Journal of Mahani Mathematical Research
Print ISSN: 2251-7952 Online ISSN: 2645-4505

$\Gamma-$BCK-ALGEBRAS

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Dedicated to sincere professor Mashaallah Mashinchi

Article type: Research Article
(Received: 16 April 2022, Received in revised form: 18 June 2022)
(Accepted: 25 July 2022, Published Online: 05 August 2022)

Abstract. We know that $\Gamma-$ring, $\Gamma-$incline, $\Gamma-$semiring, $\Gamma-$semigroup are generalizations of ring, incline, semiring and semigroup respectively. In this paper, we introduce the concept of $\Gamma-$BCK-algebras as a generalization of BCK-algebras and study $\Gamma-$BCK-algebras. We also introduce subalgebra, ideal, closed ideal, normal subalgebra, normal ideal and construct quotient of $\Gamma-$BCK-algebras. We prove that if $f : M \to L$ be a normal homomorphism of $\Gamma-$BCK-algebras M and N, then $\Gamma-$BCK-algebra M/N is isomorphic to $Im(f)$, where $N = \ker(f)$.

Keywords: $(\Gamma-)$BCK-algebra, Quotient $\Gamma-$BCK-algebra, Subalgebra, Ideal, (Closed, Normal) Ideal.
2020 MSC: 03G25, 06F35, 03E72, 08A35.

1. Introduction

In 1995, M. Murali Krishna Rao introduced the notion of a $\Gamma-$semiring as a generalization of $\Gamma-$ring, ternary semiring and semiring [11]. Let M be a semiring and $\alpha \in \Gamma$. Define a mapping $* : M \times M \to M$ such that $a * b = a\alpha b$, then $(M, +, *)$ is a semiring. It is denoted by M_α, where $\{M_\alpha | \alpha \in \Gamma\}$ is a class of semirings. As a generalization of ring, the notion of a $\Gamma-$ring introduced by Nobusawa in 1964. Sen introduced $\Gamma-$semigroups as a generalization of semigroups [15]. Murali Krishna Rao studied regular $\Gamma-$incline, $\Gamma-$field semiring and $\Gamma-$group [12, 13]. The set of all negative integers $\mathbb{Z}$ is not a semiring with respect to usual addition and multiplication but $\mathbb{Z}$ forms a $\Gamma-$semiring, where $\Gamma = \mathbb{Z}$. The vital reason for the development of $\Gamma-$semiring is a generalization of results of rings, $\Gamma-$rings, semirings, semigroups and ternary semirings.

By an algebra (groupoid), we mean a non-empty set G together with a binary multiplication and a some distinguished element 0. Such an algebra is denoted by $(G, \cdot, 0)$. Each such algebra will follow equality axioms and the rule of substitution as well as some other rules. Many of such algebras were inspired by some logical systems. For example, so-called BCK-algebras are

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DOI: 10.22103/jmmr.2022.19322.1234

Publisher: Shahid Bahonar University of Kerman
How to cite: A. Borumand Saeid, M. Murali Krishna Rao and K. Rajendra Kumar, $\Gamma-$*BCK-algebras*, J. Mahani Math. Res. 2022; 11(3): 133-145.

inspired by a BCK logic. We have BCK-algebra and BCK positive logic, BCI-algebra and BCI positive logic, positive implicative BCK-algebra and positive implicative logic, implicative BCK-algebra and implicative logic and so on. The connection between such algebras and their corresponding logics is much stronger. Therefore one can give a translation procedure which translates all well formed formulas and all theorems of a given logic, into theorems of the corresponding algebra. Two classes of abstract algebras, namely BCK and BCI-algebras were introduced by Y. Imai and K. Iseki Japanee Mathematicians in 1966 to generalize the concept of set theoretic difference and non-classical propositional calculii [7]. Every BCI-algebra M satisfy $0 * x = 0$ for all $x \in M$ is a BCK-algebra. Every abelian group is a BCK-algebra, $*$ defined as group subtraction and 0 defined as group identity. $\mathcal{P}(S)$ of S form a BCK-algebra if $A * B$ is defined as $A \setminus B$ and 0 is the empty set. BCK $*$ operation is an analogue of the set theoretical difference. Residuated lattices, Boolean algebras, MV-algebras, BE-algebras, Wajsberg algebras, BL-algebras, Hilbert algebras, Heyting algebras, NM-algebras, MTL-algebras, Weak$-R_0$ algebras etc., can be expressed as particular cases of BCK algebras. Thus these are subclasses of BCK-algebras, which have a lot of applications in computer science.

