Shahid Bahonar University of Kerman
Journal of Mahani Mathematical Research
Print ISSN: 2251-7952 Online ISSN: 2645-4505

ON SKEW POWER SERIES-WISE MCCOY RINGS

M. Zahiri ✉ and S. Zahiri

Article type: Research Article
(Received: 28 March 2022, Received in revised form: 30 May 2022)
(Accepted: 03 August 2022, Published Online: 08 August 2022)

Abstract. Let R be a ring with an endomorphism α. A ring R is a skew power series McCoy ring if whenever any non-zero power series $f(x) = \sum_{i=0}^{\infty} a_i x^i, g(x) = \sum_{j=0}^{\infty} b_j x^j \in R[[x;\alpha]]$ satisfy $f(x)g(x) = 0$, then there exists a non-zero element $c \in R$ such that $a_i c = 0$, for all $i = 0, 1, \ldots$. We generalized the results of [2] and investigate the relations between the skew power series ring and the standard ring-theoretic properties. Moreover, we obtain some characterizations for skew power series ring $R[[x;\alpha]]$, to be McCoy, zip, strongly AB and has Property (A).

Keywords: Noetherian ring, α-compatible ring, Skew Power series McCoy ring, Zip ring, Reversible ring.
2020 MSC: 16D15, 16D40, 16D70.

1. Introduction

Throughout the paper, all rings are associative with identity. Let α be a ring endomorphism of R. We denote $R[[x;\alpha]]$ to be the skew power series rings whose elements are the power series over R, the addition is defned as usual and the multiplication satisfies in the relation $xa = \alpha(a)x$ for any $a \in R$.
For notations we use $Ni\ell(R)$ and $C_{f(x)}$ for the set of all nilpotent elements of a ring R, the set in R consisting of all the "coefficients" of $f(x)$ where $f(x)$ is a power series, respectively. By $Z_\ell(R)$, $Z_r(R)$ and $Z(R)$, we mean respectively the set of all left zero-divisors of R, the set of all right zero-divisors of R and the set of all zero-divisors of R (i.e., $Z(R) = Z_\ell(R) \cup Z_r(R)$).

A ring R is *reversible* whenever it satisfies the condition $ab = 0 \Leftrightarrow ba = 0$ for $a, b \in R$. Note that for the class of reversible rings the set of all left annihilators of any element $a \in R$ coincide with set of its right annihilators. A ring R will be called *semicommutative* if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. Moreover, a ring is known *right* (resp., *left*) *duo* if every right (resp., left) ideal is an ideal. It is not hard to see that reversible as well (one-sided) duo rings are semicommutative.

Following [4], a ring R is *right McCoy* if $f(x)g(x) = 0$, then $f(x)c = 0$ for some non-zero $c \in R$, where $f(x), g(x)$ are non-zero polynomials in $R[x]$. *Left McCoy rings* are defined similary and they satisfy similar properties. A ring R is called *McCoy* if it is both left and right McCoy.

✉ m.zahiri@eghlid.ac.ir , ORCID: 0000-0001-5418-0746
DOI: 10.22103/jmmr.2022.19231.1227
Publisher: Shahid Bahonar University of Kerman

How to cite: M. Zahiri, S. Zahiri, *On skew power series-wise McCoy rings*, J. Mahani Math. Res. 2023; 12(2): 19-27.

A ring R is *skew power series-wise McCoy* if whenever any non-zero power series $f(x), g(x) \in R[[x;\alpha]]$ satisfy $f(x)g(x) = 0$, then $f(x)c = 0$ for some non-zero $c \in R$. By [5], commutative rings need not to be skew power series-wise McCoy. As a pioneer work on the relations between the zero-divisors of R and $R[[x]]$, Gilmer *et al.* in [6], Fields [5, Theorem 5], proved that if R is a commutative Noetherian ring with identity in which $(0) = Q_1 \cap Q_2 \cap \cdots \cap Q_n$ is a shortest primary representation of (0) with $\sqrt{Q_i} = P_i$, then $f(x) = \sum_{i=0}^{\infty} a_i x^i \in R[[x]]$ is a zero-divisor in $R[[x]]$ if and only if there is a non-zero element $r \in R$ which satisfies $rf(x) = 0$. Later, Camillo and Nielsen, constructed an example showing that formal power series rings over an associative noncommutative McCoy ring R, need not be McCoy in general.

Our first main aim in this paper is to provide some rich classes of skew power series McCoy rings. A detailed analysis of our main results is rewarded with the generalization of the main result of [2, 5]. Moreover, we investigate relations between skew power series McCoy property and other standard important ring-theoretic properties such as zip rings, strongly AB rings and rings with Property (A).

