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A NOTE ON SUM FORMULAS $\sum_{k=0}^{n} kx^k W_k$ AND $\sum_{k=1}^{n} kx^k W_{-k}$ OF GENERALIZED HEXANACCI NUMBERS

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Abstract. In this paper, closed forms of the sum formulas $\sum_{k=0}^{n} kx^k W_k$ and $\sum_{k=1}^{n} kx^k W_{-k}$ for generalized Hexanacci numbers are presented. As special cases, we give summation formulas of Hexanacci, Hexanacci-Lucas, and other sixth-order recurrence sequences.

Keywords: Hexanacci numbers, Hexanacci-Lucas numbers, sum formulas, summing formulas.
2020 MSC: Primary 11B37, 11B39, 11B83.

1. Introduction

The generalized Hexanacci sequence

$$\{W_n(W_0, W_1, W_2, W_3, W_4, W_5; r, s, t, u, v, y)\}_{n\geq 0}$$

(or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$(1) \quad \begin{aligned} W_n &= rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4} + vW_{n-5} + yW_{n-6}, \\ W_0 &= c_0, W_1 = c_1, W_2 = c_2, \ W_3 = c_3, W_4 = c_4, W_5 = c_5, \ n \geq 6, \end{aligned}$$

where $W_0, W_1, W_2, W_3, W_4, W_5$ are arbitrary real or complex numbers and r, s, t, u, v, y are real numbers. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{v}{y}W_{-n+1} - \frac{u}{y}W_{-n+2} - \frac{t}{y}W_{-n+3} - \frac{s}{y}W_{-n+4} - \frac{r}{y}W_{-n+5} + \frac{1}{y}W_{-n+6}$$

for $n = 1, 2, 3, ...$ when $y \neq 0$. Therefore, recurrence (1) holds for all integer n. Hexanacci sequence has been studied by many authors, see for example [28, 40, 95] and references therein.

As $\{W_n\}$ is a sixth-order recurrence sequence (difference equation), its characteristic equation is

$$(2) \qquad x^6 - rx^5 - sx^4 - tx^3 - ux^2 - vx - y = 0$$

whose roots are $\alpha, \beta, \gamma, \delta, \lambda, \mu$.

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Note that we have the following identities:
$\alpha+\beta+\gamma+\delta+\lambda+\mu=r$,
$\alpha\beta+\alpha\lambda+\alpha\gamma+\alpha\mu+\beta\lambda+\alpha\delta+\beta\gamma+\beta\mu+\lambda\gamma+\lambda\mu+\beta\delta+\lambda\delta+\gamma\mu+\gamma\delta+\mu\delta=-s$,
$\alpha\beta\lambda+\alpha\beta\gamma+\alpha\beta\mu+\alpha\lambda\gamma+\alpha\lambda\mu+\alpha\beta\delta+\alpha\lambda\delta+\alpha\gamma\mu+\beta\lambda\gamma+\beta\lambda\mu+\alpha\gamma\delta+\alpha\mu\delta+\beta\lambda\delta+\beta\gamma\mu+\lambda\gamma\mu+\beta\gamma\delta+\beta\mu\delta+\lambda\gamma\delta+\lambda\mu\delta+\gamma\mu\delta=t$,
$\alpha\beta\lambda\gamma+\alpha\beta\lambda\mu+\alpha\beta\lambda\delta+\alpha\beta\gamma\mu+\alpha\lambda\gamma\mu+\alpha\beta\gamma\delta+\alpha\beta\mu\delta+\alpha\lambda\gamma\delta+\alpha\lambda\mu\delta+\beta\lambda\gamma\mu+\alpha\gamma\mu\delta+\beta\lambda\gamma\delta+\beta\lambda\mu\delta+\beta\gamma\mu\delta+\lambda\gamma\mu\delta=-u$
$\alpha\beta\lambda\gamma\mu+\alpha\beta\lambda\gamma\delta+\alpha\beta\lambda\mu\delta+\alpha\beta\gamma\mu\delta+\alpha\lambda\gamma\mu\delta+\beta\lambda\gamma\mu\delta=v$,
$\alpha\beta\lambda\gamma\mu\delta=-y$.
Generalized Hexanacci numbers can be expressed, for all integers n, using Binet's formula.

Theorem 1.1. *[95] (Binet's formula of generalized (r,s,t,u,v,y) numbers (generalized Hexanacci numbers))*

$$
\begin{aligned}
(3)\qquad W_n &= \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)(\alpha-\mu)} \\
&+\frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)(\beta-\mu)} \\
&+\frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)(\gamma-\mu)} \\
&+\frac{p_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)(\delta-\mu)} \\
&+\frac{p_5\lambda^n}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)(\lambda-\mu)} \\
&+\frac{p_6\mu^n}{(\mu-\alpha)(\mu-\beta)(\mu-\gamma)(\mu-\delta)(\mu-\lambda)},
\end{aligned}
$$

where

$$
\begin{aligned}
p_1 &= W_5-(\beta+\gamma+\delta+\lambda+\mu)W_4 \\
&+(\beta\lambda+\beta\gamma+\beta\mu+\lambda\gamma+\lambda\mu+\beta\delta+\lambda\delta+\gamma\mu+\gamma\delta+\mu\delta)W_3 \\
&-(\beta\lambda\gamma+\beta\lambda\mu+\beta\lambda\delta+\beta\gamma\mu+\lambda\gamma\mu+\beta\gamma\delta+\beta\mu\delta+\lambda\gamma\delta+\lambda\mu\delta+\gamma\mu\delta)W_2 \\
&+(\beta\lambda\gamma\mu+\beta\lambda\gamma\delta+\beta\lambda\mu\delta+\beta\gamma\mu\delta+\lambda\gamma\mu\delta)W_1-\beta\lambda\gamma\mu\delta W_0,
\end{aligned}
$$

$$
\begin{aligned}
p_2 &= W_5-(\alpha+\gamma+\delta+\lambda+\mu)W_4 \\
&+(\alpha\lambda+\alpha\gamma+\alpha\mu+\alpha\delta+\lambda\gamma+\lambda\mu+\lambda\delta+\gamma\mu+\gamma\delta+\mu\delta)W_3 \\
&-(\alpha\lambda\gamma+\alpha\lambda\mu+\alpha\lambda\delta+\alpha\gamma\mu+\alpha\gamma\delta+\alpha\mu\delta+\lambda\gamma\mu+\lambda\gamma\delta+\lambda\mu\delta+\gamma\mu\delta)W_2 \\
&+(\alpha\lambda\gamma\mu+\alpha\lambda\gamma\delta+\alpha\lambda\mu\delta+\alpha\gamma\mu\delta+\lambda\gamma\mu\delta)W_1-\alpha\lambda\gamma\mu\delta W_0,
\end{aligned}
$$

$$\begin{aligned}
p_3 &= W_5-(\alpha+\beta+\delta+\lambda+\mu)W_4 \\
&+(\alpha\beta+\alpha\lambda+\alpha\mu+\beta\lambda+\alpha\delta+\beta\mu+\lambda\mu+\beta\delta+\lambda\delta+\mu\delta)W_3 \\
&-(\alpha\beta\lambda+\alpha\beta\mu+\alpha\lambda\mu+\alpha\beta\delta+\alpha\lambda\delta+\beta\lambda\mu+\alpha\mu\delta+\beta\lambda\delta+\beta\mu\delta+\lambda\mu\delta)W_2 \\
&+(\alpha\beta\lambda\mu+\alpha\beta\lambda\delta+\alpha\beta\mu\delta+\alpha\lambda\mu\delta+\beta\lambda\mu\delta)W_1-\alpha\beta\lambda\mu\delta W_0,
\end{aligned}$$

$$\begin{aligned}
p_4 &= W_5-(\alpha+\beta+\gamma+\lambda+\mu)W_4 \\
&+(\alpha\beta+\alpha\lambda+\alpha\gamma+\alpha\mu+\beta\lambda+\beta\gamma+\beta\mu+\lambda\gamma+\lambda\mu+\gamma\mu)W_3 \\
&-(\alpha\beta\lambda+\alpha\beta\gamma+\alpha\beta\mu+\alpha\lambda\gamma+\alpha\lambda\mu+\alpha\gamma\mu+\beta\lambda\gamma+\beta\lambda\mu+\beta\gamma\mu+\lambda\gamma\mu)W_2 \\
&+(\alpha\beta\lambda\gamma+\alpha\beta\lambda\mu+\alpha\beta\gamma\mu+\alpha\lambda\gamma\mu+\beta\lambda\gamma\mu)W_1-\alpha\beta\lambda\gamma\mu W_0,
\end{aligned}$$

