description of lattices in terms of some specific properties of those graphs. The main difficulty is figuring out what additional hypotheses the lattice or filter must satisfy to get similar results. Nevertheless, growing interest in developing the algebraic theory of lattices can be found in several papers and books (see for instance [6, 9–15]).

Beck [3] introduced the concept of the zero-divisor graph of rings. Since then, others have introduced and studied many researches in this area. One of the most important graphs which have been studied is the intersection graph. Bosak [4] defined the intersection graph of semigroups. Csákány and Pollák [8] studied the graph of subgroups of a finite group, in [8]. The intersection graph of ideals of a ring was considered by Chakrabarty, Ghosh, Mukherjee and Sen [7]. The intersection graph of ideals of rings and submodules of modules have been

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investigated by several authors (see for example [1, 16, 18, 20]).

A new kind of intersection graph of commutative rings based on non-small ideals was introduced and investigated in [11]. Since then, several kinds of intersection graph based on small object or non-small object was introduced for modules, rings and lattices (see for example [2, 17, 19]). Motivated from the structure of these graphs, we introduce the small intersection graph of filters of a lattice, to describe the structure of its filters based on its small intersection graph and vice versa (see for instance Theorems 3.20, 3.22, 4.11 and so on). Our goal is to consider algebraic properties of a lattice and its filters to investigate the structure of the corresponding graph theoretic.

Here is a brief outline of the article. Among many results in this paper, Section 2 concentrates on lattices whose graphs are empty, complete, $k$-regular and triangle-free or a tree. For example, Theorems 3.5 and 3.20 describe lattices $L$ with complete graph $\Gamma(L)$. We classify each lattice $L$ whose small intersection graph is an empty graph (Theorem 3.4), a tree (Theorem 3.17) and $k$-regular (Theorem 3.23). Theorem 3.22 describes lattices $L$ with the triangle-free graph $\Gamma(L)$. The diameter and girth of $\Gamma(L)$ is also described. The aim of Section 3 is to investigate the orthogonal vertices of $\Gamma(L)$ and their relationship with the nontrivial small filters of $L$. We collect some results on the domination number of $\Gamma(L)$ and study the conditions under which the domination number of $\Gamma(L)$ is finite. Also, we study the planar property of this graph.

## 2. Preliminaries

Let $G$ be a simple graph with vertex set $\mathcal{V}\ (G)$ and edge set $\mathcal{E}\ (G)$. For every vertex $v \in \mathcal{V}\ (G)$, the degree of $v$, denoted by $\deg_G(v)$, is defined the cardinality of the set of all vertices which are adjacent to $v$. The minimum degree of the graph $G$ is the minimum degree of its vertices and is denoted by $\delta(G)$. A graph G is said to be connected if there exists a path between any two distinct vertices, $G$ is a complete graph if every pair of distinct vertices of $G$ are adjacent and $K_n$ will stand for a complete graph with $n$ vertices. The graph $G$ is $k$-regular, if $\deg_G(v) = k < \infty$ for every $v \in \mathcal{V}\ (G)$. Let $u, v \in \mathcal{V}\ (G)$. We say that $u$ is a universal vertex of $G$ if $u$ is adjacent to all other vertices of $G$ and write $u \backsim v$ if $u$ and $v$ are adjacent. The distance $d(u, v)$ is the length of the shortest path from $u$ to $v$ if such path exists, otherwise, $d(a, b) = \infty$. The diameter of $G$ is $\mathrm{diam}(G) = \sup\{d(a, b) : a, b \in \mathcal{V}(G)\}$. The girth of a graph $G$, denoted by $\mathrm{gr}(G)$, is the length of a shortest cycle in $G$. If $G$ has no cycles, then $\mathrm{gr}(G) = \infty$. A tree is a connected graph which does not contain a cycle. A star graph is a tree consisting of one vertex adjacent to all the others. Note that a graph whose vertex set is empty is a null graph and a graph whose edge set is empty is an empty graph. We say that two distinct vertices $u$ and $v$ of the graph $G$ are orthogonal, denoted by $u \perp v$, if $u$ and $v$ are adjacent in $G$ and there is no vertex $w$ of $G$ which is adjacent to both $u$ and $v$. A graph $G$ is called complemented, if for each vertex $v$ of $G$, there is a vertex $w$ of $G$ (called a complement of $v$) such that $v \perp w$. By a dominating set $D$ in a graph $G$, we mean a subset $D$ of the vertex set $\mathcal{V}(G)$ such that every vertex in $\mathcal{V}(G) \setminus D$ is adjacent to at least one vertex in $D$. The domination number of $G$, written $\gamma(G)$, is the smallest cardinality of the cardinalities of the dominating sets of $G$. For the terminology and notation not defined here, the reader is referred to [5].

By a lattice we mean a poset $(L, \leq)$ in which every pair of elements $x, y$ has a g.l.b. (called the meet of $x$ and $y$, and written $x \wedge y$) and a l.u.b. (called the join of $x$ and $y$, and written $x \vee y$) in $L$. A lattice $L$ is complete if each subset $X$ of $L$ has a l.u.b. and a g.l.b. in $L$. Take $X := L$, we see that any non-empty complete lattice contains a least element 0 and greatest element 1 (in this case, we say that $L$ is a lattice with 0 and 1). A lattice $L$ is called a distributive lattice if $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ for all $a, b, c$ in $L$ (equivalently, $L$ is distributive if $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ for all $a, b, c$ in $L$). A non-empty subset $F$ of a lattice $L$ is called a filter, if for $a \in F$, $b \in L$, $a \leq b$ implies $b \in F$, and $x \wedge y \in F$ for all $x, y \in F$ (so if $L$ is a lattice with 1, then $1 \in F$ and $\{1\}$ is a filter of $L$). A subset $G$ of a filter $F$ is called a *subfilter* of $F$, if $G$ is a filter of $L$. A proper filter $P$ of $L$ is said to be prime if $x \vee y \in P$, then $x \in P$ or $y \in P$. A proper filter $\mathfrak{m}$ of $L$ is said to be maximal if for each filter $E$ of $L$, $\mathfrak{m} \subsetneqq E$ implies that $E = L$. The set of all maximal filters of a lattice $L$ will be denoted by $\mathrm{Max}(L)$. It is known that, if $F$ is a filter of a lattice $L$, then the radical of $F$, denoted by $\mathrm{Rad}(F)$, is the intersection of all maximal subfilters of $F$. A subfilter $G$ of a filter $F$ of $L$ is called essential in $F$ (written $G \trianglelefteq F$) if $G \cap H \neq \{1\}$ for any subfilter $H \neq \{1\}$ of $F$.

Let $H$ be subset of a lattice $L$. Then the filter generated by $H$, denoted by $T(H)$, is the intersection of all filters that is containing $H$. A filter $F$ is called finitely generated (resp. cyclic) if there is a finite subset $H$ (resp. $a \in F$) of $F$ such that $F = T(H)$ (resp. $T(\{a\})$) [6].