BCK-algebras were studied by many mathematicians and applied to group theory, functional analysis, probability theory, topology, etc. Ideal theory plays an important role in studying these algebras. Meng introduced the concept of implicative ideals in BCK-algebras and investigated its relationship with the concepts of positive implicative ideals and commutative ideals [10]. The study of BCK-algebras was motivated by classical and non-classical propositional calculii modeling logical implications. BCK-algebras are algebraic formulations of the BCK system in combinatory logic, which has applications in the language of functional programming. The purpose of this paper is to introduce subalgebra, ideal, closed ideal, normal subalgebra, normal ideal of $\Gamma-$BCK-algebras, quotient $\Gamma-$BCK-algebra, and study some of the algebraic properties.

2. **Preliminaries**

In this section, we recall the following definitions and results which are necessary for completeness.

Definition 2.1. [6] An algebra $(X, *, 0)$ is called a BCK-algebra if it satisfies the following axioms
i) $[(x * y) * (x * z)] * (z * y) = 0$,
ii) $(x * (x * y)) * y = 0$,
iii) $x * x = 0$,
iv) $0 * x = 0$,
v) $x * y = y * x = 0$ imply $x = y$ for all $x, y, z \in$ X.
We can define a partial ordering $\leq$ on X by $x \leq y$ if and only if $x * y = 0$.

Theorem 2.2. *In any BCK-algebra* $(X, *, 0)$ *the following hold,*
i) $(x * y) * (y * z) \leq z * y$,

ii) $x * [x * (x * y)] \leq x * y$,
iii) $0 \leq x$,
iv) $x * y = 0$ *if and only if* $x \leq y$,
v) $x \leq y$ *implies* $x * z \leq y * z$ *and* $z * y \leq z * x$,
vi) $(x * y) * z = (x * z) * y$,
vii) $x * y \leq z$ *if and only if* $x * z \leq y$,
viii) $0 * (x * y) = (0 * x) * (0 * y)$,
ix) $(x * y) * x = 0$,
x) $(x * z) * (y * z) \leq x * y$ *for all* $x, y, z \in M$.

Definition 2.3. [10] A BCK-algebra M is said to be commutative if $y * (y * x) = x * (x * y)$ for all $x, y \in M$.

Example 2.4. Let $M = \{0, a, b\}$. Then binary operation $*$ is defined with the following table,

$*$	0	a	b
0	0	0	0
a	a	0	0
b	b	a	0

Then $\{M, *, 0\}$ is commutative.

Definition 2.5. [10] A non-empty set I of a BCK-algebra X is called an ideal of X if i) $0 \in I$ ii) $x * y \in I$ and $y \in I$ imply $x \in I$.

Definition 2.6. [10] Let M and N be BCK-algebras. A map $f : M \to N$ is called a homomorphism if $f(x * y) = f(x) * f(y)$ for all $x, y \in M$.

Definition 2.7. [10] A non-empty subset I of a BCK-algebra M is called a subalgebra of M, if $x * y \in I$, for $x, y \in I$.

Definition 2.8. [11] Let M and Γ be two non-empty sets. Then M is called a $\Gamma-$semigroup if there exists a mapping $M \times \Gamma \times M \to M$ (the images of (x, α, y) will be denoted by $x\alpha y$, for $x, y \in M, \alpha \in \Gamma$) such that,

$$x\alpha(y\beta z) = (x\alpha y)\beta z \text{ for all } x, y, z \in M, \alpha, \beta \in \Gamma.$$

Definition 2.9. [11] Let $(M, +)$ and $(\Gamma, +)$ be commutative semigroups. Then M is said to be a $\Gamma-$semiring M if there exists a mapping $M \times \Gamma \times M \to M$ (the images of (x, α, y) will be denoted by $x\alpha y$ for $x, y \in M, \alpha \in \Gamma$) such that it satisfies,

(i) $x\alpha(y + z) = x\alpha y + x\alpha z$,
(ii) $(x + y)\alpha z = x\alpha z + y\alpha z$,
(iii) $x(\alpha + \beta)y = x\alpha y + x\beta y$.
(iv) $x\alpha(y\beta z) = (x\alpha y)\beta z$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$,

Every semiring M is a $\Gamma-$semiring with $\Gamma = M$ and ternary operation as the usual semiring multiplication.

Definition 2.10. [11] A $\Gamma-$semiring M is said to have zero element if there exists an element $0 \in M$ such that $0+x = x = x+0$ and $0\alpha x = x\alpha 0 = 0$. And is said to be a commutative $\Gamma-$semiring if $x\alpha y = y\alpha x$ for all $x, y \in M, \alpha \in \Gamma$.