$$\begin{aligned}
p_5 &= W_5-(\alpha+\beta+\gamma+\delta+\mu)W_4 \\
&+(\alpha\beta+\alpha\gamma+\alpha\mu+\alpha\delta+\beta\gamma+\beta\mu+\beta\delta+\gamma\mu+\gamma\delta+\mu\delta)W_3 \\
&-(\alpha\beta\gamma+\alpha\beta\mu+\alpha\beta\delta+\alpha\gamma\mu+\alpha\gamma\delta+\alpha\mu\delta+\beta\gamma\mu+\beta\gamma\delta+\beta\mu\delta+\gamma\mu\delta)W_2 \\
&+(\alpha\beta\gamma\mu+\alpha\beta\gamma\delta+\alpha\beta\mu\delta+\alpha\gamma\mu\delta+\beta\gamma\mu\delta)W_1-\alpha\beta\gamma\mu\delta W_0,
\end{aligned}$$

$$\begin{aligned}
p_6 &= W_5-(\alpha+\beta+\gamma+\delta+\lambda)W_4 \\
&+(\alpha\beta+\alpha\lambda+\alpha\gamma+\beta\lambda+\alpha\delta+\beta\gamma+\lambda\gamma+\beta\delta+\lambda\delta+\gamma\delta)W_3 \\
&-(\alpha\beta\lambda+\alpha\beta\gamma+\alpha\lambda\gamma+\alpha\beta\delta+\alpha\lambda\delta+\beta\lambda\gamma+\alpha\gamma\delta+\beta\lambda\delta+\beta\gamma\delta+\lambda\gamma\delta)W_2 \\
&+(\alpha\beta\lambda\gamma+\alpha\beta\lambda\delta+\alpha\beta\gamma\delta+\alpha\lambda\gamma\delta+\beta\lambda\gamma\delta)W_1-\alpha\beta\lambda\gamma\delta W_0.
\end{aligned}$$

Usually, it is customary to choose r,s,t,u,v,y so that the Equ. (2) has at least one real (say α) solution.

(3) can be written in the following form:

$$W_n = A_1\alpha^n + A_2\beta^n + A_3\gamma^n + A_4\delta^n + A_5\lambda^n + A_6\mu^n,$$

where

$$\begin{aligned}
A_1 &= \frac{p_1}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)(\alpha-\mu)}, \\
A_2 &= \frac{p_2}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)(\beta-\mu)}, \\
A_3 &= \frac{p_3}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)(\gamma-\mu)}, \\
A_4 &= \frac{p_4}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)(\delta-\mu)}, \\
A_5 &= \frac{p_5}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)(\lambda-\mu)}, \\
A_6 &= \frac{p_6}{(\mu-\alpha)(\mu-\beta)(\mu-\gamma)(\mu-\delta)(\mu-\lambda)}.
\end{aligned}$$

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_n x^n$ of the sequence $\{W_n\}$.

Lemma 1.2. *[95] Suppose that* $f_{W_n}(x)=\sum_{n=0}^{\infty}W_nx^n$ *is the ordinary generating function of the generalized* (r,s,t,u,v,y) *sequence* $\{W_n\}_{n\geq 0}$*. Then,* $\sum_{n=0}^{\infty}W_nx^n$ *is given by*

$$\sum_{n=0}^{\infty}W_nx^n=\frac{\Lambda}{1-rx-sx^2-tx^3-ux^4-vx^5-yx^6},\tag{4}$$

where

$$\begin{aligned}\Lambda &= W_0+(W_1-rW_0)x+(W_2-rW_1-sW_0)x^2\\ &\quad +(W_3-rW_2-sW_1-tW_0)x^3\\ &\quad +(W_4-rW_3-sW_2-tW_1-uW_0)x^4\\ &\quad +(W_5-rW_4-sW_3-tW_2-uW_1-vW_0)x^5.\end{aligned}$$

No	Sequences (Numbers)	Notation	Ref.
1	Gen. Hexanacci	$\{V_n\}=\{W_n(W_0,W_1,W_2,W_3,W_4,W_5;1,1,1,1,1,1)\}$	[96]
2	Gen. Sixth order Pell	$\{V_n\}=\{W_n(W_0,W_1,W_2,W_3,W_4,W_5;2,1,1,1,1,1)\}$	[97]
3	Gen. Sixth order Jacobsthal	$\{V_n\}=\{W_n(W_0,W_1,W_2,W_3,W_4,W_5;1,1,1,1,1,2)\}$	[98]
4	Gen. 6-primes	$\{V_n\}=\{W_n(W_0,W_1,W_2,W_3,W_4,W_5;2,3,5,7,11,13)\}$	[99]

Table 1. A few special case of generalized Hexanacci sequences.

For some specific values of W_0,W_1,W_2,W_3,W_4,W_5 and r,s,t,u,v,y it is worth presenting these special Hexanacci numbers in a table as a specific name. In literature, for example, the following names and notations (see Table 2) are used for the special cases of r,s,t,u,v,y and initial values.

For easy writing, from now on, we drop the superscripts from the sequences, for example we write P_n for $P_n^{(6)}$.

The following theorem presents some linear summing formulas of generalized Hexanacci numbers with positive subscripts.

Theorem 1.3. *For* $n\geq 0$ *we have the following formulas: If* $sx^2+tx^3+ux^4+vx^5+x^6y+rx-1\neq 0$*, then*

$$\sum_{k=0}^{n}x^kW_k=\frac{\Theta_1(x)}{sx^2+tx^3+ux^4+vx^5+x^6y+rx-1},$$

Sequences (Numbers)	Notation	Ref
Hexanacci	$\{H_n\} = \{W_n(0,1,1,2,4,8;1,1,1,1,1,1)\}$	[96]
Hexanacci-Lucas	$\{E_n\} = \{W_n(6,1,3,7,15,31;1,1,1,1,1,1)\}$	[96]
sixth order Pell	$\{P_n^{(6)}\} = \{W_n(0,1,2,5,13,34;2,1,1,1,1,1)\}$	[97]
sixth order Pell-Lucas	$\{Q_n^{(6)}\} = \{W_n(6,2,6,17,46,122;2,1,1,1,1,1)\}$	[97]
modified sixth order Pell	$\{E_n^{(6)}\} = \{W_n(0,1,1,3,8,21;2,1,1,1,1,1)\}$	[97]
sixth order Jacobsthal	$\{J_n^{(6)}\} = \{W_n(0,1,1,1,1,1;1,1,1,1,1,2)\}$	[6,98]
sixth order Jacobsthal-Lucas	$\{j_n^{(6)}\} = \{W_n(2,1,5,10,20,40;1,1,1,1,1,2)\}$	[6,98]
modified sixth order Jacobsthal	$\{K_n^{(6)}\} = \{W_n(3,1,3,10,20,40;1,1,1,1,1,2)\}$	[98]
sixth-order Jacobsthal Perrin	$\{Q_n^{(6)}\} = \{W_n(3,0,2,8,16,32;1,1,1,1,1,2)\}$	[98]
adjusted sixth-order Jacobsthal	$\{S_n^{(6)}\} = \{W_n(0,1,1,2,4,8;1,1,1,1,1,2)\}$	[98]
modified sixth-order Jacobsthal-Lucas	$\{R_n^{(6)}\} = \{W_n(6,1,3,7,15,31;1,1,1,1,1,2)\}$	[98]
6-primes	$\{G_n\} = \{W_n(0,0,0,0,1,2;2,3,5,7,11,13)\}$	[99]
Lucas 6-primes	$\{H_n\} = \{W_n(6,2,10,41,150,542;2,3,5,7,11,13)\}$	[99]
modified 6-primes	$\{E_n\} = \{W_n(0,0,0,0,1,1;2,3,5,7,11,13)\}$	[99]

TABLE 2. A few members of generalized Hexanacci sequences.

where

$\Theta_1(x) = x^{n+5}W_{n+5} - (rx-1)x^{n+4}W_{n+4} - (sx^2+rx-1)x^{n+3}W_{n+3} - (sx^2 + tx^3 + rx - 1)x^{n+2}W_{n+2} - (sx^2 + tx^3 + ux^4 + rx - 1)x^{n+1}W_{n+1} + yx^{n+6}W_n - x^5W_5 + x^4(rx-1)W_4 + x^3(sx^2 + rx - 1)W_3 + x^2(sx^2 + tx^3 + rx - 1)W_2 + x(sx^2 + tx^3 + ux^4 + rx - 1)W_1 + (sx^2 + tx^3 + ux^4 + vx^5 + rx - 1)W_0.$

Proof. It is given in Soykan [85, Theorem 2.1]. □

The following theorem presents some linear summing formulas of generalized Hexanacci numbers with negative subscripts.