**Definition 2.1.** [13] A lattice $L$ is called semisimple, if for each proper filter $F$ of $L$, there exists a filter $G$ of $L$ such that $L = T(F \cup G) = F \wedge G$ and $F \cap G = \{1\}$. In this case, we say that $F$ is a direct meet of $L$, and we write $L = F \odot G$. A filter $F$ of $L$ is called a semisimple filter, if every subfilter of $F$ is a direct meet. A simple filter is a filter that has no filters besides the $\{1\}$ and itself.

Let $\Lambda = \{F_i : i \in I\}$ be a set of filters of a distributive lattice $L$. Then it is easy to see that $\bigwedge_{i \in I} F_i = \{\bigwedge_{i \in I'} f_i : f_i \in F_i, I' \subset I, I' \text{ is finite}\}$ is a filter of $L$ (if $\Lambda = \emptyset$, then we set $\bigwedge_{i \in I} F_i = \{1\}$).

**Definition 2.2.** [13] Let $L$ be a distributive lattice and $\Lambda = \{F_i : i \in I\}$ be a set of filters of $L$. Then $L = \bigodot_{i \in I} F_i$ is said to be a direct decomposition of $L$ into the meet of the filters $\{F_i : i \in I\}$ if

(1) $L = \bigwedge_{i\in I} F_i$ and

(2) $\{F_i : i \in I\}$ is independent (i.e., for each $j \in I$, $F_j \cap \bigwedge_{j\neq i\in I} F_i = \{1\}$).
For each filter $F$ of $L$, $\mathrm{Soc}(F) = \bigwedge_{i\in\Lambda} F_i$, where $\{F_i\}_{i\in\Lambda}$ is the set of all simple filters of $L$ contained in $F$.

**Lemma 2.3.** [6,15] *Let $L$ be a lattice.*

(1) *A non-empty subset $F$ of $L$ is a filter of $L$ if and only if $x \vee z \in F$ and $x \wedge y \in F$ for all $x, y \in F$, $z \in L$. Moreover, since $x = x \vee (x \wedge y)$, $y = y \vee (x \wedge y)$ and $F$ is a filter, $x \wedge y \in F$ gives $x, y \in F$ for all $x, y \in L$.*

(2) *If $L$ is distributive and $F_1, F_2$ are filters of $L$, then $F_1 \wedge F_2 = \{a \wedge b : a \in F_1, b \in F_2\}$ is a filter of $L$, $F_1, F_2 \subseteq F_1 \wedge F_2$ (if $x \in F_1$, then $x = x \wedge 1 \in F_1 \wedge F_2$) and if $F_1 \subseteq F_2$, then $F_1 \wedge F_2 = F_2$.*

We need the following lemmas (Lemmas 2.4 and 2.5) proved in [9,10].

**Lemma 2.4.** *Let $A$ be an arbitrary non-empty subset of $L$. Then $T(A) = \{x \in L : a_1 \wedge a_2 \wedge \cdots \wedge a_n \leq x$ for some $a_i \in A$ $(1 \leq i \leq n)\}$. Moreover, if $F$ is a filter and $A$ is a subset of $L$ with $A \subseteq F$, then $T(A) \subseteq F$ and $T(F) = F$.*

**Lemma 2.5.** *Let $F$, $G$ and $H$ be filters of a distributive lattice $L$. Then:*

(1) *If $F \subseteq G$, then $G \cap (F \wedge H) = F \wedge (G \cap H)$;*

(2) *$T(G \cup F) = F \wedge G$;*

**Definition 2.6.** (see [10]) Let $L$ be a lattice.

(1) A filter $G$ of $L$ is called small in $L$, written $G \ll L$, if for every filter $H$ of $L$, the equality $G \wedge H = L$ implies $H = L$.

(2) A filter $F$ of $L$ is called *hollow* if $F \neq \{1\}$ and every proper subfilter of $F$ is small in $F$. A lattice $L$ is said to be *hollow*, provided that each proper filter of $L$ is small in $L$.

(3) A filter $F$ of $L$ is called *local* if it has exactly one maximal subfilter that contains all proper subfilters. A lattice $L$ is said to be *local*, it has exactly one maximal filter.

(4) A filter $F$ of $L$ is called *uniserial* if the set of all subfilters of $F$ is a chain with respect to inclusion. Similarly, a *uniserial lattice* can be defined.

We need the following lemmas proved in [10] and [13].

**Lemma 2.7.** *Let $L$ be a distributive lattice with 1. Then*

(1) *If $A \ll L$ and $C \subseteq A$, then $C \ll L$.*

(2) *If $A, B$ are filters of $L$ with $A \ll B$, then $A \ll L$.*

(3) *If $F_1, \ldots, F_n$ are small filters of $L$, then $\bigwedge_{i=1}^{n} F_i$ is also small in $L$.*

(4) *If $A, B, C$ and $D$ are filters of $L$ with $A \ll B$ and $C \ll D$, then $A \wedge C \ll B \wedge D$.*

(5) $\mathrm{Rad}(L) = \bigwedge_{G\ll L} G$.

**Lemma 2.8.** *Let $L$ be a distributive lattice with 1.*

(1) *$L$ is semisimple if and only if $L$ contains no proper essential filter. In this case, every filter of $L$ is semisimple.*

(2) *If* $L$ *has no proper essential filter, then* $\mathrm{Rad}(L) = \{1\}$.

(3) *Let* $\mathfrak{m}$ *be a maximal filter of* $L$. *If* $L = \mathfrak{m} \odot G$ *for some filter* $G$ *of* $L$, *then* $G$ *is simple.*

(4) *Let* $L$ *be a distributive lattice with* 1 *and* 0. *If* $\mathrm{Rad}(L) = \{1\}$ *and* $\mathrm{Max}(L)$ *is a finite set, then* $L$ *is semisimple.*

Similar to the definition of small intersection graph of rings, modules and lattices, we introduce the small intersection graph of filters of a lattice [2,17,19].

**Definition 2.9.** Let $L$ be a lattice. The small intersection graph $\Gamma(L)$ of $L$ is a graph whose vertices are all non-trivial filters (i.e. different from $\{1\}$ and $L$) of $L$ and two distinct filters $F$ and $G$ are adjacent if and only if $F \cap G \ll L$.

## 3. Basic properties of $\Gamma(L)$

Throughout this paper, we shall assume unless otherwise stated, that $L$ is a distributive lattice with 1 and 0. In this section, we collect some basic properties concerning the graph $\Gamma(L)$. Let us begin this section with the following easy observation:

**Lemma 3.1.** *If* $H$ *is a filter of the lattice* $L$, *then* $H \ll L$ *if and only if* $H \subseteq \mathrm{Rad}(L)$. *In particular,* $\mathrm{Rad}(L) \ll L$.