2. Annihilators in Noetherian power series rings

Following [1, Definition 2.18] and [3, Definition 2.1], skew power series-wise McCoy ring is defined as in the following:

Definition 2.1. A ring R is *skew power series-wise McCoy* if whenever any non-zero power series $f(x), g(x) \in R[[x;\alpha]]$ satisfy $f(x)g(x) = 0$, then $f(x)c = 0$ for some non-zero $c \in R$.

As mentioned, McCoy's theorem fails in the skew power series ring $R[[x;\alpha]]$ over either commutative or noncommutative ring R. However, if we assume the "Noetherian" hypothesis on the coefficient ring R, we obtain stronger conditions on the coefficients of power series $f(x) \in R[[x;\alpha]]$. The crucial for some of our results is the following lemma, which might be useful in some other studies.

The poof of our results are inspired by the techniques developed in [2].

Lemma 2.2. *Let R be a reversible right Noetherian α-compatible ring. If $f(x) = \sum_{i=0}^{\infty} a_i x^i$ and $g(x) = \sum_{j=0}^{\infty} b_j x^j$ be non-zero elements in $R[[x;\alpha]]$ such that $f(x)g(x) = 0$, then there exist non-negative integers $\ell_0, \ell_1, \ell_2, \ldots$ and $t_0, t_1, t_2, \ldots$ such that*

$$a_0^{\ell_0} a_1^{\ell_1} \cdots a_{m-1}^{\ell_{m-1}} a_m^{\ell_m} g(x) \neq 0 = a_0^{\ell_0} a_1^{\ell_1} \cdots a_{m-1}^{\ell_{m-1}} a_m^{\ell_m+1} g(x)$$

and also

$$f(x) b_n^{t_n} b_{n-1}^{t_{n-1}} \cdots b_1^{t_1} b_0^{t_0} \neq 0 = f(x) b_n^{t_n+1} b_{n-1}^{t_{n-1}} \cdots b_1^{t_1} b_0^{t_0}.$$

Proof. Suppose $f(x) = \sum_{i=0}^{\infty} a_i x^i$ and $g(x) = \sum_{j=0}^{\infty} b_j x^j$ be non-zero elements in $R[[x;\alpha]]$ such that $f(x)g(x) = 0$. First we show by induction that there exists non-negative integers $\ell_i (i \geq 0)$ such that $a_0^{\ell_0} a_1^{\ell_1} \cdots a_{m-1}^{\ell_{m-1}} a_m^{\ell_m} g(x) \neq 0$ but $a_0^{\ell_0} a_1^{\ell_1} \cdots a_{m-1}^{\ell_{m-1}} a_m^{\ell_m+1} g(x) = 0$. Since the ring R is right Noetherian, then the right ideals $C_{f(x)}R$ and $C_{g(x)}R$ are finitely generated and hence there exist finite subsets $\{a_0, a_1, \ldots, a_m\} = T \subseteq C_{f(x)}$ and $\{b_0, b_1, \ldots, b_n\} = S \subseteq C_{g(x)}$ such that $C_{f(x)}R = TR$ and $C_{g(x)}R = SR$. We claim that $a_0^{j+1} b_j = 0$ for each $0 \leq j \leq n$. By looking at the degree i term in the equation $f(x)g(x) = 0$, we get $\sum_{j=0}^{i} a_{i-j} \alpha^{i-j}(b_j) = 0$, for each $i \geq 0$. In particular, the degree zero term implies that $a_0 b_0 = 0$ and then by reversible property of a ring R we have $a_0 a_1 b_0 = 0$, since R is α-compatible we have $a_0 a_1 \alpha(b_0) = 0$. Then multiplying $a_0 b_1 + a_1 \alpha(b_0) = 0$ on the left-hand side by a_0, we get $a_0^2 b_1 = 0$. Now, suppose by induction that $a_0^{j+1} b_j = 0$ for each $j < k$. In particular, $a_0^k b_j = 0$, since R is reversible and α-compatible, we have $a_0^k a_{k-j} \alpha^{k-j}(b_j) = 0$ for each $j < k$. Multypling the degree k term in $f(x)g(x) = 0$ on the left-hand side by a_0^k, yields that $a_0^{k+1} b_k = -\sum_{j=0}^{k} a_0^k a_{k-j} \alpha^{k-j}(b_j) = 0$, which finishes the proof of our internal induction and the claim. Now, since $C_{g(x)}R = SR$, this implies that $a_0^{n+1} g(x) = 0$ but $a_0^n g(x) \neq 0$, which shows the existence of the ℓ_0 for the base case of our iduction. Suppose, by induction, we have constructed $\ell_0, \ell_1, \ldots, \ell_{m-1}$ satisfying the mentioned condition and set $r = a_{m-1}^{\ell_{m-1}} \cdots a_1^{\ell_1} a_0^{\ell_0}$. Then $rg(x) \neq 0$, since R is reversible and α-compatible, we get $g(x)r \neq 0$. Now by taking $g'(x) := g(x)r$, we have $f(x)g'(x) = 0$ and by the base case, there exists a non-negative integer ℓ_m such that $a_m^{\ell_m} g'(x) \neq 0$ but $a_m^{\ell_m+1} g'(x) = 0$. Using again reversibility and α-compatiblity of a ring R, we get $a_0^{\ell_0} a_1^{\ell_1} \cdots a_{m-1}^{\ell_{m-1}} a_m^{\ell_m} g(x) \neq 0$ but $a_0^{\ell_0} a_1^{\ell_1} \cdots a_{m-1}^{\ell_{m-1}} a_m^{\ell_m+1} g(x) = 0$, where $\ell_0, \ell_1, \ldots, \ell_m$ are non-negative integers, which completes our induction. The same proof, with only small changes, works for the other statement. □