Theorem 1.4. *Let x be a complex number. For $n \geq 1$ we have the following formulas: If $y + rx^5 + sx^4 + tx^3 + ux^2 + vx - x^6 \neq 0$, then*

$$\sum_{k=1}^{n} x^k W_{-k} = \frac{\Theta_4(x)}{y + rx^5 + sx^4 + tx^3 + ux^2 + vx - x^6},$$

where

$\Theta_4(x) = -x^{n+1}W_{-n+5} + (r-x)x^{n+1}W_{-n+4} + (s + rx - x^2)x^{n+1}W_{-n+3} + (t + rx^2 + sx - x^3)x^{n+1}W_{-n+2} + (u + rx^3 + sx^2 + tx - x^4)x^{n+1}W_{-n+1} + (v +$

$rx^4+sx^3+tx^2+ux-x^5)x^{n+1}W_{-n}+xW_5-x(r-x)W_4+x(-s-rx+x^2)W_3+x(-t-rx^2-sx+x^3)W_2+x(-u-rx^3-sx^2-tx+x^4)W_1+x(-v-rx^4-sx^3-tx^2-ux+x^5)W_0.$

Proof. It is given in Soykan [85, Theorem 4.1]. □

In this work, we investigate summation formulas of generalized Hexanacci numbers.

2. An Application of the Sum of the Numbers

An application of the sum of the numbers is circulant matrix. Computations of the Frobenius norm, spectral norm, maximum column length norm and maximum row length norm of circulant (r-circulant, geometric circulant, semicirculant) matrices with the generalized m-step Fibonacci sequences require the sum of the squares of the numbers of the sequences. For generalized m-step Fibonacci sequences see for example Soykan [50]. If $m=2, m=3$ and $m=4$, we get the generalized Fibonacci sequence, generalized Tribonacci sequence and generalized Tetranacci sequence, respectively. Next, we recall some information on circulant (r-circulant, geometric circulant) matrices and Frobenius norm, spectral norm, maximum column length norm and maximum row length norm.

Circulant matrices have been around for a long time and have been extensively used in many scientific areas. In some scientific areas such as image processing, coding theory and signal processing we often encounter circulant matrices. These matrices also have many applications in numerical analysis, optimization, digital image processing, mathematical statistics and modern technology.

Let $n \geq 2$ be an integer and r be any real or complex number. An $n \times n$ matrix C_r is called a r-circulant matrix if it of the form

$$C_r = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ rc_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ rc_{n-2} & rc_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ rc_1 & rc_2 & rc_3 & \cdots & rc_{n-1} & c_0 \end{pmatrix}_{n\times n},$$

and the r-circulant matrix C_r is denoted by

$$C_r = Circ_r(c_0, c_1, ..., c_{n-1}).$$

If $r=1$ then 1-circulant matrix is called as circulant matrix and denoted by $C = Circ(c_0, c_1, ..., c_{n-1})$. Circulant matrixs were first proposed by Davis in [8]. These matrixs have many interesting properties, and it is one of the most important research subject in the field of the computational and pure mathematics (see for example references given in Table 3). For instance, Shen and Cen [46] studied on the norms of r-circulant matrices with Fibonacci and

Lucas numbers. Then, later Kızılateş and Tuglu [24] defined a new geometric circulant matrix as follows:

$$C_{r^*}=\begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ rc_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ r^2c_{n-2} & rc_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ r^{n-1}c_1 & r^{n-2}c_2 & r^{n-3}c_3 & \cdots & rc_{n-1} & c_0 \end{pmatrix}_{n\times n},$$

and then they obtained the bounds for the spectral norms of geometric circulant matrices with the generalized Fibonacci number and Lucas numbers. When the parameter satisfies $r=1$, we get the classical circulant matrix. See also Polatlı [33] for the spectral norms of r-circulant matrices with a type of Catalan triangle numbers.

The Frobenius (or Euclidean) norm and spectral norm of a matrix $A=(a_{ij})_{m\times n}\in M_{m\times n}(\mathbb{C})$ are defined respectively as follows:

$$\left\|A\right\|_F=\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\left|a_{ij}\right|^2\right)^{1/2} \quad \text{and} \quad \left\|A\right\|_2=\left(\max_{1\leq i\leq n}\left|\lambda_i\right|\right)^{1/2},$$

where λ_i 's are the eigenvalues of the matrix A^*A and A^* is the conjugate of transpose of the matrix A . The maximum column length norm $c_1(.)$ and the maximum row length norm $r_1(.)$ of an matrix of order $n\times n$ are defined as follows:

$$c_1(A)=\max_{1\leq j\leq n}\left(\sum_{i=1}^{n}\left|a_{ij}\right|^2\right)^{1/2} \text{ and } r_1(A)=\max_{1\leq i\leq n}\left(\sum_{j=1}^{n}\left|a_{ij}\right|^2\right)^{1/2}.$$

The following inequality holds for any matrix $A=M_{n\times n}(\mathbb{C})$:

$$\frac{1}{\sqrt{n}}\left\|A\right\|_F\leq\left\|A\right\|_2\leq\left\|A\right\|_F.$$

In literature there are other types of norms of matrices. The maximum column sum matrix norm of $n\times n$ matrix $A=(a_{ij})$ is $\left\|A\right\|_1=\max\limits_{1\leq j\leq n}\sum_{i=1}^{n}\left|a_{ij}\right|$ and the maximum row sum matrix norm is $\left\|A\right\|_\infty=\max\limits_{1\leq i\leq n}\sum_{j=1}^{n}\left|a_{ij}\right|$.

Calculations of the above norms $\left\|A\right\|_F$, $\left\|A\right\|_2$, $c_1(A)$ and $r_1(A)$ require the sum of the squares of the numbers a_{ij} and calculations of the above norms $\left\|A\right\|_1$ and $\left\|A\right\|_\infty$ require the linear sum the numbers a_{ij}. We also note that the sum of entries of (a_{ij}) require the linear sum the numbers a_{ij}. As in our case, the numbers a_{ij} can be chosen as elements of second, third or higher order linear recurrence sequences.

In Table 3, we present a few special study on the Frobenius norm, spectral norm, maximum column length norm and maximum row length norm of circulant (r-circulant, geometric circulant, semicirculant) matrices with the generalized m-step Fibonacci sequences which require sum formulas of second powers of numbers in m-step Fibonacci sequences ($m = 2, 3, 4$).

Name of sequence	Papers
second order↓	second order↓
Fibonacci, Lucas	[9, 10, 20, 24, 27, 38, 43–49, 101]
Pell, Pell-Lucas	[2, 102]
Jacobsthal, Jacobsthal-Lucas	[36, 103–105]
third order↓	third order↓
Tribonacci, Tribonacci-Lucas	[23, 37, 39, 94]
Padovan, Perrin	[7, 32, 42]
Third-Order Pell Numbers	[93]
fourth order↓	fourth order↓
Tetranacci, Tetranacci-Lucas	[29]

Table 3. Papers on the norms.

Linear summing formulas of the generalized m-step Fibonacci sequences are required for the computation of various norms of circulant matrices with the generalized m-step Fibonacci sequences. We present some works on summing formulas of the numbers in Table 4.

Name of sequence	Papers which deal with summing formulas
Pell and Pell-Lucas	[1, 18, 25, 26, 30]
Generalized Fibonacci	[19, 63–67, 69, 82]
Generalized Tribonacci	[13, 16, 31, 68, 81, 83]
Generalized Tetranacci	[70, 75, 106]
Generalized Pentanacci	[71, 72, 84]
Generalized Hexanacci	[73, 74, 85]

Table 4. A few special study of sum formulas.

Also, the sum of the squares of the generalized m-step Fibonacci sequences are required for the computation of various norms of circulant matrices with the generalized m-step Fibonacci sequences. We present some works on sum formulas of powers of the numbers in Table 5

Name of sequence	sums of second powers	sums of third powers	sums of powers
Generalized Fibonacci	[3, 4, 17, 21, 22, 51, 57, 60, 61, 78, 89–92]	[15, 52, 54, 55, 58, 59, 79, 80, 86–88, 107]	[5, 14, 34]
Generalized Tribonacci	[39, 53, 56, 76]		
Generalized Tetranacci	[35, 41, 62, 77]		

TABLE 5. A few special study on sum formulas of second, third and arbitrary powers.

3. Sum Formulas of Generalized Hexanacci Numbers with Positive Subscripts

The following theorem presents some summing formulas of generalized Hexanacci numbers with positive subscripts.