Definition 2.11. [11] Let M be a $\Gamma-$semiring. An element $a \in M$ is said to be an idempotent of M if there exists $\alpha \in \Gamma$, such that $a = a\alpha a$ and a is said to be α idempotent. And an element $a \in M$ is said to be a regular element of M if there exist $x \in M, \alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$.

3. $\Gamma-$BCK-algebra

In this section, we introduce the concept of $\Gamma-$BCK-algebra and study its properties.

Definition 3.1. Let M be a set with an element 0 and Γ be a non-empty set. If there exists a mapping $M \times \Gamma \times M \to M$ (images to be denoted by $x\alpha y$, for all $x, y \in M$ and $\alpha \in \Gamma$) satisfies the following axioms:
i)$[(x\alpha y)\beta(x\alpha z)]\beta(z\alpha y) = 0$,
ii) $x\alpha y = y\alpha x = 0 \Rightarrow x = y$,
iii) $x\alpha x = 0$,
iv) $0\alpha x = 0$ for all $\alpha, \beta \in \Gamma$, $x, y, z \in M$.
Then M is called a $\Gamma-$BCK-algebra.

Note: Let M be a $\Gamma-$BCK-algebra and $\alpha \in \Gamma$. Define a mapping $* : M \times M \to M$ such that $a * b = a\alpha b$ for all $a, b \in M$. Then $(M, *, 0)$ is a BCK-algebra and it is denoted by M_α.

Example 3.2. Let $M = \{0, a, b, c, d, e\}$ and $\Gamma = \{\alpha, \beta, \gamma, \delta, \psi\}$. The ternary operation is defined by the following tables

α	0	a	b	c	d	e
0	0	0	0	0	0	0
a	a	0	a	a	a	a
b	b	b	0	b	b	b
c	c	c	c	0	c	c
d	d	d	d	d	0	d
e	e	e	e	e	e	0

β	0	a	b	c	d	e
0	0	0	0	0	0	0
a	a	0	b	b	b	b
b	b	c	0	c	c	c
c	c	d	d	0	d	d
d	d	e	e	e	0	e
e	e	a	a	a	a	0

γ	0	a	b	c	d	e
0	0	0	0	0	0	0
a	a	0	c	c	c	c
b	b	d	0	d	d	d
c	c	e	e	0	e	e
d	d	a	a	a	0	a
e	e	b	b	b	b	0

δ	0	a	b	c	d	e
0	0	0	0	0	0	0
a	a	0	d	d	d	d
b	b	e	0	e	e	e
c	c	a	a	0	a	a
d	d	b	b	b	0	b
e	e	c	c	c	c	0

ψ	0	a	b	c	d	e
0	0	0	0	0	0	0
a	a	0	e	e	e	e
b	b	a	0	a	a	a
c	c	b	b	0	b	b
d	d	c	c	c	0	c
e	e	d	d	d	d	0

Let $N = \{0, 1, 2, 3\}$ and $\Gamma = \{\alpha, \beta\}$. The ternary operation is defined by the following tables

α	0	1	2	3
0	0	0	0	0
1	1	0	2	2
2	2	3	0	3
3	3	1	1	0

β	0	1	2	3
0	0	0	0	0
1	1	0	1	1
2	2	2	0	2
3	3	3	3	0

Then M and N are $\Gamma-$BCK-algebras.

Example 3.3. Let $M = \{0, x, y\}$ and $\Gamma = \{\alpha, \beta\}$. The ternary operation is defined by the following tables

α	0	x	y
0	0	0	0
x	x	0	0
y	y	x	0

β	0	x	y
0	0	0	0
x	x	0	y
y	y	0	0

Then M is a $\Gamma-$BCK-algebra.

Example 3.4. Any BCK-algebra $(M, *, 0)$ can be considered as $\Gamma-$BCK-algebra if we choose $\Gamma = \{0\}$ and the ternary operation $x0y$ is defined as $(x * 0) * y$ for all $x, y \in M$.
Let $M = \{0, b_1, b_2, b_3\}$. The ternary operation is defined by the following table

$*$	0	b_1	b_2	b_3
0	0	0	0	0
b_1	b_1	0	0	0
b_2	b_2	b_1	0	0
b_3	b_3	b_3	b_3	0

Then M is a BCK-algebra.
If $\Gamma = \{0\}$ the ternary operation is defined by the following table

0	0	b_1	b_2	b_3
0	0	0	0	0
b_1	b_1	0	0	0
b_2	b_2	b_1	0	0
b_3	b_3	b_3	b_3	0

Then M is a $\Gamma-$BCK-algebra when $\Gamma = \{0\}$.