*Proof.* It is straightforward. □

**Lemma 3.2.** *For the lattice* $L$, *the following conditions hold:*

(1) *If* $\Gamma(L)$ *has a universal vertex* $G$ *which is not small in* $L$, *then* $G$ *is a maximal filter.*

(2) *If* $\Gamma(L)$ *has a unique universal vertex* $S$, *then* $S$ *is a simple filter.*

*Proof.* (1) Let $\mathfrak{m}$ be a maximal filter of $L$ such that $G \subseteq \mathfrak{m}$. Then $G$ is a universal vertex which gives $G = G \cap \mathfrak{m}$ is small in $L$ that is impossible. Thus $G = \mathfrak{m}$.

(2) If $S$ is not simple, then there exists a proper filter $K$ of $L$ such that $\{1\} \subsetneqq K \subsetneqq S$. Since $S$ is an universal vertex, we deduce that $K = K \cap S \ll L$ and so for each non-trivial filter $F$ of $L$, $F \cap K \subseteq K$ gives $K \cap F \ll L$ by Lemma 2.7 (1); hence $K$ is a universal vertex of $\Gamma(L)$ which is impossible. Thus $S$ is simple. □

**Proposition 3.3.** *The graph* $\Gamma(L)$ *is a null graph if and only if* $L$ *is a simple lattice.*

*Proof.* The proof is clear.

□

Henceforth, we will assume that all considered lattices $L$ are not simple, since all definitions of graph theory are for non-null graphs [5]. In the next theorem, we classify all lattices $L$ whose small intersection graph is an empty graph.

**Theorem 3.4.** *If $L$ has at least one simple filter, then $\Gamma(L)$ is an empty graph if and only if $L$ has exactly one small simple filter.*

*Proof.* One side is clear. Let $\Gamma(L)$ be an empty graph. At first, we show that $L$ has exactly one simple filter. If $S_1$ and $S_2$ are two arbitrary distinct simple filters of $L$, then $S_1 \cap S_2 = \{1\} \ll L$ and so $S_1$ and $S_2$ are adjacent in $\Gamma(L)$ which is a contradiction. So suppose that $S$ is the unique simple filter of $L$. Let $S \wedge H = L$ for some filter $H$ of $L$. If $H \cap S = \{1\} \ll L$, then $H$ and $S$ are adjacent in $\Gamma(L)$ which is impossible. Thus $S \subseteq H$ which implies that $H = L$. Hence $S \ll L$. We claim that $S$ is the unique non-trivial filter of $L$. Assume to the contrary, that $F$ be a non-trivial filter of $L$ with $F \neq S$. Since $\Gamma(L)$ is an empty graph, $S \cap F \neq \{1\}$; so $S \subseteq F$. It follows that $F \cap S = S \ll L$ which implies that $F$ and $S$ are adjacent in $\Gamma(L)$, a contradiction. This completes the proof. □

**Theorem 3.5.** *Let $L$ be a lattice. Then the graph $\Gamma(L)$ is complete if one of the following conditions hold:*

(1) *$L$ is a hollow lattice;*

(2) *$L = S_1 \odot S_2$, where $S_1,\ S_2$ are simple filters.*

*Proof.* (1) Assume that $L$ is a hollow lattice and let $F$ and $G$ be two distinct vertices of the graph $\Gamma(L)$ (so $F, G \ll L$). Then $F \cap G \subseteq F$ gives $G \cap F \ll L$ by Lemma 2.7 (1), as needed.

(2) Suppose that $G$ is a non-trivial filter of $L$ and let $1 \neq x \in G$. So $x = s_1 \wedge s_2$ for some $s_1 \in S_1$ and $s_2 \in S_2$. Since $G$ is a filter, $s_1, s_2 \in G$ by Lemma 2.3. If $s_1 \neq 1$ and $s_2 \neq 1$, then $S_1 \subseteq G$ and $S_2 \subseteq G$ which gives $G = L$, a contradiction. Without loss of generality, let $s_1 \neq 1$ and $s_2 = 1$. Then $S_1 \subseteq G$ and $S_2 \cap G = \{1\}$. By Lemma 2.5 (1), $G = G \cap (S_1 \wedge S_2) = S_1 \wedge (G \cap S_2) = S_1$. Thus every non-trivial filter of $L$ is a simple filter. Let $F$ and $H$ be two distinct vertices of the graph $\Gamma(L)$. If $F \cap H \neq \{1\}$, then $F \cap H \subseteq F, H$ gives $F \cap H = F = H$ which is a contradiction. Hence $F \cap H = \{1\} \ll L$. Therefore $\Gamma(L)$ is a complete graph.

□

In the following example, it is shown that the converse of Theorem 3.5 is not necessarily true.

**Example 3.6.** *Let $L = \{0, a, b, c, 1\}$ be a lattice with $0 \leq a \leq c \leq 1$, $0 \leq b \leq c \leq 1$, $a \vee b = c$ and $a \wedge b = 0$. An inspection shows that the non-trivial filters of $L$ are $S_1 = \{1, a, c\}$, $S_2 = \{1, b, c\}$ and $S_3 = \{1, c\}$ with $S_3 \ll L$ but $S_1, S_2$ are not small in $L$, since $S_1 \wedge S_2 = L$. Moreover, $S_3$ is simple and $S_1, S_2$ are maximal filters of $L$ which are not simples. Since $S_1 \cap S_2 = S_1 \cap S_3 = S_2 \cap S_3 = S_3 \ll L$, we get that $\Gamma(L) = K_3$ is a complete graph.*

*Remark* 3.7. (1) By [10, Remark 2.19], uniserial and local filters are hollow; so if $L$ is uniserial (resp. is local), then $\Gamma(L)$ is a complete graph by Theorem 3.5.

(2) Since every small filter $F \neq \{1\}$ is a universal vertex, we have the subgraph induced by vertices which are small as filters of $L$ is a complete graph.

**Proposition 3.8.** *If the graph $\Gamma(L)$ is complete, then one of the following conditions hold:*
(1) *$L$ is a hollow lattice;*
(2) *$L = S_1 \wedge S_2$, where $S_1, S_2$ are different maximal filters.*

*Proof.* Assume that $\Gamma(L)$ is a complete graph and $L$ is not hollow. So there exists a non-trivial filter $S_1$ of $L$ such that it is not small in $L$ which implies that $S_1$ is a maximal filter by Lemma 3.2 (1). $S_1$ is not small gives there is a proper nonsmall maximal filter $S_2 \neq \{1\}$ of $L$ such that $L = S_1 \wedge S_2$ with $S_1 \neq S_2$, as required.