Theorem 3.1. *Let x be a real (or complex) number. For $n \geq 0$ we have the following formula; If $sx^2 + tx^3 + ux^4 + vx^5 + x^6y + rx - 1 \neq 0$, then*

$$\sum_{k=0}^{n} kx^k W_k = \frac{\Omega_1}{(sx^2 + tx^3 + ux^4 + vx^5 + x^6y + rx - 1)^2},$$

where

$\Omega_1 = x^{n+5}(n(sx^2 + tx^3 + ux^4 + vx^5 + x^6y + rx - 1) - 5 + 3sx^2 + 2tx^3 + ux^4 - x^6y + 4rx)W_{n+5} + x^{n+4}(n(1 - rx)(sx^2 + tx^3 + ux^4 + vx^5 + x^6y + rx - 1) - 4 + 2sx^2 + tx^3 - vx^5 - 2x^6y - 4r^2x^2 + 8rx - 3rsx^3 - 2rtx^4 - rux^5 + rx^7y)W_{n+4} + x^{n+3}(-n(sx^2 + rx - 1)(sx^2 + tx^3 + ux^4 + vx^5 + x^6y + rx - 1) - 3 + 6sx^2 - ux^4 - 2vx^5 - 3x^6y - 3r^2x^2 - 3s^2x^4 + 6rx - 6rsx^3 - rtx^4 - 2stx^5 + rvx^6 - sux^6 + 2r x^7y + sx^8y)W_{n+3} + x^{n+2}(-n(sx^2 + tx^3 + rx - 1)(sx^2 + tx^3 + ux^4 + vx^5 + x^6y + rx - 1) - 2 + 4sx^2 + 4tx^3 - 2ux^4 - 3vx^5 - 4x^6y - 2r^2x^2 - 2s^2x^4 - 2t^2x^6 + 4rx - 4rsx^3 - 4rtx^4 + rux^5 - 4stx^5 + 2rvx^6 + svx^7 - tux^7 + 3rx^7y + 2sx^8y + tx^9y)W_{n+2} + x^{n+1}(-n(sx^2 + tx^3 + ux^4 + rx - 1)(sx^2 + tx^3 + ux^4 + vx^5 + x^6y + rx - 1) - 1 + 2sx^2 + 2tx^3 + 2ux^4 - 4vx^5 - 5x^6y - r^2x^2 - s^2x^4 - t^2x^6 - u^2x^8 + 2rx - 2rsx^3 - 2rtx^4 - 2rux^5 - 2stx^5 + 3rvx^6 - 2sux^6 + 2svx^7 - 2tux^7 + tvx^8 + 4rx^7y + 3sx^8y + 2tx^9y + ux^{10}y)W_{n+1} + yx^{n+6}(n(sx^2 + tx^3 + ux^4 + vx^5 + x^6y + rx - 1) - 6 + 4sx^2 + 3tx^3 + 2ux^4 + vx^5 + 5rx)W_n + x^5(yx^6 - ux^4 - 2tx^3 - 3sx^2 - 4rx + 5)W_5 + x^4(-2sx^2 - tx^3 + vx^5 + 2x^6y + 4r^2x^2 - 8rx + 3rsx^3 + 2rtx^4 + rux^5 - rx^7y + 4) W_4 + x^3(-6sx^2 + ux^4 + 2vx^5 + 3x^6y + 3r^2x^2 + 3s^2x^4 - 6rx + 6rsx^3 + rtx^4 + 2stx^5 - rvx^6 + sux^6 - 2rx^7y - sx^8y + 3)W_3 + x^2(-4sx^2 - 4tx^3 + 2ux^4 + 3vx^5 + 4x^6y + 2r^2x^2 + 2s^2x^4 + 2t^2x^6 - 4rx + 4rsx^3 + 4rtx^4 - rux^5 + 4stx^5 - 2rvx^6 - svx^7 + tux^7 - 3rx^7y - 2sx^8y - tx^9y + 2)W_2 + x(-2sx^2 - 2tx^3 - 2ux^4 + 4vx^5 + 5x^6y + r^2x^2 + s^2x^4 + t^2x^6 + u^2x^8 - 2rx + 2rsx^3 + 2rtx^4 + 2rux^5 + 2stx^5 -$

$3rvx^6+2sux^6-2svx^7+2tux^7-tvx^8-4rx^7y-3sx^8y-2tx^9y-ux^{10}y+1)W_1$
$+\,yx^6(-vx^5-2ux^4-3tx^3-4sx^2-5rx+6)W_0.$

Proof. Using the recurrence relation

$$W_n = rW_{n-1}+sW_{n-2}+tW_{n-3}+uW_{n-4}+vW_{n-5}+yW_{n-6}$$

i.e.,

$$yW_{n-6} = W_n-rW_{n-1}-sW_{n-2}-tW_{n-3}-uW_{n-4}-vW_{n-5},$$

we obtain

$$\begin{array}{rcl}
y\times 0\times x^0W_0 &=& 0\times x^0W_6-r\times 0\times x^0W_5-s\times 0\times x^0W_4\\
&& -t\times 0\times x^0W_3-u\times 0\times x^0W_2-v\times 0\times x^0W_1\\
y\times 1\times x^1W_1 &=& 1\times x^1W_7-r\times 1\times x^1W_6-s\times 1\times x^1W_5\\
&& -t\times 1\times x^1W_4-u\times 1\times x^1W_3-v\times 1\times x^1W_2\\
y\times 2\times x^2W_2 &=& 2\times x^2W_8-r\times 2\times x^2W_7-s\times 2\times x^2W_6\\
&& -t\times 2\times x^2W_5-u\times 2\times x^2W_4-v\times 2\times x^2W_3\\
&& \vdots
\end{array}$$

$$\begin{array}{rcl}
y(n-2)x^{n-2}W_{n-2} &=& (n-2)x^{n-2}W_{n+4}-r(n-2)x^{n-2}W_{n+3}\\
&& -s(n-2)x^{n-2}W_{n+2}-t(n-2)x^{n-2}W_{n+1}\\
&& -u(n-2)x^{n-2}W_n-v(n-2)x^{n-2}W_{n-1}\\
y(n-1)x^{n-1}W_{n-1} &=& (n-1)x^{n-1}W_{n+5}-r(n-1)x^{n-1}W_{n+4}\\
&& -s(n-1)x^{n-1}W_{n+3}-t(n-1)x^{n-1}W_{n+2}\\
&& -u(n-1)x^{n-1}W_{n+1}-v(n-1)x^{n-1}W_n\\
y\times n\times x^nW_n &=& n\times x^nW_{n+6}-r\times n\times x^nW_{n+5}-s\times n\times x^nW_{n+4}\\
&& -t\times n\times x^nW_{n+3}-u\times n\times x^nW_{n+2}\\
&& -v\times n\times x^nW_{n+1}.
\end{array}$$

If we add the equations side by side we obtain

$$
\begin{aligned}
y\sum_{k=0}^{n}kx^{k}W_{k} &= (nx^{n}W_{n+6}+(n-1)x^{n-1}W_{n+5}+(n-2)x^{n-2}W_{n+4}\\
&+(n-3)x^{n-3}W_{n+3}+(n-4)x^{n-4}W_{n+2}+(n-5)x^{n-5}W_{n+1}\\
&-(-1)x^{-1}W_{5}-(-2)x^{-2}W_{4}-(-3)x^{-3}W_{3}-(-4)x^{-4}W_{2}-(-5)x^{-5}W_{1}\\
&-(-6)x^{-6}W_{0}+\sum_{k=0}^{n}(k-6)x^{k-6}W_{k})-r(nx^{n}W_{n+5}+(n-1)x^{n-1}W_{n+4}\\
&+(n-2)x^{n-2}W_{n+3}+(n-3)x^{n-3}W_{n+2}+(n-4)x^{n-4}W_{n+1}\\
&-(-1)x^{-1}W_{4}-(-2)x^{-2}W_{3}-(-3)x^{-3}W_{2}-(-4)x^{-4}W_{1}-(-5)x^{-5}W_{0}\\
&+\sum_{k=0}^{n}(k-5)x^{k-5}W_{k})-s(nx^{n}W_{n+4}+(n-1)x^{n-1}W_{n+3}\\
&+(n-2)x^{n-2}W_{n+2}+(n-3)x^{n-3}W_{n+1}-(-1)x^{-1}W_{3}-(-2)x^{-2}W_{2}\\
&-(-3)x^{-3}W_{1}-(-4)x^{-4}W_{0}+\sum_{k=0}^{n}(k-4)x^{k-4}W_{k})-t(nx^{n}W_{n+3}\\
&+(n-1)x^{n-1}W_{n+2}+(n-2)x^{n-2}W_{n+1}-(-1)x^{-1}W_{2}-(-2)x^{-2}W_{1}\\
&-(-3)x^{-3}W_{0}+\sum_{k=0}^{n}(k-3)x^{k-3}W_{k})-u(nx^{n}W_{n+2}+(n-1)x^{n-1}W_{n+1}\\
&-(-1)x^{-1}W_{1}-(-2)x^{-2}W_{0}+\sum_{k=0}^{n}(k-2)x^{k-2}W_{k})\\
&-v(nx^{n}W_{n+1}-(-1)x^{-1}W_{0}+\sum_{k=0}^{n}(k-1)x^{k-1}W_{k}).
\end{aligned}
$$

Then, using Theorem 1.3, we get the required result. □

4. Special Cases

In this section, for the special cases of x, we present the closed form solutions (identities) of the sums $\sum_{k=0}^{n}kx^{k}W_{k}$, $\sum_{k=0}^{n}kx^{k}W_{2k}$ and $\sum_{k=0}^{n}kx^{k}W_{2k+1}$ for the specific case of sequence $\{W_n\}$.