Example 3.5. Let $\Gamma = \{0, x\}$ and $M = \{0, y\}$. Then ternary operation is defined by the following tables

0	0	x
0	0	0
x	x	0

y	0	x
0	0	0
x	x	0

Then M is a $\Gamma-$BCK-algebra.

Example 3.6. Let $M = \Gamma = \{0, x\}$. Then ternary operation is defined by the following tables

0	0	x
0	0	0
x	x	0

x	0	x
0	0	0
x	x	0

Then M is a $\Gamma-$BCK-algebra.

Example 3.7. Let $M = \{0, a, b, c\}$ and $\Gamma = \{\alpha, \beta, \gamma\}$. The ternary operation is defined by the following tables

α	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	b	0	b
c	c	c	c	0

β	0	a	b	c
0	0	0	0	0
a	a	0	b	b
b	b	c	0	c
c	c	a	a	0

γ	0	a	b	c
0	0	0	0	0
a	a	0	c	c
b	b	a	0	a
c	c	b	b	0

Then M is a $\Gamma-$BCK-algebra.

Definition 3.8. A non-empty subset A of a $\Gamma-$BCK-algebra M is said to be subalgebra if $a\alpha b \in A$ for all $a, b \in A$ and $\alpha \in \Gamma$.
A non-empty subset N of a $\Gamma-$BCK-algebra M is said to be a normal subalgebra if $(x\alpha a)\alpha(y\alpha b) \in N$ for any $x\alpha y,\ a\alpha b \in N$, $\alpha \in \Gamma$.

Example 3.9. Let $M = \{0, 1, 2, 3\}$ and $\Gamma = \{\alpha\}$. The ternary operation is defined by the following table

α	0	1	2	3
0	0	0	0	0
1	1	0	1	1
2	2	2	0	2
3	3	3	3	0

$I = \{0, 1\}$ is a normal subalgebra.

Definition 3.10. A $\Gamma-$BCK-algebra M is said to be commutative if $y\alpha(y\beta x) = x\alpha(x\beta y)$ for all $x, y \in M$, $\alpha, \beta \in \Gamma$.
A $\Gamma-$BCK-algebra M can be partially ordered by $x \leq y$ if and only if $x\alpha y = 0$ for all $\alpha \in \Gamma$. This ordering is called a $\Gamma-$BCK ordering.

Example 3.11. Let $M = \{0, x, y, z\}$, $\Gamma = \{\alpha, \beta\}$. Then ternary operation is defined with the following tables

α	0	x	y	z
0	0	0	0	0
x	x	0	x	x
y	y	y	0	y
z	z	z	z	0

β	0	x	y	z
0	0	0	0	0
x	x	0	x	z
y	y	x	0	x
z	z	y	x	0

Then the $\Gamma-$BCK-algebra M is commutative.

Example 3.12. Let $M = \{0, x, y\}$, $\Gamma = \{\alpha, \beta\}$. Then ternary operation is defined by the following tables

α	0	y	z	β	0	y	z
0	0	0	0	0	0	0	0
y	y	0	0	y	y	0	0
z	z	y	0	z	z	y	0

Then the $\Gamma-$BCK-algebra M is commutative.

Definition 3.13. Let M and N be $\Gamma-$BCK-algebras. A map $f : M \to N$ is called a homomorphism if $f(x\alpha y) = f(x)\alpha f(y)$ for all $x, y \in M, \alpha \in \Gamma$

Definition 3.14. Let M and N be $\Gamma-$BCK-algebras and $f : M \to N$ be a homomorphism. Then the set $\{x \in M/f(x) = 0\}$ is called the kernel of f and it is denoted by $\ker(f)$ and the set $\{f(x)/x \in M\}$ is called the image of f and is denoted by $Im(f)$.

Theorem 3.15. *Any* $\Gamma-$*BCK-algebra M satisfies the following condition,* $x\alpha(x\beta y)\alpha y = 0$ *for all* $x, y \in M, \alpha, \beta \in \Gamma$.

Proof. We have by definition of $\Gamma-$BCK-algebra,
$((x\beta y)\alpha(x\beta z))\alpha(z\beta y) = 0$ for all $x, y, z \in M, \alpha, \beta \in \Gamma$. (1)
Put $y = 0, z = y,$ *in* (1). Then $x\alpha(x\beta y)\alpha y = 0$. □

Theorem 3.16. *Let M be a* $\Gamma-$*BCK-algebra. Then the following are equivalent*
(i) *M is commutative,*
(ii) $x \leq y \Rightarrow x = y\alpha(y\beta x)$ *for all* $x, y \in M, \alpha, \beta \in \Gamma$.