Theorem 2.3. *Let R be a reversible right Noetherian ring. If R is an α-compatible ring, then R is a skew right power series-wise McCoy ring.*

Proof. First notice that, as $Ni\ell(R)$ is an ideal of a reversible ring R, then by Levitzki's Theorem, $Ni\ell(R)$ is a nilpotent ideal of reversible right Noetherian ring R. Hence there exists an integer t such that $(Ni\ell(R))^t = 0$ but $Ni\ell(R))^{t-1} \neq 0$. Now, let $f(x) = \sum_{i=0}^{\infty} a_i x^i$ and $g(x) = \sum_{j=0}^{\infty} b_j x^j$ be non-zero elements in $R[[x;\alpha]]$ such that $f(x)g(x) = 0$. Now, we have the following two cases:

Case (1): If $f(x) = \sum_{i=0}^{\infty} a_i x^i \in Ni\ell(R)[[x;\alpha]]$, then $a_i \in Ni\ell(R), i \geq 0$. Using $(Ni\ell(R))^t = 0$, then we have $a_i (Ni\ell(R))^{t-1} = 0$. Now, taking a non-zero element $c \in (Ni\ell(R))^{t-1}$, so $a_i c = 0$. since R is α-compatible, we get $f(x)c = 0$, as desired.

Case (2): If $f(x) \notin Ni\ell(R)[[x;\alpha]]$, then $C_f \setminus S \subseteq Ni\ell(R)$, where

$$S = \{a_{l0}, a_{l1}, a_{l2}, \ldots \mid a_{li} \notin Ni\ell(R), li < l(i+1), \text{ for all } i = 0, 1, 2, \ldots\}.$$

It is not hard to see that there exists a non-negative integer p such that we have $g(x)(Ni\ell(R))^p \neq 0$ but $g(x)(Ni\ell(R))^{p+1} = 0$. Let $0 \neq g'(x) = \sum_{k=0}^{\infty} d_k x^k \in$

$g(x)(Ni\ell(R))^p$. Then we have $f(x)g'(x) = 0$, since R is reversible and α-compatible by using $g(x)(Ni\ell(R))^{p+1} = 0$, we get $(\sum_{i=0}^{\infty} a_{li}x^{li})g'(x) = 0$. As R is right Noetherian, then there exists a finite subset $T = \{a_{l0}, a_{l1}, a_{l2}, \dots, a_{lt}\}$ of S such that $SR = TR$ and t is the smallest non-negative integer relative to this property. The proof is complete if we show that there exists a non-zero $c = ab$ for $a \in R$ and $b \in (Ni\ell(R))^p$ such that $Tc = 0$. Now by Lemma 2.2, there exists a non-negative integer n_{l0} such that $g'(x)a_{l0}^{n_{l0}} \neq 0$ but $g'(x)a_{l0}^{n_{l0}+1} = 0$. Then $d_k\alpha^k(a_{l0})^{nl_0+1} = 0$ for every $k \geq 0$. Since R is α-compatible we have $d_k(a_{l0})^{nl_0+1} = 0$ for every $k \geq 0$. By reversible property of R, we get $a_{l0}d_ka_{l0}^{n_{l0}} = 0$, since R is α-compatible we have $a_{l0}g'(x)a_{l0}^{n_{l0}} = 0$.