4.1. **The case $x=1$.** In this subsection we consider the special case $x=1$. The case $x=1$ of Theorem 3.1 is given in Soykan [100].

4.2. **The case $x=-1$.** In this subsection we consider the special case $x=-1$ and we present the closed form solutions (identities) of the sums $\sum_{k=0}^{n}k(-1)^{k}W_{k}$, $\sum_{k=0}^{n}k(-1)^{k}W_{2k}$ and $\sum_{k=0}^{n}k(-1)^{k}W_{2k+1}$ for the specific case of the sequence $\{W_n\}$.

Taking $r=s=t=u=v=y=1$ in Theorem 3.1, we obtain the following proposition.

Proposition 4.1. *If $r = s = t = u = v = y = 1$, then for $n \geq 0$ we have the following formula:*
$\sum_{k=0}^{n} k(-1)^k W_k = (-1)^n ((n+8)W_{n+5} - (2n+15)W_{n+4} + (n+5)W_{n+3} - (2n+12)W_{n+2} + (n+2)W_{n+1} - (n+9)W_n) - 8W_5 + 15W_4 - 5W_3 + 12W_2 - 2W_1 + 9W_0$.

From the above proposition, we have the following corollary which gives sum formulas of Hexanacci and Hexanacci-Lucas numbers (take $W_n = H_n$ with $H_0 = 0, H_1 = 1, H_2 = 1, H_3 = 2, H_4 = 4, H_5 = 8$ and take $W_n = E_n$ with $E_0 = 6, E_1 = 1, E_2 = 3, E_3 = 7, E_4 = 15, E_5 = 31$, respectively).

Corollary 4.2. *For $n \geq 1$, we have the following properties:*

- **(a):** $\sum_{k=0}^{n} k(-1)^k H_k = (-1)^n ((n+8)H_{n+5} - (2n+15)H_{n+4} + (n+5)H_{n+3} - (2n+12)H_{n+2} + (n+2)H_{n+1} - (n+9)H_n) - 4$.
- **(b):** $\sum_{k=0}^{n} k(-1)^k E_k = (-1)^n ((n+8)E_{n+5} - (2n+15)E_{n+4} + (n+5)E_{n+3} - (2n+12)E_{n+2} + (n+2)E_{n+1} - (n+9)E_n) + 30$.

We present the next result as an example of the above corollary for $n = 8$.

Example 4.3. *For $n = 8$, we have the followings:*

- **(a):** $\sum_{k=0}^{8} k(-1)^k H_k = 347$.
- **(b):** $\sum_{k=0}^{8} k(-1)^k E_k = 1339$.

Taking $r = 2, s = t = u = v = y = 1$ in Theorem 3.1, we obtain the following proposition.

Proposition 4.4. *If $r = 2, s = t = u = v = y = 1$, then for $n \geq 1$ we have the following formula:*
$\sum_{k=0}^{n} k(-1)^k W_k = \frac{1}{2}((-1)^n ((n+6)W_{n+5} - (3n+17)W_{n+4} + (2n+8)W_{n+3} - (3n+12)W_{n+2} + (2n+3)W_{n+1} - (n+7)W_n) - 6W_5 + 17W_4 - 8W_3 + 12W_2 - 3W_1 + 7W_0)$.

From the above proposition, we have the following corollary which gives sum formulas of sixth-order Pell and sixth-order Pell-Lucas numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 5, P_4 = 13, P_5 = 34$ and take $W_n = Q_n$ with $Q_0 = 6, Q_1 = 2, Q_2 = 6, Q_3 = 17, Q_4 = 46, Q_5 = 122$, respectively).

Corollary 4.5. *For $n \geq 1$, we have the following properties:*

- **(a):** $\sum_{k=0}^{n} k(-1)^k P_k = \frac{1}{2}((-1)^n ((n+6)P_{n+5} - (3n+17)P_{n+4} + (2n+8)P_{n+3} - (3n+12)P_{n+2} + (2n+3)P_{n+1} - (n+7)P_n) - 2)$.
- **(b):** $\sum_{k=0}^{n} k(-1)^k Q_k = \frac{1}{2}((-1)^n ((n+6)Q_{n+5} - (3n+17)Q_{n+4} + (2n+8)Q_{n+3} - (3n+12)Q_{n+2} + (2n+3)Q_{n+1} - (n+7)Q_n) + 22)$.

We present the next result as an example of the above corollary for $n = 8$.

Example 4.6. *For $n = 8$, we have the followings:*

- **(a):** $\sum_{k=0}^{8} k(-1)^k P_k = 3645$.
- **(b):** $\sum_{k=0}^{8} k(-1)^k Q_k = 13070$.

Observe that setting $x = -1, r = 1, s = 1, t = 1, u = 1, v = 1, y = 2$ (i.e., for the generalized sixth order Jacobsthal case) in Theorem 3.1, makes the right hand side of the sum formulas to be an indeterminate form. Application of L'Hospital rule however provides the evaluation of the sum formulas.

Theorem 4.7. *If $r = 1, s = 1, t = 1, u = 1, v = 2, y = 2$, then for $n \geq 1$ we have the following formulas:*

$\sum_{k=0}^{n} k(-1)^k W_k = \frac{1}{54}((-1)^n(-(3n^2 + 11n - 118)W_{n+5} + 2(3n^2 + 8n - 122)W_{n+4} - (3n + 17)(n - 8)W_{n+3} + 2(3n^2 - n - 122)W_{n+2} - (3n^2 - 25n - 118)W_{n+1} + 2(3n^2 + 17n - 104)W_n) - 118W_5 + 244W_4 - 136W_3 + 244W_2 - 118W_1 + 208W_0)$.

Proof. We use Theorem 3.1. If we set $r = 1, s = 1, t = 1, u = 2$ in Theorem 3.1, then we have

$$\sum_{k=0}^{n} kx^k W_k = \frac{g_1(x)}{(2x-1)^2(x+1)^2(-x+x^2+1)^2(x+x^2+1)^2},$$

where

$g_1(x) = x^{n+5}(4x + n(2x^6 + x^5 + x^4 + x^3 + x^2 + x - 1) + 3x^2 + 2x^3 + x^4 - 2x^6 - 5)W_{n+5} - x^{n+4}(2x^2 - 8x + 2x^3 + 2x^4 + 2x^5 + 4x^6 - 2x^7 + n(x-1)(2x^6 + x^5 + x^4 + x^3 + x^2 + x - 1) + 4)W_{n+4} - x^{n+3}(6x^3 - 3x^2 - 6x + 5x^4 + 4x^5 + 6x^6 - 4x^7 - 2x^8 + n(x^2 + x - 1)(2x^6 + x^5 + x^4 + x^3 + x^2 + x - 1) + 3)W_{n+3} + x^{n+2}(4x + 2x^2 - 8x^4 - 6x^5 - 8x^6 + 6x^7 + 4x^8 + 2x^9 - n(x^3 + x^2 + x - 1)(2x^6 + x^5 + x^4 + x^3 + x^2 + x - 1) - 2)W_{n+2} + x^{n+1}(2x + x^2 - x^4 - 8x^5 - 10x^6 + 8x^7 + 6x^8 + 4x^9 + 2x^{10} - n(x^4 + x^3 + x^2 + x - 1)(2x^6 + x^5 + x^4 + x^3 + x^2 + x - 1) - 1)W_{n+1} + 2x^{n+6}(5x + n(2x^6 + x^5 + x^4 + x^3 + x^2 + x - 1) + 4x^2 + 3x^3 + 2x^4 + x^5 - 6)W_n - x^5(-2x^6 + x^4 + 2x^3 + 3x^2 + 4x - 5)W_5 + x^4(-2x^7 + 4x^6 + 2x^5 + 2x^4 + 2x^3 + 2x^2 - 8x + 4)W_4 + x^3(-2x^8 - 4x^7 + 6x^6 + 4x^5 + 5x^4 + 6x^3 - 3x^2 - 6x + 3)W_3 - x^2(2x^9 + 4x^8 + 6x^7 - 8x^6 - 6x^5 - 8x^4 + 2x^2 + 4x - 2)W_2 - x(2x^{10} + 4x^9 + 6x^8 + 8x^7 - 10x^6 - 8x^5 - x^4 + x^2 + 2x - 1)W_1 - 2x^6(x^5 + 2x^4 + 3x^3 + 4x^2 + 5x - 6)W_0$.