Proof. (i)⇒ (ii): Suppose that M is commutative. Then $x\alpha(x\beta y) = y\alpha(y\beta x)$. for all $x, y \in M, \alpha, \beta \in \Gamma$. That implies $x\alpha 0 = y\alpha(y\beta x)$, hence $x = y\alpha(y\beta x)$.
(ii)⇒ (i): Suppose $y \leq x$ then $x = y\alpha(y\beta x)$. On the other hand $x \leq y$, then $x\alpha y = 0$. Then $x = x\alpha 0 = x\alpha(x\beta y)$. Hence $x\alpha(x\beta y) = y\alpha(y\beta x)$. □

Lemma 3.17. *Let M be a* $\Gamma-$*BCK-algebra. Then* $0 \leq x$ *for all* $x \in M$.

Proof. We have $0\alpha x = 0$ for all $\alpha \in \Gamma$, then $0 \leq x$. Hence 0 is the least element of the $\Gamma-$BCK-algebra M. □

Theorem 3.18. *If* $f : M \to N$ *is a homomorphism of* $\Gamma-$*BCK-algebras M and N, then* $Im(f)$ *is a subalgebra of N.*

Proof. Let $f : M \to N$ be a homomorphism of $\Gamma-$BCK-algebras M and N, $x, y \in Im(f)$. Then there exist $u, v \in M$ such that $f(u) = x,\ f(v) = y$. That implies $f(u)\alpha f(v) = x\alpha y$ for all $\alpha \in \Gamma$. That implies $f(u\alpha v) = x\alpha y$. Then $x\alpha y \in Im(f)$. Hence $Im(f)$ is a subalgebra of N. □

Lemma 3.19. *Let* $f : M \to N$ *be a homomorphism of* $\Gamma-$*BCK-algebras M and N. Then* $\ker(f)$ *is a subalgebra of M.*

Proof. Let $f : M \to N$ be a homomorphism of $\Gamma-$BCK-algebras M and N, $x, y \in \ker(f)$. Then $f(x) = f(y) = 0$ and so $f(x\alpha y) = f(x)\alpha f(y) = 0\alpha 0 = 0$. Hence $x\alpha y \in \ker(f)$. Therefore $\ker(f)$ is a subalgebra of M. □

Lemma 3.20. *Let $f : M \to N$ be a homomorphism of $\Gamma-$BCK-algebras, M and N. Then $(i) f(0) = 0, (ii) if\ x\alpha y = 0, then\ f(x)\alpha f(y) = 0$.*

Proof. (i) Now $f(0) = f(0\alpha 0) = f(0)\alpha f(0) = 0$.
(ii) Suppose $x\alpha y = 0$. Then $f(x\alpha y) = f(0)$ implies $f(x)\alpha f(y) = 0$. □

Theorem 3.21. *Let $f : M \to N$ be a homomorphism of $\Gamma-$BCK-algebras, M and N. Then f is injective if and only if $ker(f) = \{0\}$.*

Proof. Suppose $ker(f) = \{0\}$ and $f(x) = f(y)$, for some $x, y \in M$. Then $f(x\alpha y) = f(x)\alpha f(y) = 0$, that implies $x\alpha y \in ker(f) = \{0\}$, implies $x\alpha y = 0$ for all $\alpha \in \Gamma$.
Similarly, $y\alpha x = 0$ for all $\alpha \in \Gamma$. Therefore $x = y$.
Conversely, suppose f is injective and $x \in \ker(f)$. Then $f(x) = 0 = f(0)$, that implies $x = 0$, implies $\ker(f) = 0$. □

Lemma 3.22. *Let N be a normal subalgebra of a $\Gamma-$BCK-algebra M. If $x\alpha y \in N$, for all $x, y \in M$, then $y\alpha x \in N, \alpha \in \Gamma$.*

Proof. Let $x, y \in M$ and $x\alpha y \in N$. We have $y\alpha y = 0 \in N$ for all $\alpha \in \Gamma$. Then $y\alpha x = (y\alpha x)\alpha(0) = (y\alpha x)\alpha(y\alpha y) \in N, \alpha \in \Gamma$, since N is a normal subalgebra of M. Therefore $y\alpha x \in N$.

□

Let N be a normal subalgebra of a $\Gamma-$BCK-algebra M. Define a relation " $\sim_N$ " on M by $x \sim_N y$ if and only if $x\alpha y \in N$ for any $x, y \in M, \alpha \in \Gamma$.