□

**Example 3.9.** *The collection of ideals of $\mathbb{Z}$, the ring of integers, forms a lattice under set inclusion which we shall denote by $L(\mathbb{Z})$ with respect to the following definitions: $m\mathbb{Z} \vee n\mathbb{Z} = (m, n)\mathbb{Z}$ and $m\mathbb{Z} \wedge n\mathbb{Z} = [m, n]\mathbb{Z}$ for all ideals $m\mathbb{Z}$ and $n\mathbb{Z}$ of $\mathbb{Z}$, where $(m, n)$ and $[m, n]$ are greatest common divisor and least common multiple of $m, n$, respectively. Note that, $L(\mathbb{Z})$ is a distributive complete lattice with least element the zero ideal and the greatest element $\mathbb{Z}$. By [15, Theorem 2.9 (ii)], $L(\mathbb{Z}) \setminus \{0\}$ is the only maximal filter of $L(\mathbb{Z})$ and so $L(\mathbb{Z})$ is a local lattice. It follows that $\Gamma(L(\mathbb{Z}))$ is a complete graph by Theorem 3.5 and Remark 3.7.*

**Theorem 3.10.** *The following statements are equivalent for a lattice $L$:*
(1) *The graph $\Gamma(L)$ is disconnected;*
(2) *$Rad(L) = \{1\}$ and $L$ has a non-trivial essential filter;*
(3) *The graph $\Gamma(L)$ has an isolated vertex.*

*Proof.* (1) ⇒ (2) Let $Rad(L) \neq \{1\}$. Since $\Gamma(L)$ is disconnected, $Rad(L) \neq L$. If $F_1$ and $F_2$ are two vertices of $\Gamma(L)$ such that there is not a path between them, then for each $1 \neq l \in Rad(L)$, $F_1 \backsim T(\{l\}) \backsim F_2$ is a path between $F_1$ and $F_2$, a contradiction. Therefore, we have $Rad(L) = \{1\}$. If $L$ has no non-trivial essential filter, then $F_1$ and $F_2$ are not essential filters. Hence $F_1 \cap H = \{1\}$ and $F_2 \cap K = \{1\}$ for some filters $K$ and $H$ of $L$.

If $K \cap H = \{1\}$, then $F_1 \backsim H \backsim K \backsim F_2$ is a path connecting $F_1$ and $F_2$, a contradiction. If $H \cap K \neq \{1\}$, then $F_1 \backsim H \cap K \backsim N_2$ is a path, a contradiction. Therefore $L$ has a non-trivial essential submodule.

(2) ⇒ (3) If $N$ is a non-trivial essential filter of $L$, then $N \cap K \neq \{1\}$ for any proper filter $K$ of $L$. We have $Rad(L) = \{1\}$; hence any two filters $J$ and $K$ are adjacent if and only if $J \cap K = \{1\}$. Thus $N$ cannot be adjacent to any filter of $L$ and so $N$ is an isolated vertex in $\Gamma(L)$.

(3) ⇒ (1) It is obvious. □

**Theorem 3.11.** *For a lattice $L$ the following conditions hold:*

(1) *If there exists a non-trivial filter of* $L$ *contained in* $\mathrm{Rad}(L)$*, then* $\Gamma(L)$ *is connected with* $\mathrm{diam}(\Gamma(L)) \leq 2$*;*

(2) *If* $\delta(\Gamma(L)) \geq 1$*, then* $\Gamma(L)$ *is connected with* $\mathrm{diam}(\Gamma(L)) \leq 3$.

*Proof.* (1) It is clear by Theorem 3.10.

(2) Let $F$ and $G$ be two non-adjacent vertices of $\Gamma(L)$. By assumption, there exist vertices $H, K$ such that $G \cap H \ll L$ and $F \cap K \ll L$. If $H \cap K \ll L$, then $F \backsim K \backsim H \backsim G$ is a path of length 3 in $\Gamma(L)$. If $H \cap K$ is not small in $L$, $F \backsim H \cap K \backsim G$ is a path of length 2 in $\Gamma(L)$. Hence $\Gamma(L)$ is connected with $\mathrm{diam}(\Gamma(L)) \leq 3$.

□

In the next example, we show that the condition "there exists a non-trivial filter of $L$ contained in $\mathrm{Rad}(L)$ or $L$ has at least one small non-trivial filter" is not superfluous in Theorem 3.11 (1).

**Example 3.12.** *Let* $D = \{a, b, c\}$*. Then* $L(D) = \{X : X \subseteq D\}$ *forms a distributive lattice under the set inclusion with the greatest element* $D$ *and the least element* $\emptyset$ *(note that if* $X, Y \in L(D)$*, then* $X \vee Y = X \cup Y$ *and* $X \wedge Y = X \cap Y$*). It can be easily seen that the set of all proper filters* $L(D)$ *is* $\{\{D\}, F_1, F_2, F_3, F_4, F_5, F_6\}$*, where* $F_1 = \{D, \{a, b\}\}$*,* $F_2 = \{D, \{a, c\}\}$*,* $F_3 = \{D, \{b, c\}\}$*,*

$$F_4 = \{D, \{a, c\}, \{a, b\}\{a\}\},$$

$F_5 = \{D, \{b, c\}, \{a, b\}\{b\}\}$ *and* $F_6 = \{D, \{a, c\}, \{c, b\}\{c\}\}$*. Then* $F_2 \odot F_5 = L(D)$*,* $F_1 \odot F_6 = L(D)$ *and* $F_3 \odot F_4 = L(D)$ *give* $L(D)$ *has no small non-trivial filter and* $\mathrm{d}(F_5, F_3) = 3$*. Moreover,* $L(D)$ *is semisimple with simple filters* $F_1, F_2$ *and* $F_3$ *and* $\Gamma(L(D))$ *is not a complete graph.*

**Proposition 3.13.** *Let* $L$ *be a lattice. If* $\Gamma(L)$ *is a connected graph, then* $diam(\Gamma(L)) \leq 3$.

*Proof.* Assume that $\Gamma(L)$ is connected. By Theorem 3.10, we have either $Rad(L) \neq \{1\}$ or $L$ is semisimple. Let $F_1$ and $F_2$ be two non-adjacent vertices of $\Gamma(L)$.

If $Rad(L) \neq \{1\}$, then there exists $1 \neq x \in Rad(L)$ such that $T(\{x\}) \ll L$. Hence $F_1 \cap T(\{x\})$ and $F_2 \cap T(\{x\})$ are small in $L$. Therefore $F_1 \backsim T(\{x\}) \backsim F_2$ is a path between $F_1$ and $F_2$ and so $d(F_1, F_2) = 2$.

If $L$ is semisimple, then $F_1 \cap H = \{1\}$ and $F_2 \cap K = \{1\}$ for some filters $K, H$ of $M$. If $H \cap L = \{1\}$, then $F_1 \backsim H \backsim K \backsim F_2$ is a path connecting $F_1$ and $F_2$ and so $d(F_1, F_2) = 3$. If $H \cap L \neq \{1\}$, then $F_1 \backsim H \cap K \backsim F_2$ is a path between $F_1$ and $F_2$ and hence $d(F_1, F_2) = 2$. □

**Corollary 3.14.** *Let* $L$ *be a semisimple lattice. Then the following conditions hold:*

(1) $\Gamma(L)$ *has no isolated vertex;*

(2) $\Gamma(L)$ *is connected with* $\mathrm{diam}(\Gamma(L)) \leq 3$.