Multiplying $f(x)g'(x) = 0$ on the right-hand side by $a_{l0}^{n_{l0}}$, we get

$$0 = f(x)g'(x)a_{l0}^{n_{l0}} = (a_{l1}x^{l1} + a_{l2}x^{l2} + \cdots)g'(x)a_{l0}^{n_{l0}}.$$

Again by using Lemma 2.2, there is an integer n_{l1} such that $a_{l1}^{n_{l1}}g'(x)a_{l0}^{n_0} \neq 0$ but $a_{l1}^{n_{l1}+1}g'(x)a_{l0}^{n_{l0}} = 0$. Hence by using reversibility and α-compatiblity of R, we get $g'(x)a_{l0}^{n_{l0}}a_{l1}^{n_{l1}} \neq 0$ but $a_{l1}g'(x)a_{l0}^{n_{l0}}a_{l1}^{n_{l1}} = 0$. Now, multiplying $0 = f(x)g'(x)a_{l0}^{n_{l0}}$ on the right-hand side by $a_{l1}^{n_{l1}}$, we get

$$0 = f(x)g'(x)a_{l0}^{n_{l0}}a_{l1}^{n_{l1}} = (a_{l2}x^{l1} + a_{l3}x^{l2} + \cdots)g'(x)a_{l0}^{n_{l0}}a_{l1}^{n_{l1}}.$$

Continuing this process finite number of times, we can find integers $n_{l0}, n_{l1}, \dots, n_{lt}$ such that $a_{lt}^{n_{lt}}g'(x)a_{l0}^{n_{l0}}a_{l1}^{n_{l1}} \cdots a_{l(t-1)}^{n_{l(t-1)}} \neq 0$ but $a_{lt}^{n_{lt}+1}g'(x)a_{l0}^{n_{l0}}a_{l1}^{n_{l1}} \cdots a_{l(t-1)}^{n_{l(t-1)}} = 0$. Hence by using reversibility and and α-compatiblity of R, we get the information that $g'(x)a_{l0}^{n_{l0}}a_{l1}^{n_{l1}} \cdots a_{l(t-1)}^{n_{l(t-1)}}a_{lt}^{n_{lt}} \neq 0 = a_{lt}g'(x)a_{l0}^{n_{l0}}a_{l1}^{n_{l1}} \cdots a_{l(t-1)}^{n_{l(t-1)}}a_{lt}^{n_{lt}}$. Now, letting $0 \neq c \in C_{g'(x)a_{l0}^{n_{l0}}a_{l1}^{n_{l1}} \cdots a_{l(t-1)}^{n_{l(t-1)}}a_{lt}^{n_{lt}}}$, we get $Tc = 0$. Since R is α-compatible $T\alpha^i(c) = 0$ and hence $f(x)c = 0$, showing that R is right power series-wise McCoy ring. □

Corollary 2.4. *[2, Theorem 2.2] Let R be a reversible right Noetherian ring. Then R is a right power series-wise McCoy ring.*

Theorem 2.5. *Let R be a Noetherian reversible ring. If R is an α-compatible ring, then for each $f(x) = \sum\limits_{i=1}^{\infty} a_ix^i \in R[[x;\alpha]]$, these conditions are equivalent:*

1. *$f(x)$ is a zero-divisor in $R[[x;\alpha]]$;*
2. *$f(x) \in P[[x;\alpha]]$ for some prime ideal P of R;*
3. *There is a non-zero element $c \in R$ such that $f(x)c = 0$.*

Proof. (1) $\Longrightarrow$ (2) We have $Z(R) = \bigcup_{i=1}^{n} P_i$, where $P_i = ann_R(a_i)$ for some $a_i \in R$. Suppose $f(x) \in Z(R[[x;\alpha]])$. Then by Theorem 2.3, there exists a non-zero element $r \in R$ such that $f(x)r = 0 = rf(x)$. Taking $I = \langle C_f \rangle$, then we get $I \subseteq Z(R) = \bigcup_{i=1}^{n} P_i$. Hence by [7, Theorem 4], there exsists an integer $1 \leq k \leq n$ such that $I \subseteq P_k$. So $I[[x;\alpha]] \subseteq P_k[[x;\alpha]]$ and therefore by the definition of I, we get $f(x) \in P_k[[x;\alpha]]$, as desired.

(2) $\Longrightarrow$ (3) Since by [7, Theorem 5], we have $P_i = ann_R(a_i)$ for some $a_i \in R$, then the result follows.