For $x = 1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule. Then we get the required result using

$$\begin{aligned}\sum_{k=0}^{n} k(-1)^k W_k &= \left.\frac{\frac{d^2}{dx^2}(g_1(x))}{\frac{d^2}{dx^2}((2x-1)^2(x+1)^2(-x+x^2+1)^2(x+x^2+1)^2)}\right|_{x=-1} \\ &= \frac{1}{54}((-1)^n(-(3n^2 + 11n - 118)W_{n+5} \\ &\quad +2(3n^2 + 8n - 122)W_{n+4} - (3n + 17)(n - 8)W_{n+3} \\ &\quad +2(3n^2 - n - 122)W_{n+2} - (3n^2 - 25n - 118)W_{n+1} \\ &\quad +2(3n^2 + 17n - 104)W_n) - 118W_5 \\ &\quad +244W_4 - 136W_3 + 244W_2 - 118W_1 + 208W_0).\end{aligned}$$

□

Taking, respectively,

$W_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 1, J_3 = 1, J_4 = 1, J_5 = 1$ (sixth-order Jacobsthal numbers),

$W_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5, j_3 = 10, j_4 = 20, j_5 = 40$ (sixth order Jacobsthal-Lucas numbers),

$W_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3, K_3 = 10, K_4 = 20, K_5 = 40$ (modified sixth order Jacobsthal numbers),

$W_n = Q_n$ with $Q_0 = 3, Q_1 = 0, Q_2 = 2, Q_3 = 8, Q_4 = 16, Q_5 = 32$ (sixth-order Jacobsthal Perrin numbers),

$W_n = S_n$ with $S_0 = 0, S_1 = 1, S_2 = 1, S_3 = 2, S_4 = 4, S_5 = 8$ (adjusted sixth-order Jacobsthal numbers),

$W_n = R_n$ with $R_0 = 6, R_1 = 1, R_2 = 3, R_3 = 7, R_4 = 15, R_5 = 31$ (modified sixth-order Jacobsthal-Lucas numbers),

in the last Theorem, we have the following corollary.

Corollary 4.8. *For $n \geq 0$, we have the following properties:*

- **(a):** $\sum_{k=0}^{n} k(-1)^k J_k = \frac{1}{54}((-1)^n(-(3n^2+11n-118)J_{n+5}+2(3n^2+8n-122)J_{n+4}-(3n+17)(n-8)J_{n+3}+2(3n^2-n-122)J_{n+2}-(3n^2-25n-118)J_{n+1}+2(3n^2+17n-104)J_n)+116)$.
- **(b):** $\sum_{k=0}^{n} k(-1)^k j_k = \frac{1}{54}((-1)^n(-(3n^2+11n-118)j_{n+5}+2(3n^2+8n-122)j_{n+4}-(3n+17)(n-8)j_{n+3}+2(3n^2-n-122)j_{n+2}-(3n^2-25n-118)j_{n+1}+2(3n^2+17n-104)j_n)+318)$.
- **(c):** $\sum_{k=0}^{n} k(-1)^k K_k = \frac{1}{54}((-1)^n(-(3n^2+11n-118)K_{n+5}+2(3n^2+8n-122)K_{n+4}-(3n+17)(n-8)K_{n+3}+2(3n^2-n-122)K_{n+2}-(3n^2-25n-118)K_{n+1}+2(3n^2+17n-104)K_n)+38)$.
- **(d):** $\sum_{k=0}^{n} k(-1)^k Q_k = \frac{1}{54}((-1)^n(-(3n^2+11n-118)Q_{n+5}+2(3n^2+8n-122)Q_{n+4}-(3n+17)(n-8)Q_{n+3}+2(3n^2-n-122)Q_{n+2}-(3n^2-25n-118)Q_{n+1}+2(3n^2+17n-104)Q_n)+152)$.
- **(e):** $\sum_{k=0}^{n} k(-1)^k S_k = \frac{1}{54}((-1)^n(-(3n^2+11n-118)S_{n+5}+2(3n^2+8n-122)S_{n+4}-(3n+17)(n-8)S_{n+3}+2(3n^2-n-122)S_{n+2}-(3n^2-25n-118)S_{n+1}+2(3n^2+17n-104)S_n)-114)$.
- **(f):** $\sum_{k=0}^{n} k(-1)^k R_k = \frac{1}{54}((-1)^n(-(3n^2+11n-118)R_{n+5}+2(3n^2+8n-122)R_{n+4}-(3n+17)(n-8)R_{n+3}+2(3n^2-n-122)R_{n+2}-(3n^2-25n-118)R_{n+1}+2(3n^2+17n-104)R_n)+912)$.

We present the next result as an example of the above corollary for $n = 8$.

Example 4.9. *For $n = 8$, we have the followings:*

- **(a):** $\sum_{k=0}^{8} k(-1)^k J_k = 118$.
- **(b):** $\sum_{k=0}^{8} k(-1)^k j_k = 1776$.
- **(c):** $\sum_{k=0}^{8} k(-1)^k K_k = 1738$.
- **(d):** $\sum_{k=0}^{8} k(-1)^k Q_k = 1382$.
- **(e):** $\sum_{k=0}^{8} k(-1)^k S_k = 356$.
- **(f):** $\sum_{k=0}^{8} k(-1)^k R_k = 1454$.

Taking $r = 2, s = 3, t = 5, u = 7, v = 11, y = 13$ in Theorem 3.1, we obtain the following proposition.

Proposition 4.10. *If $r = 2, s = 3, t = 5, u = 7, v = 11, y = 13$, then for $n \geq 1$ we have the following formula:*
$\sum_{k=0}^{n} k(-1)^k W_k = \frac{1}{4}((-1)^n(-(n-5)W_{n+5} + (3n-16)W_{n+4} + 4W_{n+3} + (5n-29)W_{n+2} + (2n-1)W_{n+1} + 13(n-4)W_n) - 5W_5 + 16W_4 - 4W_3 + 29W_2 + W_1 + 52W_0)$.

From the above proposition, we have the following corollary which gives sum formulas of 6-primes, Lucas 6-primes and modified 6-primes numbers (take $W_n = G_n$ with $G_0 = 0, G_1 = 0, G_2 = 0, G_3 = 0, G_4 = 1, G_5 = 2$ and take $W_n = H_n$ with $H_0 = 6, H_1 = 2, H_2 = 10, H_3 = 41, H_4 = 150, H_5 = 542$ and take $W_n = E_n$ with $E_0 = 0, E_1 = 0, E_2 = 0, E_3 = 0, E_4 = 1, E_5 = 1$, respectively).

Corollary 4.11. *For $n \geq 1$, we have the following properties:*

(a): $\sum_{k=0}^{n} k(-1)^k G_k = \frac{1}{4}((-1)^n(-(n-5)G_{n+5} + (3n-16)G_{n+4} + 4G_{n+3} + (5n-29)G_{n+2} + (2n-1)G_{n+1} + 13(n-4)G_n) + 6)$.

(b): $\sum_{k=0}^{n} k(-1)^k H_k = \frac{1}{4}((-1)^n(-(n-5)H_{n+5} + (3n-16)H_{n+4} + 4H_{n+3} + (5n-29)H_{n+2} + (2n-1)H_{n+1} + 13(n-4)H_n) + 130)$.

(c): $\sum_{k=0}^{n} k(-1)^k E_k = \frac{1}{4}((-1)^n(-(n-5)E_{n+5} + (3n-16)E_{n+4} + 4E_{n+3} + (5n-29)E_{n+2} + (2n-1)E_{n+1} + 13(n-4)E_n) + 11)$.

We present the next result as an example of the above corollary for $n = 8$.

Example 4.12. *For $n = 8$, we have the followings:*

(a): $\sum_{k=0}^{8} k(-1)^k G_k = 565$.