*Proof.* (1) Let $G$ be a vertex of the graph $\Gamma(L)$. By assumption, there is a filter $F$ of $L$ such that $L = G \wedge F$ and $G \cap F = \{1\} \ll L$ which gives there is an edge between the vertex $G$ of $\Gamma(L)$ and another vertex of the graph. Thus $G$ is not an isolated vertex.

(2) This is a consequence of (1) and Proposition 3.13. □

The set of all non-trivial small filters of $L$ is denoted by $\mathcal{S}(L)$.

**Proposition 3.15.** *If* $|\mathcal{S}(L)| \geq 2$ *and* $|\mathcal{V}(\Gamma(L))| \geq 3$, *then* $\Gamma(L)$ *contains at least one cycle and* $\mathrm{gr}(\Gamma(L)) = 3$.

*Proof.* Let $F, G \in \mathcal{S}(L)$ and $H$ be a non-small filter of $L$. By Lemma 2.7, $F \cap G \ll L$, $H \cap G \ll L$ and $F \cap H \ll L$. Hence $F \backsim H \backsim G \backsim F$ is a cycle of length 3. □

In the following example, it is shown that the converse of Proposition 3.15 is not true.

**Example 3.16.** *Let $L$ be the lattice $L(D)$ and $\{\{D\}, F_1, F_2, F_3, F_4, F_5, F_6\}$ be the set of its proper filters as in Example 3.12. Then $L$ has no non-trivial small filter. However* $\mathrm{gr}(\Gamma(L)) = 3$ *(see Figure 1).*

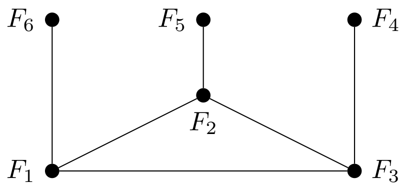


FIGURE 1. $\Gamma(L)$

**Theorem 3.17.** *If* $\Gamma(L)$ *is a tree, then* $|\mathcal{S}(L)| = 1$ *and either* $\Gamma(L) \cong K_1$ *or* $\Gamma(L)$ *is a star graph.*

*Proof.* Let $\Gamma(L)$ be a tree. We claim that $|\mathcal{S}(L)| = 1$. Assume to the contrary, that $G$ and $F$ are two non-trivial small filters of $L$ with $F \neq G$. As $F \cap G \ll L$ by Lemma 2.7 (1), $F \backsim F \cap G \backsim G \backsim F$ is a cycle of length 3 which is a contradiction. Thus $\mathcal{S}(L)$ has only one element $H$ with $H \neq \{1\}$. Let $K$ be an arbitrary vertex of $\Gamma(L)$. If $K = H$, then $\Gamma(L) \cong K_1$. If $K \neq H$, then $H \cap K \ll L$, as $H \cap K \subseteq H$. Let $\Delta = \{K_i : K_i \neq H\}_{i \in \Lambda}$. If $K_i, K_j$ are two distinct elements of $\Delta$, then $K_i \cap K_j \neq H$ gives $K_i \cap K_j$ is not small in $L$; so $K_i$ and $K_j$ are not adjacent in $\Gamma(L)$ and for each $i \neq j$, $K_i \backsim H \backsim K_j$ is a path in $\Gamma(L)$. Hence $\Gamma(L)$ is a star graph. □

In the following example, it is shown that the converse of Theorem 3.17 is not necessary true. It exhibits an example of a lattice $L$ with $|\mathcal{S}(L)| = 1$ such that $\Gamma(L)$ is not a tree.

**Example 3.18.** *Let $L$ be the lattice as in Example 3.6. Then $|\mathcal{S}(L)| = 1$. However $\Gamma(L) = K_3$ is not a tree.*

**Lemma 3.19.** (1) *If $L$ is a semisimple lattice, then $L$ has no small non-trivial filter. In particular, $L$ is not a hollow lattice.*
(2) *If $G$ is a direct meet of $L$ such that $G \ll L$, then $G = \{1\}$.*

*Proof.* (1) Assume to the contrary, that $G$ is a non-trivial filter of $L$ such that $G \ll L$. By assumption, $L = G \wedge K$ and $G \cap K = \{1\}$ for some filter $K$ of $L$. It follows that $K = L$; so $G = G \cap K = \{1\}$ which is impossible. The last assertion is clear.
(2) A similar argument in the proof of (1) shows that $G = \{1\}$. □

The next theorem gives a more explicit description of lattices with a complete small intersection graph.

**Theorem 3.20.** *Let $L$ be a semisimple lattice. Then $\Gamma(L)$ is a complete graph if and only if $L = S_1 \odot S_2$, where $S_1, S_2$ are simple filters.*

*Proof.* One side is clear by Theorem 3.5 and Lemma 3.19. Let $\Gamma(L)$ be a complete graph. Then by Proposition 3.8 and Lemma 3.19, $L = S_1 \wedge S_2$, where $S_1, S_2$ are different maximal filters. Since $L$ is semisimple, $L = S_2 \odot S$ for some filter $S$ of $L$. If $s \in S$, then $s = a \wedge b$ for some $a \in S_1$ and $b \in S_2$. Now $S$ is a filter gives $b \in S \cap S_2 = \{1\}$ and so $s = a \in S_1$. Hence $S \subseteq S_1$. If $S \subsetneqq S_1$, then $S = S \cap S_1 \ll L$ gives $S_2 = L$ which is a contradiction. Thus $S = S_1$ and $L = S_1 \odot S_2$. If $\{1\} \subsetneqq F \subsetneqq S_1$, then $S_2 \subsetneqq S_2 \wedge F \subseteq L$ gives $L = S_2 \wedge F$; so $S_2 = L$, a contradiction by $F = F \cap S_1 \ll L$. Thus $S_1$ is simple. Similarly, $S_2$ is simple, as needed. □

**Example 3.21.** *Let $L = \{0, a, b, 1\}$ be a lattice with $0 \leq a \leq 1$, $0 \leq b \leq 1$, $a \vee b = 1$ and $a \wedge b = 0$. It is clear that the non-trivial filters of $L$ are $S_1 = \{1, a\}$ and $S_2 = \{1, b\}$. Moreover, $L = S_1 \odot S_2$ is a semisimple lattice and $\Gamma(L) = K_2$ is a complete graph.*

**Theorem 3.22.** *Let $\text{Max}(L) = \{\mathfrak{m}_1, \mathfrak{m}_2\}$. If $\Gamma(L)$ is triangle-free, then $L$ is semisimple and $\Gamma(L) \cong K_2$ the complete graph with two vertices.*