Theorem 3.23. *Let N be a normal subalgebra of a $\Gamma-$BCK-algebra M. Then " $\sim_N$ " is a congruence relation.*

Proof. Let $x \in M, \alpha \in \Gamma$. Then the relation $\sim_N$ is reflexive, since $x\alpha x = 0 \in N$. The relation $\sim_N$ is symmetric, follows from Lemma 3.22.
Suppose $x \sim_N y$,$y \sim_N z \in N$. Then $x\alpha y \in N$ and $y\alpha z \in N$. By Lemma 3.22 $y\alpha z \in N$, thus $(x\alpha z)\alpha(y\alpha y) = (x\alpha z)\alpha 0 = (x\alpha z) \in N$, since N is normal subalgebra. Hence $x \sim_N z$. Then " $\sim_N$ " is an equivalence relation. Let $x \sim_N y$ and $p \sim_N q$ for any x, y, p and $q \in M$. Then $x\alpha y \in N$, $p\alpha q \in N$, we have $(x\alpha p)\alpha(y\alpha q) \in N$. Therefore $x\alpha p \sim_N y\alpha q$, since N is a normal subalgebra. Hence $\sim_N$ is a congruence relation. □

Definition 3.24. Let N be a congruence relation on a $\Gamma-$BCK-algebra M. Denote $M/N = \{[x]_N / x \in M, \}$, where $[x]_N = \{y \in M / x \sim_N y.\}$
Define $[x]_N \alpha [y]_N = [x\alpha y]_N, \alpha \in \Gamma$. M/N is a $\Gamma-$BCK algebra. Then $\Gamma-$BCK-algebra M/N is called the quotient $\Gamma-$BCK-algebra.

Example 3.25. Let $M = \{0, 1, 2, 3\}$ and $\Gamma = \{\alpha, \beta\}$. The ternary operation is defined by the following tables

α	0	1	2	3	β	0	1	2	3
0	0	0	0	0	0	0	0	0	0
1	1	0	1	1	1	1	0	2	2
2	2	2	0	2	2	2	3	0	3
3	3	3	3	0	3	3	1	1	0

$N = \{0, 1\}$ is a normal subalgebra.
Denote $M/N = \{[x]_N | x \in M\}$. Define $[x]_N \alpha [y]_N = [x\alpha y]_N, \alpha \in \Gamma$. Then M/N is a quotient Γ−BCK-algebra.

Theorem 3.26. *Let N be a normal subalgebra of a Γ−BCK-algebra M. Then the mapping $f : M \to M/N$ defined by $f(x) = [x]_N$ is a surjective homomorphism and* $\ker(f) = N$.

Proof. $f(x\alpha y) = [x\alpha y]_N = [x]_N \alpha [y]_N = f(x)\alpha f(y)$
$f(M) = \{f(x)/x \in M\} = \{[x]_N/x \in M\} = M/N$. Therefore f is surjective.
$\ker(f) = \{x \in M/f(x) = N\} = \{x \in M/[x]_N = N\}$
$= \{x \in M/[x]_N = [0]_N\} = \{x \in M/x \in N\} = N$ □

4. Ideals in Γ−BCK-algebras

In this section, we introduce the notion of ideal, closed ideal, normal ideal of Γ−BCK-algebras and study some of the properties of Γ−BCK-algebras.

Definition 4.1. Let M be a Γ−BCK-algebra and I be a non-empty subset of M. Then I is called an ideal of M if
i) $0 \in I$ ii) $x\alpha y \in I$, $\alpha \in \Gamma$, $y \in I \Rightarrow x \in I$.
M and $\{0\}$ are trivial ideals.
An ideal I is proper if $I \neq M$.

Example 4.2. Let $M = \{0, a, b, c\}$ and $\Gamma = \{\alpha, \beta\}$. The ternary operation is defined by the following tables

α	0	a	b	c	β	0	a	b	c
0	0	0	0	0	0	0	0	0	0
a	a	0	a	b	a	a	0	a	b
b	b	b	0	b	b	b	b	0	b
c	c	b	c	0	c	c	c	c	0

$I_1 = \{0, a, c\}$ is an ideal of a Γ−BCK-algebra M.
$I_2 = \{0, a, b\}$ is not an ideal and is a subalgebra of the Γ−BCK-algebra M.