(b): $\sum_{k=0}^{8} k(-1)^k H_k = 149336$.

(c): $\sum_{k=0}^{8} k(-1)^k E_k = 407$.

4.3. **The case $x = i$.** In this subsection we consider the special case $x = i$.

Taking $x = i, r = s = t = u = v = y = 1$ in Theorem 3.1, we obtain the following proposition.

Proposition 4.13. *If $r = s = t = u = v = y = 1$, then for $n \geq 0$ we have the following formula:*
$\sum_{k=0}^{n} k i^k W_k = \frac{1}{3-4i}(i^n(-((1+2i)n + (2+6i))W_{n+5} - ((1-3i)n + (2-7i))W_{n+4} + ((4+3i)n + (6+7i))W_{n+3} + ((4-2i)n + (6+4i))W_{n+2} + ((8+4i) - (1+2i)n)W_{n+1} + ((2-i)n + (8-3i))W_n) + (2+6i)W_5 + (2-7i)W_4 - (6+7i)W_3 - (6+4i)W_2 - (8+4i)W_1 - (8-3i)W_0)$.

From the above proposition, we have the following corollary which gives sum formulas of Hexanacci and Hexanacci-Lucas numbers (take $W_n = H_n$ with $H_0 = 0, H_1 = 1, H_2 = 1, H_3 = 2, H_4 = 4, H_5 = 8$ and take $H_n = E_n$ with $E_0 = 6, E_1 = 1, E_2 = 3, E_3 = 7, E_4 = 15, E_5 = 31$, respectively).

Corollary 4.14. *For $n \geq 0$, we have the following properties:*

(a): $\sum_{k=0}^{n} ki^k H_k = \frac{1}{3-4i}(i^n(-((1+2i)n+(2+6i))H_{n+5} - ((1-3i)n + (2-7i))H_{n+4} + ((4+3i)n+(6+7i))H_{n+3} + ((4-2i)n+(6+4i)) H_{n+2} + ((8+4i)-(1+2i)n)H_{n+1} + ((2-i)n+(8-3i))H_n) - (2+2i))$.

(b): $\sum_{k=0}^{n} ki^k E_k = \frac{1}{3-4i}(i^n(-((1+2i)n+(2+6i))E_{n+5} - ((1-3i)n + (2-7i))E_{n+4} + ((4+3i)n+(6+7i))E_{n+3} + ((4-2i)n+(6+4i)) E_{n+2} + ((8+4i)-(1+2i)n)E_{n+1} + ((2-i)n+(8-3i))E_n) + (-24+34i))$.

Corresponding sums of the other sixth order generalized Hexanacci numbers can be calculated similarly.

5. Sum Formulas of Generalized Hexanacci Numbers with Negative Subscripts

The following tTheorem presents some summing formulas of generalized Hexanacci numbers with negative subscripts.

Theorem 5.1. *Let x be a real (or complex) number. For $n \geq 1$ we have the following formulas: If $y + rx^5 + sx^4 + tx^3 + ux^2 + vx - x^6 \neq 0$, then*

$$\sum_{k=1}^{n} kx^k W_{-k} = \frac{\Omega_2}{(y + rx^5 + sx^4 + tx^3 + ux^2 + vx - x^6)^2},$$

where

$\Omega_2 = x^{n+1}(n(-y - rx^5 - sx^4 - tx^3 - ux^2 - vx + x^6) + 4rx^5 + 3sx^4 + 2tx^3 + ux^2 - y - 5x^6)W_{-n+5} + x^{n+1}(n(r-x)(y + rx^5 + sx^4 + tx^3 + ux^2 + vx - x^6) + 8rx^6 + 2sx^5 + tx^4 - vx^2 - 4r^2x^5 + ry - 2xy - 4x^7 - 3rsx^4 - 2rtx^3 - rux^2)W_{-n+4} + x^{n+1}(n(s + rx - x^2)(y + rx^5 + sx^4 + tx^3 + ux^2 + vx - x^6) + 6rx^7 + 6sx^6 - ux^4 - 2vx^3 - 3x^2y - 3r^2x^6 - 3s^2x^4 + sy - 3x^8 - 6rsx^5 - rtx^4 - 2stx^3 + rvx^2 - sux^2 + 2rxy)W_{-n+3} + x^{n+1}(n(t + rx^2 + sx - x^3)(y + rx^5 + sx^4 + tx^3 + ux^2 + vx - x^6) + 4rx^8 + 4sx^7 + 4tx^6 - 2ux^5 - 3vx^4 - 4x^3y - 2r^2x^7 - 2s^2x^5 - 2t^2x^3 + ty - 2x^9 - 4rsx^6 - 4rtx^5 + rux^4 - 4stx^4 + 2rvx^3 + svx^2 - tux^2 + 3rx^2y + 2sxy)W_{-n+2} + x^{n+1}(n(u + rx^3 + sx^2 + tx - x^4)(y + rx^5 + sx^4 + tx^3 + ux^2 + vx - x^6) + 2rx^9 + 2sx^8 + 2tx^7 + 2ux^6 - 4vx^5 - 5x^4y - r^2x^8 - s^2x^6 - t^2x^4 - u^2x^2 + uy - x^{10} - 2rsx^7 - 2rtx^6 - 2rux^5 - 2stx^5 + 3rvx^4 - 2sux^4 + 2svx^3 - 2tux^3 + tvx^2 + 4rx^3y + 3sx^2y + 2txy)W_{-n+1} + x^{n+1}(n(v + rx^4 + sx^3 + tx^2 + ux - x^5)(y + rx^5 + sx^4 + tx^3 + ux^2 + vx - x^6) - 6x^5y + vy + 5rx^4y + 4sx^3y + 3tx^2y + 2uxy)W_{-n} + x(y - 4rx^5 - 3sx^4 - 2tx^3 - ux^2 + 5x^6)W_5 + x(-8rx^6 - 2sx^5 - tx^4 + vx^2 + 4r^2x^5 - ry + 2xy + 4x^7 + 3rsx^4 + 2rtx^3 + rux^2)W_4 + x(-6rx^7 - 6sx^6 + ux^4 + 2vx^3 + 3x^2y + 3r^2x^6 + 3s^2x^4 - sy + 3x^8 + 6rsx^5 + rtx^4 + 2stx^3 - rvx^2 + sux^2 - 2rxy) W_3 + x(-4rx^8 - 4sx^7 - 4tx^6 + 2ux^5 + 3vx^4 + 4x^3y + 2r^2x^7 + 2s^2x^5 + 2t^2x^3 - ty + 2x^9 + 4rsx^6 + 4rtx^5 - rux^4 + 4stx^4 - 2rvx^3 - svx^2 + tux^2 - 3rx^2y - 2sxy) W_2 + x(-2rx^9 - 2sx^8 - 2tx^7 - 2ux^6 + 4vx^5 + 5x^4y + r^2x^8 + s^2x^6 + t^2x^4 + u^2x^2 - uy + x^{10} + 2rsx^7 + 2rtx^6 + 2rux^5 + 2stx^5 - 3rvx^4 + 2sux^4 - 2svx^3 + 2tux^3 - tvx^2 - 4rx^3y - 3sx^2y - 2txy)W_1 + xy(-v - 5rx^4 - 4sx^3 - 3tx^2 - 2ux + 6x^5)W_0.$