*Proof.* By Theorem 2.8 (4), it suffices to show that $\text{Rad}(L) = \{1\}$. As $\mathfrak{m}_1 \neq \mathfrak{m}_2$, $\mathfrak{m}_1 \wedge \mathfrak{m}_2 = L$ which implies that $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are not small in $L$. Assume to the contrary, that $\text{Rad}(L) \neq \{1\}$. If $\text{Rad}(L) \ll L$, then $\mathfrak{m}_1 \cap \mathfrak{m}_2 \ll L$ gives $\mathfrak{m}_1, \mathfrak{m}_2$ and $\text{Rad}(L)$ would form a triangle, a contradiction. Hence $\text{Rad}(L)$ is not small in $L$. Now $\Gamma(L)$ is a connected triangle-free graph shows that $\text{Rad}(L) \cap F$ is small for some non-trivial filter $F$ of $L$. We may assume that $F \subseteq \mathfrak{m}_1$. Then $F \cap \mathfrak{m}_2$, $F$ and $\text{Rad}(L)$ would form a triangle which is impossible. Thus,

$\mathrm{Rad}(L) = \{1\}$ and so $L$ is semisimple. Finally, it is enough to show that $L$ has exactly two simple filters. Clearly, $\mathfrak{m}_1 \wedge \mathfrak{m}_2 = L$ and $\mathfrak{m}_1 \cap \mathfrak{m}_2 = \{1\}$. If $\mathfrak{m}_1$ is not simple, then there exists a non-trivial filter $S$ of $L$ such that $S \subsetneqq \mathfrak{m}_1$ with $S \nsubseteq \mathfrak{m}_2$. It follows that $S \wedge \mathfrak{m}_2 = L$. If $x \in \mathfrak{m}_1$, then $x = x \wedge 1 \in \mathfrak{m}_1 \wedge \mathfrak{m}_2 = S \wedge \mathfrak{m}_2$ which implies that $x = s \wedge b$ for some $s \in S$ and $b \in \mathfrak{m}_2$. Now $\mathfrak{m}_1$ is a filter gives $b \in \mathfrak{m}_1 \cap \mathfrak{m}_2 = \{1\}$ and so $x = s \in S$; hence $\mathfrak{m}_1 = S$, a contradiction. Similarly, $\mathfrak{m}_2$ is simple. Let $S$ be any simple filter of $L$. We show that $S \in \{\mathfrak{m}_1, \mathfrak{m}_2\}$. Suppose to the contrary, that $S \neq \mathfrak{m}_i$ $(1 \leq i \leq 2)$; hence $S \cap \mathfrak{m}_i = \{1\}$. If $x \in S$ and $x \neq 1$, then $x = x \wedge 1 \in L$ gives $x = s_1 \wedge s_2$ for some $s_1 \in \mathfrak{m}_1$ and $s_2 \in \mathfrak{m}_2$. Therefore, without loss of generality, we can assume that $s_1 \neq 1$. Then $S$ is a filter gives $s_1 \in \mathfrak{m}_1 \cap S = \{1\}$ which is impossible. This completes the proof. □

**Theorem 3.23.** *$\Gamma(L)$ is a $k$-regular graph for some positive integer $k$ if and only if $\Gamma(L)$ is a finite complete graph.*

*Proof.* One side is clear. Let $\Gamma(L)$ be a $k$-regular graph. Suppose $L$ contains a non-trivial small filter $F$. Then $F$ is adjacent to all other vertices of $\Gamma(L)$ by Lemma 2.7 (1) which implies that $k+1$ is the number of vertices of $\Gamma(L)$. Since $\Gamma(L)$ is $k$-regular, we deduce that $\Gamma(L)$ is a complete graph. So we may assume that $L$ does not have non-trivial small filter. We want to show that $\Gamma(L) \cong K_2$ the complete graph with two vertices. Let $\mathfrak{m}$ be a maximal filter of $L$. If $S$ is a simple filter of $L$ with $S \nsubseteq \mathfrak{m}$, then $\mathfrak{m} \subsetneqq S \wedge \mathfrak{m} \subseteq L$ and $\mathfrak{m}$ is maximal gives $S \wedge \mathfrak{m} = L$ and $S \cap \mathfrak{m} = \{1\} \ll L$. So $\mathfrak{m}$ and $S$ are adjacent in $\Gamma(L)$. As $\deg_{\Gamma(L)}(\mathfrak{m}) = k$, $k$ is the number of simple filters $S$ of $L$ such that $S \nsubseteq \mathfrak{m}$. Let $S_1, \ldots, S_k$ be such simple filters. If $K \subseteq \mathfrak{m}$, then $K \cap S_1 \subseteq \mathfrak{m} \cap S_1 = \{1\} \ll L$; so $S_1$ and $K$ are adjacent. It follows that $S_1$ is adjacent to $S_2, \ldots, S_k$ and to arbitrary proper filter of $L$ contained in $\mathfrak{m}$ which is a contradiction, since $\Gamma(L)$ is $k$-regular; hence $L$ has no proper filters contained in $\mathfrak{m}$. Therefore $S_1 = \mathfrak{m}$ is simple. Hence $L = S \odot S_1$, where $S, S_1$ are simple filters. Finally, it is enough to show that $L$ has exactly two simple filters. Let $S'$ be any simple filter of $L$. We show that $S' \in \{S, S_1\}$. Suppose to the contrary, that $S \neq S'$ and $S_1 \neq S'$; hence $S \cap S' = S_1 \cap S' = \{1\}$. If $1 \neq x \in S'$, then $x = x \wedge 1 \in L$ gives $x = a \wedge b$ for some $a \in S$ and $b \in S_1$. Therefore, without loss of generality, we can assume that $a \neq 1$. Then $S'$ is a filter gives $a \in S \cap S' = \{1\}$, a contradiction. This completes the proof. □

## 4. **Orthogonal vertices, domination number and planar property of $\Gamma(L)$**

In this section, we study the orthogonal vertices, the domination number, the planar property and the conditions under which the graph $\Gamma(L)$ is complemented. The set of all triangles of the graph $\Gamma(L)$ is denoted by $\mathcal{O}(L)$. Let us begin the following proposition:

**Proposition 4.1.** *If $|\mathcal{S}(L)| \geq 2$ and $|\mathcal{V}(\Gamma(L))| \geq 3$, then the following conditions hold:*

(1) *Every two small non-trivial filters of $L$ cannot be orthogonal vertices to each other;*

(2) $\mathcal{O}(L) \neq \emptyset$.

*Proof.* (1) If $S_1, S_2$ are two non-trivial small filters of $L$, then $S_1 \cap S_2 \ll L$ and for any vertex $S$ of $\Gamma(L)$, $S \cap S_1 \subseteq S_1$ and $S \cap S_2 \subseteq S_2$ gives $S \cap S_1 \ll L$ and $S \cap S_2 \ll L$, by Lemma 2.7 (1). Hence $S_1$ and $S_2$ cannot be orthogonal to each other.