Example 4.3. Let $M = \{0, 1, 2, 3\}$ and $\Gamma = \{\alpha, \beta\}$. The ternary operation is defined by the following tables

α	0	1	2	3
0	0	0	0	0
1	1	0	1	1
2	2	2	0	2
3	3	3	3	0

β	0	1	2	3
0	0	0	0	0
1	1	0	2	2
2	2	3	0	3
3	3	1	1	0

$I_1 = \{0, 1, 2\}$ is an ideal of the $\Gamma-$BCK-algebra M.

Theorem 4.4. *Let I be an ideal of a $\Gamma-BCK$-algebra M, if $y \in M$, $x \in I$ and $y \leq x$, then $y \in I$.*

Proof. Suppose $y \leq x$, $x \in I$, $y \in M$, then $y\alpha x = 0$ for all $\alpha \in \Gamma$. So $y\alpha x \in I$, thus $y \in I$, since $x \in I$. □

Definition 4.5. A non-empty subset I of a $\Gamma-$BCK-algebra M is called an implicative ideal of M if it satisfies
i) $0 \in I$,
ii) If $(x\alpha(y\beta x))\gamma z \in I$ and $z \in I$ imply $x \in I$ for all $x, y \in M, \alpha, \beta, \gamma \in \Gamma$.

Theorem 4.6. *Every implicative ideal of a $\Gamma-BCK$-algebra M is an ideal of M.*

Proof. Suppose I is an implicative ideal of M.
Let $x\gamma z \in I, z \in I, x \in M, \alpha, \gamma \in \Gamma$. Then
$x\gamma z = (x\alpha 0)\gamma z = [x\alpha(x\beta x)]\gamma z \in I$.
Therefore $x \in I$.

□

Definition 4.7. Let I be a non-empty subset of a $\Gamma-$BCK-algebra M. Then I is called a closed ideal if I is both an ideal and a subalgebra of M.

Example 4.8. Let $M = \{0, a, b\}$, $\Gamma = \{\alpha, \beta\}$. Then ternary operation is defined by the following tables.

α	0	a	b
0	0	0	0
a	a	0	a
b	b	b	0

β	0	a	b
0	0	0	0
a	a	0	b
b	b	a	0

Then $\{0, a\}$ is a closed ideal of the $\Gamma-$BCK-algebra M.

Lemma 4.9. *Let N be a normal subalgebra of a $\Gamma-BCK$-algebra M. Then $[0]_N$ is a closed ideal of M.*

Proof. We have $[0]_N = \{x \in M/x \sim_N 0\} = \{x \in M/x\alpha 0 \in N\}$ $= \{x \in M/x \in N\} = N$. Suppose that $x\alpha y, y \in [0]_N$. Then $x\alpha y \sim_N 0$ and $y \sim_N 0$. Then $(x\alpha y)\alpha 0,\ y\alpha 0 \in N$. That implies $x\alpha y, y\alpha 0 \in N$.
That implies $x\alpha y, 0\alpha y \in N$. Implies that $(x\alpha 0)\alpha(y\alpha y) \in N$. That implies

$x\alpha 0 \in N$. Therefore $x \in [0]_N$. Hence $[0]_N$ is an ideal of M.
Let $x, y \in [0]_N$. Then $x \sim_N 0$ and $y \sim_N 0$, thus $x\alpha y \sim_N 0$, hence $x\alpha y \in [0]_N$. Therefore $[0]_N$ is a closed ideal of M. □

Theorem 4.10. *Let $f : M \to N$ be a homomorphism of $\Gamma-$BCK-algebras M and N. Then $ker(f)$ is a closed ideal of M.*

Proof. Let $f : M \to N$ be a homomorphism and $0 \in M$. Then $f(0\alpha 0) = f(0)\alpha f(0) \Rightarrow f(0) = 0$, that implies $0 \in ker(f)$.
Let $x\alpha y \in ker(f),\ \alpha \in \Gamma,\ x, y \in M$ and $y \in ker(f)$. Then
$0 = f(x\alpha y) = f(x)\alpha f(y) = f(x)\alpha 0 = f(x)$ so $x \in ker(f)$. Hence $ker(f)$ is an ideal of M. Suppose $x, y \in ker(f)$ and $\alpha \in \Gamma$. Then $f(x\alpha y) = f(x)\alpha f(y)$, then $f(x\alpha y) = 0\alpha 0 = 0$, implies $x\alpha y \in ker(f)$. Hence $ker(f)$ is a closed ideal of M. □

Definition 4.11. Let I be an ideal of a $\Gamma-$BCK-algebra M. Then I is called normal ideal of M if it is a normal subalgebra.