(3) $\Longrightarrow$ (1) Obvious. $\square$

Corollary 2.6. *[2, Theorem 2.3] Let R be a Noetherian reversible ring. Then for each* $f(x) = \sum_{i=1}^{\infty} a_i x^i \in R[[x]]$*, these conditions are equivalent:*

1. *$f(x)$ is a zero-divisor in $R[[x]]$;*
2. *$f(x) \in P_k[[x]]$ for some $1 \leq k \leq n$, with P_k prime;*
3. *There is a non-zero element $c \in R$ such that $f(x)c = 0$.*

Theorem 2.7. *Let R be a right duo right Noetherian ring. If R is an α-compatible ring, then R is a skew right power series-wise McCoy ring.*

Proof. First notice that $Ni\ell(R)$ is a nilpotent ideal of a right duo right Noetherian ring R and then there exists an integer t such that $(Ni\ell(R))^t = 0$ but $(Ni\ell(R))^{t-1} \neq 0$. Now, let $f(x) = \sum_{i=0}^{\infty} a_i x^i$ and $g(x) = \sum_{j=0}^{\infty} b_j x^j$ be non-zero elements in $R[[x;\alpha]]$ such that $f(x)g(x) = 0$. Without loss of generality we can assume that $b_0 \neq 0$. We only consider the case where $f(x) \notin Ni\ell(R)[[x;\alpha]]$. Since R is right Noetherian, then the right ideal $C_f R$ is finitely generated and hence there exists a finite subset $\{a_0, a_1, \ldots, a_t\} = T \subseteq C_f$ such that we have $C_f R = TR$ whereas $C_f R \neq T'R$ for $\{a_0, a_1, \ldots, a_{t-1}\} = T' \subsetneq T$. We claim that there exists a non-zero $c \in R$ such that $b_j c \neq 0$ for some $b_j \in C_{g(x)}$ but $f(x) b_j c = 0$. We will proceeed by induction on $|T|$(the cardinality of T). If $|T| = 1$, then $T = \{a_0\}$ and hence $a_0 R = C_f R$. On the other hand since $a_0 b_0 = 0$ as a constant term in the equation $f(x)g(x) = 0$. Now from semi-commutative property of R we get $a_0 r b_0 = 0$ for each $r \in R$. Then $C_f b_0 = 0$, since R is α-compatible we have $a_i \alpha^i(b_0) = 0, i \geq 0$ and hence $f(x)b_0 = 0$, as desired. So the base case of our induction is stablished.

So, we may assume lenght(T)= $k \geq 2$ and that the claim is true for all smaller values by inductive assumption. Now, we consider the following two cases:

Case (1): If $a_k g(x) = 0$. since R is α-compatible we have $a_k \alpha^k(b_j) = 0, j \geq 0$. Then by semicommutative property of R, we have $a_k r_k g(x) = 0$ for each $r_k \in R$. Now, since lenght(T)= $k \geq 2$, by the definition of the subset T we have $a_{k+l} = a'_{k+l} + a_k t_k$ for each $l = 1, 2, \ldots$, where a'_{k+l} is an element of a right ideal generated by $\{a_0, a_1, \ldots, a_{k-1}\}$ and $t_k \in R$. Now let $f'(x) = \sum_{i=1}^{k-1} a_i x^i + \sum_{j=k+1}^{\infty} a'_j x^j$. Clearly $f'(x)g(x) = 0$ and $T_1 = \{a_0, a_1, \ldots, a_{k-1}\}$ is a generator set for the right ideal $C_{f'}R$. Since lenght$(T_1) = k-1$, then by inductive assumption there exist $c_1 \in R$ and $b_{j_1} \in C_{g(x)}$ such that $b_{j_1} c_1 \neq 0$ but $a_i \alpha^i((b_{j_1} c_1)) = 0, a_i \in C_{f'(x)}$. Now from $a_k r_k g(x) = 0$ we have $a_k r_k \alpha^k((b_{j_1} c_1)) = 0$ and hence we get $f(x)(b_{j_1} c_1) = 0$, as needed.