(2) Assume that $S_1, S_2$ are two small non-trivial filters of $L$ and $S$ is a vertex of $\Gamma(L)$ such that $S_1 \neq S$ and $S_2 \neq S$, as $|\mathcal{V}(\Gamma(L))| \geq 3$. Then by an argument like that as above, $S_1, S$ and $S_2$ is a triangle. □

**Theorem 4.2.** *For the lattice $L$, the following statements are equivalent:*

(1) *The graph $\Gamma(L)$ has no triangle;*

(2) *Every two adjacent vertices of the graph $\Gamma(L)$ are orthogonal vertices;*

(3) *The lattice $L$ has at most one small non-trivial filter such that the intersection of every pair of the non-small non-trivial filters of $L$ is non-small.*

*Proof.* (1) $\Rightarrow$ (2) Assume to the contrary, that $F$ and $G$ are two adjacent vertices in $\Gamma(L)$ which are not orthogonal vertices. Then there exists a vertex $H \neq F, G$ of $\Gamma(L)$ such that $F \cap H \ll L$ and $H \cap G \ll L$; hence $F, H$ and $G$ would form a triangle which is impossible.

(2) $\Rightarrow$ (3) If there exist at least two small non-trivial filters of $L$, then they cannot be orthogonal vertices to each other by Proposition 4.1 (1) which is a contradiction.

(3) $\Rightarrow$ (1) At first, suppose that $L$ has no small non-trivial filter. By (3), the intersection of every pair of the non-small non-trivial filters of $L$ is non-small, we get that $\Gamma(L)$ has no triangle. If S is the only small non-trivial filters of $L$, then for every three arbitrary vertices $S_1, S_2$ and $S_3$ of $\Gamma(L)$ at least two of them are non-small. Set $S = S_3$. Then $S_1 \cap S_2$ is not small in $L$ gives $S_1 \backsim S \backsim S_2$ is a path in $\Gamma(L)$. If $S \notin \{S_1, S_2, S_3\}$, then $S_i \cap S_j$ is not small in $L$ for $i \neq j$, $i, j = 1, 2, 3$. Hence there is no triangle in the graph $\Gamma(L)$. □

**Corollary 4.3.** *If $L = G \odot K$ with $\mathrm{Rad}(G) \neq \{1\}$ and $\mathrm{Rad}(K) \neq \{1\}$, then the vertices $G$ and $K$ are not orthogonal in $\Gamma(L)$.*

*Proof.* By Lemma 3.1, $\mathrm{Rad}(L) \ll L$. By Lemma 2.7 (1), $G \cap \mathrm{Rad}(L) \ll L$ and $K \cap \mathrm{Rad}(L) \ll L$. Then $G, \mathrm{Rad}(L)$ and $K$ would form a triangle in $\Gamma(L)$, which is impossible, by Theorem 4.2. □

**Theorem 4.4.** *For the lattice $L$, the following conditions hold:*

(1) *If $L$ is a semisimple lattice, then the graph $\Gamma(L)$ is complemented;*

(2) *If $\mathrm{Rad}(L) \neq \{1\}$, then the graph $\Gamma(L)$ is not complemented.*

(3) *If $L = G \odot K$ with $\mathrm{Rad}(G) \neq \{1\}$ and $\mathrm{Rad}(K) \neq \{1\}$, then $\Gamma(L)$ is not a complemented graph.*

*Proof.* (1) Let $G$ be a vertex of $\Gamma(L)$. Then $L = G \wedge H$ and $G \cap H = \{1\} \ll L$ for some filter $H$ of $L$. By Lemma 3.19, since $L$ has no small non-trivial filter, there is no vertex $K$ of $\Gamma(L)$ such that $G \cap K \ll L$ and $H \cap K \ll L$. Thus $\Gamma(L)$ is complemented.

(2) By Lemma 3.1, $\{1\} \neq \mathrm{Rad}(L) \ll L$. We split the proof into two cases.

**Case 1.**: $\mathrm{Rad}(L)$ is simple. Since $\mathrm{Rad}(L) = \bigcap_{i\in\Lambda} \mathfrak{m}_i$, where $\mathfrak{m}_i$ is maximal filter of $L$ for each $i \in I$, we set $H = \bigcap_{j\neq i\in\Lambda\setminus\{j\}} \mathfrak{m}_i$. Then $\mathfrak{m}_j, H$ and $\mathrm{Rad}(L)$ would form a triangle; hence $\Gamma(L)$ is not complemented.

**Case 1.**: $\mathrm{Rad}(L)$ is not simple. Then there is a non-trivial filter $F$ of $L$ such that $F \subsetneqq \mathrm{Rad}(L)$; so $F \ll L$ by Lemma 3.1. It follows that for each vertex $K$ of $\Gamma(L)$, we would have a triangle with vertices $F, K$ and $\mathrm{Rad}(L)$. This completes the proof.

(3) This is a direct consequence of (2), if we take $L = G \odot K$. □

The following example shows that the converse of Theorem 4.4 (1) is not true, in general.

**Example 4.5.** *Let $L = \{0, a, b, 1\}$ be a lattice with $0 \leq a \leq b \leq 1$. Then non-trivial filters of $L$ are $\{1, a, b\}$ and $\{1, a\}$. Hence $\Gamma(L) = K_2$ is complemented. But, $L$ is not semisimple.*

**Theorem 4.6.** *Let $L = S_1 \odot S_2$, where $S_1$, $S_2$ are two simple filters. Then*

$$\gamma(\Gamma(L)) = 1.$$

*Proof.* Assume that $L = S_1 \odot S_2$, where $S_1, S_2$ are two simple filters. Then by Theorem 3.5, $\Gamma(L)$ is a complete graph. Let $G \in \mathcal{V}(\Gamma(L))$. Then for any $H \in \mathcal{V}(\Gamma(L)) \setminus \{G\}$, $G \cap H \ll L$; hence $\{G\}$ is a minimal dominating set in $\Gamma(L)$. Thus $\gamma(\Gamma(L)) = 1$. □

**Corollary 4.7.** *Assume that $|\mathcal{V}(\Gamma(L))| \geq 2$ and let $\mathrm{Rad}(L)$ be a non-trivial filter of $L$. If $L$ is a uniserial, local or hollow lattice, then every subset of $\mathcal{V}(\Gamma(L))$ is a dominating set in $\Gamma(L)$ and $\gamma(\Gamma(L)) = 1$.*

*Proof.* By Remark 3.7, $\Gamma(L)$ is a complete graph. Now the assertion follows from Theorem 4.6. □

*Remark* 4.8. Let $|\mathcal{V}(\Gamma(L))| \geq 2$ and $D \subseteq \mathcal{V}(\Gamma(L))$. Then:

(1) If $D$ either contains at least one small filter of $L$ or there is a vertex $S \in D$ such that $S \cap G = \{1\}$ for every vertex $G \in \mathcal{V}(\Gamma(L)) \setminus D$, then $D$ is a dominating set in $\Gamma(L)$.

(2) If $|\mathcal{S}(L)| \geq 1$, then for each small non-trivial filters $G$ of $L$, $\{G\}$ is a minimal dominating set; so $\gamma(\Gamma(L)) = 1$.