Example 4.12. Let $M = \{0, 1, 2, 3\}$ and $\Gamma = \{\alpha, \beta\}$. The ternary operation is defined by the following tables

α	0	1	2	3
0	0	0	0	0
1	1	0	1	1
2	2	2	0	2
3	3	3	3	0

β	0	1	2	3
0	0	0	0	0
1	1	0	2	2
2	2	3	0	3
3	3	1	1	0

$I = \{0, 1\}$ is a normal ideal of the $\Gamma-$BCK-algebra M.

Theorem 4.13. *Let I be a normal ideal of a $\Gamma-$BCK-algebra M. Then I is a subalgebra of M.*

Proof. Let I be a normal ideal of M and $x, y \in I$, $\alpha, \beta \in \Gamma$.
Then $x\alpha x = 0 \in I$ and $y\alpha 0 = y \in I$. Therefore $(x\alpha y)\beta(x\alpha 0) = (x\alpha y)\beta x \in I$, since I is a normal ideal. Therefore $x\alpha y \in I$. □

Theorem 4.14. *If I is a normal subalgebra of a $\Gamma-$BCK-algebra M, then I is a normal ideal of M.*

Proof. Let I be a normal subalgebra of M. Suppose $x\alpha y \in I, y \in I$ and $\alpha, \beta \in \Gamma$. We have $0\alpha y = 0 \in I$. Then $(x\alpha 0)\beta(y\alpha y) \in I$, that implies $x\beta 0 \in I$, implies $x \in I$. Therefore $x \in I$. □

Definition 4.15. A homomorphism $f : M \to N$, where M, N are $\Gamma-$BCK-algebras, is said to be a normal homomorphism, if $\ker f$ is a normal ideal of M.

Example 4.16. Let $M = \{0, 1, 2, 3\}$ and $\Gamma = \{\alpha, \beta\}$. The ternary operation is defined by the following tables

α	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	2	1	0

β	0	1	2	3
0	0	0	0	0
1	1	0	2	2
2	2	3	0	3
3	3	3	3	0

Then M is a $\Gamma-$BCK-algebra. Define $f : M \to M$ by $f : 0 \to 0, 1 \to 0,$ $2 \to 0, 3 \to 3$. Then f is a normal homomorphism.

Theorem 4.17. *Let $f : M \to L$ be a normal homomorphism of $\Gamma-$BCK-algebras M and N. Then $\Gamma-$BCK-algebra M/N is isomorphic to $Im(f)$ where $N = \ker(f)$.*

Proof. Let $f : M \to L$ be a normal homomorphism of $\Gamma-$BCK-algebras of M and N. Then by definition $\ker(f)$ is a normal ideal of M. Let $N = \ker(f)$. Therefore $\ker(f)$ is a normal subalgebra of M. Define a mapping $\phi : M/N \to Im(f)$ by $\phi([x]_N) = f(x)$ for all $x \in M$. Let $[x]_N = [y]_N$. Implies $x \sim_N y$, i.e., $x\alpha y \in N$ and $y\alpha x \in N$, that implies$f(x)\alpha f(y) = [0]_L = f(y)\alpha f(x)$, implies $f(x) = f(y)$, implies $\phi([x]_N = \phi([y]_N)$. Therefore ϕ is well defined.

$$\begin{aligned}\phi([x]_N\alpha[y]_N) &= \phi([x\alpha y]_N)\\ &= f(x\alpha y)\\ &= f(x)\alpha f(y)\\ &= \phi([x]_N)\alpha\phi([y]_N)\end{aligned}$$

Hence ϕ is a homomorphism from $M/\ker(f) \to Im(f)$

$$\begin{aligned}&\phi([x]_N) = 0_{Im(f)}, \text{ that implies } f(x) = 0_{Im(f)},\\ &\Rightarrow x \in \ker(f), \text{implies } x \in N.\\ &\text{Therefore } [x]_N = [0]_N.\end{aligned}$$

Therefore ϕ is one-one. Hence ϕ is an isomorphism from $M/\ker(f)$ onto $Im(f)$. □

5. Conclusion:

In this paper, we extended the concepts of BCK-algebra to $\Gamma-$BCK-algebras, quotient $\Gamma-$BCK-algebra. We studied closed ideal, normal ideal and some of properties were investigated. We proved that a normal ideal of a $\Gamma-$BCK-algebra is a subalgebra, if $f : M \to N$ is a homomorphism of $\Gamma-$BCK algebras, then $\ker(f)$ is a closed ideal. We further study fuzzy implicative ideals of $\Gamma-$BCK-algebras.

Acknowledgements. We wish to thank the reviewers for excellent suggestions that have been incorporated into the paper.

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