Case (2): If $a_k g(x) \neq 0$. Now, take the set $T = \{a_0, a_1, \ldots, a_{k-1}, a_k\}$. It suffices to prove that $r_R(T) \cap b_j R \neq 0$ for some $b_j \in C_{g(x)}$. By Lemma 2.2, there exists an integer n_0 such that $a_0^{n_0} g(x) \neq 0$ but $a_0^{n_0+1} g(x) = 0$. Let j be a minimal integer such that $a_0^{n_0}b_j \neq 0$. Since R is right duo, there exists $r_0 \in R$ such that $a_0^{n_0}b_j = b_jr_0$. Now taking $0 \neq g_0(x) = g(x)r_0$, we get $f(x)g_0(x) = 0$. Since R is right Noetherian, $C_{g(x)}R$ has a finite generator set and hence by doing above process finite times, we may assume that $a_0g'_0(x) = 0$, where $0 \neq g'_0(x) = g(x)d$ for $d \in R$. Then $(\sum_{i=1}^{\infty} a_ix^i)g'_0(x) = 0$. Now, by a similar way as above, we can find an element $r_1 \in R$ such that $0 \neq g'_1(x) = g'_0(x)r_1$ but $a_1g'_1(x) = 0$ and hence $(\sum_{i=2}^{\infty} a_ix^i)g'_0(x) = 0$. After doing finite times in this way, we can find $0 \neq g'_k(x) = g(x)v$ where $v \in R$ such that $a_ig'_k(x) = 0$ for each $0 \leq i \leq k$. Now taking $0 \neq b_jv \in C_{g'_k(x)}$, then $Sb_jv = 0$, as needed. □

Corollary 2.8. *[2, Theorem 2.4] Let R be a right duo right Noetherian ring. Then R is a right power series-wise McCoy ring.*

Theorem 2.9. *Let R be a Noetherian duo ring. If R is an α-compatible ring. Then for each $f(x) = \sum\limits_{i=1}^{\infty} a_ix^i \in R[[x;\alpha]]$, these conditions are equivalent:*

(1) *$f(x)$ is a zero-divisor in $R[[x;\alpha]]$;*
(2) *$f(x) \in P_k[[x;\alpha]]$ for some $1 \leq k \leq n$, with P_k prime;*
(3) *There is a non-zero element $c \in R$ such that $f(x)c = 0$.*

Proof. It is similar to the proof of Theorem 2.5. □

Corollary 2.10. *[2, Theorem 2.5] Let R be a Noetherian duo ring. Then for each $f(x) = \sum\limits_{i=1}^{\infty} a_ix^i \in R[[x]]$, these conditions are equivalent:*

(1) *$f(x)$ is a zero-divisor in $R[[x]]$;*
(2) *$f(x) \in P_k[[x]]$ for some $1 \leq k \leq n$, with P_k prime;*
(3) *There is a non-zero element $c \in R$ such that $f(x)c = 0$.*

Proposition 2.11. *Let R be a skew right power series-wise McCoy Noetherian ring. If R is an α-compatible ring, then the ring R is right zip if and only if $R[[x;\alpha]]$ is a zip ring.*

Proof. First suppose that R is right zip, $S = R[[x;\alpha]]$ and let $X \subseteq S$ such that $r_S(X) = 0$. For an element $f(x) = \sum_{i=0}^{\infty} a_ix^i \in S$, let C_f denotes the set of all coefficients of $f(x)$, and for a subset V of S, suppose that C_V denotes $\cup_{f\in V} C_f$. Thus $r_R(C_X) = 0$. Since R is right zip, there exists a finite subset $Y_0 = \{c_0, c_1, \ldots, c_l\}$ of C_X such that $r_R(Y_0) = 0$. For each $c_i \in Y_0$, there exists $f_{c_i}(x) \in X$ such that some of the coefficients of $f_{c_i}(x)$ are c_i, $0 \leq i \leq l$. For every $0 \leq i \leq l$, there exists $g_i \in R[[x;\alpha]]$ such that $f_{c_i}(x) = c_ix^{t_i} + g_i$, $0 \leq i \leq l$.
Put $Y = \{f_{c_0}, f_{c_1}, \ldots, f_{c_l}\}$. We claim that $r_S(Y) = 0$. Assume on the contrary that $0 \neq g(x) \in r_S(Y)$. Then for each $0 \leq i \leq l$, we have $f_{c_i}(x)g(x) = 0$. Now, define $F(x) = f_{c_0}(x) + x^{t_0}f_{c_1}(x) + x^{t_0+t_1}f_{c_2}(x) + \cdots + x^{t_0+t_1+\cdots+t_{l-1}}f_{c_l}(x)$. Then $F(x)g(x) = 0$ in $R[[x;\alpha]]$. As R is skew right power series-wise McCoy ring, there exists $0 \neq d \in R$ such that $F(x)d = 0$. On the other hand, $C_{F(x)} =$

$\{C_{f_{c_0}(x)}, \alpha^{t_0}(C_{f_{c_0}(x)}), \dots, \alpha^{t_0+t_1+\cdots+t_{l-1}}(C_{f_{c_l}(x)})\}$. As R is α-compatible, we get $d \in r_R(Y_0) = 0$, a contradiction. Thus $r_S(Y) = 0$, as desired.