**Theorem 4.9.** *Let $D \subseteq \mathcal{V}(\Gamma(L))$. If $\mathrm{Rad}(L) \neq \{1\}$, $\mathrm{Soc}(L) \neq \{1\}$ and $\mathrm{Soc}(\mathrm{Rad}(L)) \in D$, then $D$ is a dominating set in $\Gamma(L)$ and $\gamma(\Gamma(L)) = 1$.*

*Proof.* By part (2) of Remark 4.8, it suffices to show that $\mathrm{Soc}(\mathrm{Rad}(L)) \ll L$. Put $H = \mathrm{Soc}(\mathrm{Rad}(L))$ and assume $L = H \wedge K$ for some filter $K$ of $L$. Take

$G := H \cap K$, we obtain $H = G \odot G'$ for some subfilter $G'$ of $H$ and $L = H \wedge K = (G \wedge G') \wedge K = G' \wedge K$; hence $L = K \odot G'$ (since if $x \in G' \cap K$, then $x \in K \cap H = G$ gives $x \in G \cap G' = \{1\}$). We claim that $G' = \{1\}$. Assume to the contrary, that $S$ is a simple subfilter of $G'$. Then $G' = S \odot S'$ for some subfilter $S'$ of $G'$ which implies that $L = K \odot S \odot S'$. Moreover, $S \ll L$, as $S \subseteq H \subseteq \mathrm{Rad}(L)$ by Lemma 3.1. Now the simple filter $S$ is a direct meet of $L$ and is small in $L$ and hence is $\{1\}$ by Lemma 3.19 (2) which is a contradiction. Therefore $G' = \{1\}$ and so $K = L$, showing $H \ll L$. □

**Corollary 4.10.** *Let $D \subseteq \mathcal{V}(\Gamma(L))$ and $\mathrm{Rad}(L) \neq \{1\}$. Then $D$ is a dominating set in $\Gamma(L)$ and $\gamma(\Gamma(L)) = 1$ if one of the following conditions hold:*

(1) *$\mathrm{Rad}(L) \in D$;*

(2) *There is a non-trivial filter $G$ of $L$ which is a direct meet of $L$ with $G \cap \mathrm{Rad}(L) \ll L$ and $\mathrm{Rad}(G) \in D$;*

(3) *There is a non-trivial filter $G$ of $L$ such that $G \subseteq \mathrm{Rad}(L)$.*

*Proof.* (1) By Lemma 3.1, $\mathrm{Rad}(L) \ll L$. As $\mathrm{Rad}(L) \in D$, the assertion follows from part 2 of Remark 4.8.

(2) By assumption, $L = G \odot G'$ for some filter $G'$ of $L$; so $\mathrm{Rad}(L) = \mathrm{Rad}(G) \odot \mathrm{Rad}(G')$. Since $G \cap \mathrm{Rad}(G') \subseteq G \cap G' = \{1\}$, we get $G \cap \mathrm{Rad}(L) = G \cap (\mathrm{Rad}(G) \wedge \mathrm{Rad}(G')) = \mathrm{Rad}(G) \wedge (G \cap \mathrm{Rad}(G')) = \mathrm{Rad}(G) \ll L$. Then part 2 of Remark 4.8 shows that (2) holds, as $\mathrm{Rad}(G) \in D$.

(3) By Lemma 3.1, $G \ll L$. Now the assertion follows from part 2 of Remark 4.8. □

A graph is said to be planar if it can be drawn in the plane so that its edges intersect only at their ends. A subdivision of a graph is a graph obtained from it by replacing edges with pairwise internally-disjoint paths. A remarkably simple characterization of planar graphs was given by Kuratowski in 1930, that says a graph is planar if and only if it contains no subdivision of $K_5$ or $K_{3,3}$ [5].

**Theorem 4.11.** *For the lattice $L$, the following conditions hold:*

(1) *If $|\mathcal{S}(L)| = 1$ or $|\mathcal{S}(L)| = 2$ and the intersection of every pair of non-small filters of $L$ is a non-small filter, then $\Gamma(L)$ is a planar graph;*

(2) *If $|\mathcal{S}(L)| \geq 3$ and $|\mathcal{V}(G(L)) \setminus \mathcal{S}(L)| \geq 3$, then $\Gamma(L)$ is not a planar graph.*

*Proof.* (1) By hypothesis, if $|\mathcal{S}(L)| = 1$, then $\Gamma(L)$ is a star graph which is planar and if $|\mathcal{S}(L)| = 2$, then $\Gamma(L)$ is planar, as the definition of a planar graph.

(2) Assume that $|\mathcal{S}(L)| \geq 3$ and $|\mathcal{V}(G(L)) \setminus \mathcal{S}(L)| \geq 3$, An inspection shows that $\mathcal{S}(L)$ and $\mathcal{V}(G(L)) \setminus \mathcal{S}(L)$ makes $K_{3,3}$ as a subgraph of $\Gamma(L)$ which is a contradiction. Thus $\Gamma(L)$ is not a planar graph. □

**Theorem 4.12.** *If $L = G \odot K$ with $\mathrm{Rad}(G) \neq \{1\}$ and $\mathrm{Rad}(K) \neq \{1\}$, then $\Gamma(L)$ is not a planar graph.*

*Proof.* Let $L = G \odot K$; so $G \cap K = \{1\} \ll L$. By [10, Proposition 2.16], $\mathrm{Rad}(L) = \mathrm{Rad}(G) \odot \mathrm{Rad}(K)$. It follows that $\mathrm{Rad}(G) \cap \mathrm{Rad}(K) = \{1\} \ll L$

and $G \cap \mathrm{Rad}(K) \subseteq G \cap K = \{1\}$, as $\mathrm{Rad}(K) \subseteq K$. By Lemma 2.5 (1), $G \cap \mathrm{Rad}(L) = G \cap (\mathrm{Rad}(G) \wedge \mathrm{Rad}(K)) = \mathrm{Rad}(G) \wedge (G \cap \mathrm{Rad}(K)) = \mathrm{Rad}(G) \ll L$, as $\mathrm{Rad}(L) \ll L$ by Lemma 3.1. Similarly, $\mathrm{Rad}(K) = K \cap \mathrm{Rad}(L) \ll L$. Hence $\{G, K, \mathrm{Rad}(G), \mathrm{Rad}(K), \mathrm{Rad}(L)\}$ makes $K_5$ as a subgraph $\Gamma(L)$ which is a contradiction. Thus $\Gamma(L)$ is not a planar graph. □

## 5. Conclusion

We give some closed connections between algebraic properties of a lattice and the graph theoretical properties of its small intersection graph. We proved that if $L$ is a semisimple lattice, then $\Gamma(L)$ is a complete graph if and only if $L = S_1 \odot S_2$, where $S_1, S_2$ are simple filters. Moreover, it is shown that every two adjacent vertices of the graph $\Gamma(L)$ are orthogonal vertices if and only if the lattice $L$ has at most one small non-trivial filter such that the intersection of every pair of the non-small non-trivial filters of $L$ is non-small. Further, we use the small filters to consider the planar property of $\Gamma(L)$.

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