Conversely, suppose that $S = R[[x;\alpha]]$ is right zip. Let X be a subset of R such that $r_R(X) = 0$. If $f(x) = \sum_{i=0}^{\infty} a_i x^i \in r_S(X)$, then $a_i \in r_R(X) = 0$ and so $f(x) = 0$. Hence $r_S(X) = 0$ and since S is right zip, there exists a finite set $Y \subseteq X$ such that $r_S(Y) = 0$. Hence $r_R(Y) = r_S(Y) \cap R = 0$. Therefore R is right zip ring. □

Corollary 2.12. (a) *Let R be a reversible one-sided Noetherian ring. If R is an α-compatible ring, then the ring R is right zip if and only if $R[[x;\alpha]]$ is a right zip.*

(b) *Let R be a one-sided duo one-sided Noetherian ring. If R is an α-compatible ring, then the ring R is right zip if and only if $R[[x;\alpha]]$ is right zip.*

Theorem 2.13. *Let R be a Noetherian skew power series-wise McCoy ring. Then R is strongly AB if and only if $R[[x;\alpha]]$ is strongly AB.*

Proof. First suppose R is strongly right AB and $J \subseteq R[[x;\alpha]]$ such that $r_{R[[x;\alpha]]}(J) \neq 0$. Take non-zero $g(x) \in r_{R[[x;\alpha]]}(J)$. For an element $f(x) \in R[[x;\alpha]]$, let C_f denotes the set of all coefficients of $f(x)$, and for a subset V of S, suppose that C_V denotes $\cup_{f \in V} C_f$. It is easy to see that $r_R(C_J) = r_R(RC_J)$, where RC_J is a left ideal generated by C_J. As R is a Noetherian ring, there exists finite subset $\{f_0, f_1, \dots, f_k\} \subseteq J$ such that $RC_J = \sum_{i=0}^{k} RC_{f_i}$. As R is a Noetherian ring, there exists finite subset $\{a_0^i, a_1^i, \dots, a_{t_i}^i\} \subseteq C_{f_i}$, $0 \leq i \leq k$, such that $RC_{f_i} = \sum_{j=0}^{t_i} Ra_j^i$. Hence

$$r_R(C_J) = r_R(RC_J) = r_R(\sum_{i=0}^{k} \sum_{j=0}^{t_i} Ra_j^i) = \cap_{i=0}^{k} \cap_{j=0}^{t_i} r_R(a_j^i). \quad (*)$$

We claim that $r_R(\sum_{i=0}^{k} \sum_{j=0}^{t_i} Ra_j^i) \neq 0$. Put $F(x) = f_0 + x^{t_0} f_1 + x^{t_0+t_1} f_2 + \cdots + x^{t_0+t_1+\cdots+t_{k-1}} f_k$. Then F is contained in the left ideal of $R[[x:\alpha]]$ generated by J. Hence $F(x)g(x) = 0$, because $g(x) \in r_{R[[x;\alpha]]}(J)$. Since R is skew power series-wise McCoy, there exists $0 \neq d \in R$ such that $F(x)d = 0$. Since

$$\{a_0^0, a_1^0, \dots, a_{t_0}^0, \alpha^{t_0}(a_0^1), \alpha^{t_0}(a_1^1), \dots, \alpha^{t_0}(a_{t_1}^1), \alpha^{t_0+t_1}(a_0^2), \dots,$$
$$\alpha^{t_0+t_1+\dots+t_{k-1}}(a_0^k), \dots, \alpha^{t_0+t_1+\dots+t_{k-1}}(a_{t_k}^k)\} \subseteq C_{F(x)}$$

and R is α-compatible, we get $[\sum_{i=0}^{k} \sum_{j=0}^{t_i} Ra_j^i]d = 0$.
As R is strongly right AB we get $[\sum_{i=0}^{k} \sum_{j=0}^{t_i} Ra_j^i]Rr = 0$ for some $0 \neq r \in R$. Using this and $(*)$ we obtain that $RC_J Rr = 0$. As R is α-compatible, we have $RC_J R[[x;\alpha]]r = 0$ and so $JR[[x;\alpha]]r = 0$, which implies that $R[[x;\alpha]]$ is strongly right AB.

Conversely, suppose $R[[x;\alpha]]$ is strongly right AB and $J \subseteq R$ such that $r_R(J) \neq 0$. Since $r_R(J) = r_{R[[x;\alpha]]}(J) \cap R$ and $r_R(J) \neq 0$, we have $r_{R[[x;\alpha]]}(J) \neq$

0. Then there exists a non-zero ideal K of $R[[x;\alpha]]$ such that $K \subseteq r_{R[[x;\alpha]]}(J)$, since $R[[x;\alpha]]$ is strongly right AB. Then $JC_K = 0$, where C_K is the non-zero ideal of R consisting of all coefficients of all elements in K. Therefore R is strongly right AB. The proof of the left case is similar. □

Corollary 2.14. *[2, Lemma 3.5] Let R be a Noetherian power series-wise McCoy ring. Then R is strongly AB if and only if $R[[x]]$ is strongly AB.*

Corollary 2.15. (a) *Let R be a reversible Noetherian ring. If R is an α-compatible ring, then R is strongly AB if and only if $R[[x;\alpha]]$ is strongly AB.*

(b) *Let R be a duo Noetherian ring. If R is an α-compatible ring, then R is strongly AB if and only if $R[[x;\alpha]]$ is strongly AB.*

Proposition 2.16. *Let R be a right Noetherian ring such that $R[[x;\alpha]]$ is strongly right AB. If R is an α-compatible ring, then the ring R is right skew power series-wise McCoy.*

Proof. One follows the proof of Theorem 2.3, but since the needed changes are not obvious we clarify them here. Notice that in Theorem 2.3, $Ni\ell(R)$ was an ideal of R for a reversible ring R, whereas when $R[[x;\alpha]]$ is strongly right AB, then $Ni\ell(R)$ need not be an ideal of R. Hence replacing either the lower nilradical $Ni\ell_*(R)$ or the upper nilradical $Ni\ell^*(R)$ by $Ni\ell(R)$ and using Levitzki's Theorem [9, Theorem 10.30], we get the desired result. □

Corollary 2.17. *[2, Proposition 3.8] Let R be a right Noetherian ring such that $R[[x]]$ is strongly right AB. Then the ring R is right power series-wise McCoy.*

Corollary 2.18. *Let R be a right Noetherian, α-compatible ring. Then the power series ring $R[[x,\alpha]]$ is strongly right AB if and only if the ring R is skew right power series-wise McCoy and strongly right AB.*

Problem: Let α be an automorphism of R. Under what conditions reversible (duo) right Noetherian rings are skew right power series-wise McCoy ?

3. Aknowledgement

We wish to thank the reviewers for excellent suggestions that have been incorporated into the paper.

References

[1] A. Alhevaz and E. Hashemi , *An alternative perspective on skew generalized power series ring*, Mediterr. J. Math.13 (2016), 4723–4744.

[2] A. Alhevaz, E. Hashemi and R. Mohammad, *On transfer of annihilator conditions of rings*, J. Algebra Appl 17, No 10(2018), 185–199.

[3] A. Alhevaz and D. Kiani, *McCoy property of skew Laurent polynomials and power series rings*, J. Algebra Appl 13(2)(2014), 1350083 (23 pages).

[4] V. Camillo and P.P. Nielsen, *McCoy rings and zero-divisors*, J. Pure Appl. Algebra., 212, No 3 (2008), 599–615.

[5] D.E. Fields, *Zero divisors and nilpotent elements in power series rings*, Proc. Amer. Math. Soc., 27 (1971), 427–433.

[6] R. Gilmer, A. Grams and T. Parker, *Zero divisors in power series rings*, J. Reine Angew. Math., 278/279 (1975), 145–164.

[7] E. Hashemi, A. As. Estaji and M. Ziembowski, *Answers to some questions concerning rings with Property (A) Proc. Edinb.* Math. Soc., in prees.

[8] T.K. Kwak and Y. Lee, *Zero divisors in skew power series rings*, Comm. Algebra., 43, No 10(2015), 4427–4445.

[9] T.Y. Lam, *A First Course in Noncommutative Rings*, Second edition. Graduate Texts in Mathematics, 131. Springer-Verlag, New York, 2001.

Masoome Zahiri
Orcid number: 0000-0001-5418-0746
Department of Mathematics
Higher Education center of Eghlid
Eghlid, Iran
Email address: m.zahiri@eghlid.ac.ir, m.zahiri86@gmail.com

Saeide Zahiri
Orcid number: 0000-0003-1342-0365
Department of Mathematics
Higher Education center of Eghlid
Eghlid, Iran
Email address: s.zahiri@eghlid.ac.ir, saeede.zahiri@yahoo.com