Proof. Using the recurrence relation

$$W_{-n} = \frac{1}{y}W_{-n+6} - \frac{v}{y}W_{-n+1} - \frac{u}{y}W_{-n+2} - \frac{t}{y}W_{-n+3} - \frac{s}{y}W_{-n+4} - \frac{r}{y}W_{-n+5}$$

i.e.,

$$yW_{-n} = W_{-n+6} - rW_{-n+5} - sW_{-n+4} - tW_{-n+3} - uW_{-n+2} - vW_{-n+1},$$

we obtain

$$\begin{aligned}
y \times n \times x^n W_{-n} &= n \times x^n W_{-n+6} - r \times n \times x^n W_{-n+5} \\
&\quad - s \times n \times x^n W_{-n+4} - t \times n \times x^n W_{-n+3} \\
&\quad - u \times n \times x^n W_{-n+2} - v \times n \times x^n W_{-n+1} \\
y(n-1)x^{n-1}W_{-n+1} &= (n-1)x^{n-1}W_{-n+7} - r(n-1)x^{n-1}W_{-n+6} \\
&\quad - s(n-1)x^{n-1}W_{-n+5} - t(n-1)x^{n-1}W_{-n+4} \\
&\quad - u(n-1)x^{n-1}W_{-n+3} - v(n-1)x^{n-1}W_{-n+2} \\
y(n-2)x^{n-2}W_{-n+2} &= (n-2)x^{n-2}W_{-n+8} - r(n-2)x^{n-2}W_{-n+7} \\
&\quad - s(n-2)x^{n-2}W_{-n+6} - t(n-2)x^{n-2}W_{-n+5} \\
&\quad - u(n-2)x^{n-2}W_{-n+4} - v(n-2)x^{n-2}W_{-n+3} \\
&\quad \vdots
\end{aligned}$$

$$\begin{aligned}
y \times 3 \times x^3 W_{-3} &= 3 \times x^3 W_3 - r \times 3 \times x^3 W_2 - s \times 3 \times x^3 W_1 \\
&\quad - t \times 3 \times x^3 W_0 - u \times 3 \times x^3 W_{-1} - v \times 3 \times x^3 W_{-2} \\
y \times 2 \times x^2 W_{-2} &= 2 \times x^2 W_4 - r \times 2 \times x^2 W_3 - s \times 2 \times x^2 W_2 \\
&\quad - t \times 2 \times x^2 W_1 - u \times 2 \times x^2 W_0 - v \times 2 \times x^2 W_{-1} \\
y \times 1 \times x^1 W_{-1} &= 1 \times x^1 W_5 - r \times 1 \times x^1 W_4 - s \times 1 \times x^1 W_3 \\
&\quad - t \times 1 \times x^1 W_2 - u \times 1 \times x^1 W_1 - v \times 1 \times x^1 W_0.
\end{aligned}$$

If we add the equations side by side we obtain

$$
\begin{aligned}
y\sum_{k=1}^{n}kx^{k}W_{-k} &= (-(n+1)x^{n+1}W_{-n+5}-(n+2)x^{n+2}W_{-n+4} \\
&-(n+3)x^{n+3}W_{-n+3}-(n+4)x^{n+4}W_{-n+2} \\
&-(n+5)x^{n+5}W_{-n+1}-(n+6)x^{n+6}W_{-n}+1\times x^{1}W_{5} \\
&+2x^{2}W_{4}+3x^{3}W_{3}+4x^{4}W_{2}+5x^{5}W_{1}+6x^{6}W_{0} \\
&+\sum_{k=1}^{n}(k+6)x^{k+6}W_{-k})-r(-(n+1)x^{n+1}W_{-n+4} \\
&-(n+2)x^{n+2}W_{-n+3}-(n+3)x^{n+3}W_{-n+2} \\
&-(n+4)x^{n+4}W_{-n+1}-(n+5)x^{n+5}W_{-n}+1\times x^{1}W_{4} \\
&+2x^{2}W_{3}+3x^{3}W_{2}+4x^{4}W_{1}+5x^{5}W_{0}+\sum_{k=1}^{n}(k+5)x^{k+5}W_{-k}) \\
&-s(-(n+1)x^{n+1}W_{-n+3}-(n+2)x^{n+2}W_{-n+2} \\
&-(n+3)x^{n+3}W_{-n+1}-(n+4)x^{n+4}W_{-n}+1\times x^{1}W_{3} \\
&+2x^{2}W_{2}+3x^{3}W_{1}+4x^{4}W_{0}+\sum_{k=1}^{n}(k+4)x^{k+4}W_{-k}) \\
&-t(-(n+1)x^{n+1}W_{-n+2}-(n+2)x^{n+2}W_{-n+1} \\
&-(n+3)x^{n+3}W_{-n}+1\times x^{1}W_{2}+2x^{2}W_{1}+3x^{3}W_{0} \\
&+\sum_{k=1}^{n}(k+3)x^{k+3}W_{-k})-u(-(n+1)x^{n+1}W_{-n+1} \\
&-(n+2)x^{n+2}W_{-n}+1\times x^{1}W_{1}+2x^{2}W_{0} \\
&+\sum_{k=1}^{n}(k+2)x^{k+2}W_{-k})-v(-(n+1)x^{n+1}W_{-n} \\
&+1\times x^{1}W_{0}+\sum_{k=1}^{n}(k+1)x^{k+1}W_{-k})
\end{aligned}
$$

Then, using Theorem 1.4, we get the required result. □

6. **Specific Cases**

In this section, for the specific cases of x, we present the closed form solutions (identities) of the sums $\sum_{k=1}^{n}kx^{k}W_{-k}$, $\sum_{k=1}^{n}kx^{k}W_{-2k}$ and $\sum_{k=1}^{n}kx^{k}W_{-2k+1}$ for the specific case of sequence $\{W_n\}$.

6.1. **The case $x=1$.**

In this subsection we consider the special case $x=1$. The case $x=1$ of Theorem 5.1 is given in Soykan [100].

6.2. **The case $x = -1$.** In this subsection we consider the special case $x = -1$.

Taking $r = s = t = u = v = y = 1$ in Theorem 5.1, we obtain the following proposition.

Proposition 6.1. *If $r = s = t = u = v = y = 1$, then for $n \geq 0$ we have the following formula:*
$\sum_{k=1}^{n} k(-1)^k W_{-k} = (-1)^n (-(n-8)W_{-n+5} + (2n-15)W_{-n+4} - (n-5)W_{-n+3} + (2n-12)W_{-n+2} - (n-2)W_{-n+1} + (2n-9)W_{-n}) - 8W_5 + 15W_4 - 5W_3 + 12W_2 - 2W_1 + 9W_0.$

From the above proposition, we have the following corollary which gives sum formulas of Hexanacci and Hexanacci-Lucas numbers (take $W_n = H_n$ with $H_0 = 0, H_1 = 1, H_2 = 1, H_3 = 2, H_4 = 4, H_5 = 8$ and take $W_n = E_n$ with $E_0 = 6, E_1 = 1, E_2 = 3, E_3 = 7, E_4 = 15, E_5 = 31$, respectively).

Corollary 6.2. *For $n \geq 1$, we have the following properties:*

(a): $\sum_{k=1}^{n} k(-1)^k H_{-k} = (-1)^n (-(n-8)H_{-n+5} + (2n-15)H_{-n+4} - (n-5)H_{-n+3} + (2n-12)H_{-n+2} - (n-2)H_{-n+1} + (2n-9)H_{-n}) - 4.$

(b): $\sum_{k=1}^{n} k(-1)^k E_{-k} = (-1)^n (-(n-8)E_{-n+5} + (2n-15)E_{-n+4} - (n-5)E_{-n+3} + (2n-12)E_{-n+2} - (n-2)E_{-n+1} + (2n-9)E_{-n}) + 30.$

Taking $r = 2, s = t = u = v = y = 1$ in Theorem 5.1, we obtain the following Proposition.

Proposition 6.3. *If $r = 2, s = t = u = v = y = 1$, then for $n \geq 1$ we have the following formula:*
$\sum_{k=1}^{n} k(-1)^k W_{-k} = \frac{1}{2}((-1)^n (-(n-6)W_{-n+5} + (3n-17)W_{-n+4} - (2n-8)W_{-n+3} + (3n-12)W_{-n+2} - (2n-3)W_{-n+1} + (3n-7)W_{-n}) - 6W_5 + 17W_4 - 8W_3 + 12W_2 - 3W_1 + 7W_0).$

From the above proposition, we have the following corollary which gives sum formulas of sixth-order Pell and sixth-order Pell-Lucas numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 5, P_4 = 13, P_5 = 34$ and take $W_n = Q_n$ with $Q_0 = 6, Q_1 = 2, Q_2 = 6, Q_3 = 17, Q_4 = 46, Q_5 = 122$, respectively).

Corollary 6.4. *For $n \geq 1$, we have the following properties:*

(a): $\sum_{k=1}^{n} k(-1)^k P_{-k} = \frac{1}{2}((-1)^n (-(n-6)P_{-n+5} + (3n-17)P_{-n+4} - (2n-8)P_{-n+3} + (3n-12)P_{-n+2} - (2n-3)P_{-n+1} + (3n-7)P_{-n}) - 2).$

(b): $\sum_{k=1}^{n} k(-1)^k Q_{-k} = \frac{1}{2}((-1)^n (-(n-6)Q_{-n+5} + (3n-17)Q_{-n+4} - (2n-8)Q_{-n+3} + (3n-12)Q_{-n+2} - (2n-3)Q_{-n+1} + (3n-7)Q_{-n}) + 22).$

Observe that setting $x = -1, r = 1, s = 1, t = 1, u = 1, v = 1, y = 2$ (i.e., for the generalized sixth order Jacobsthal case) in Theorem 5.1, makes the right hand side of the sum formulas to be an indeterminate form. Application of L'Hospital rule however provides the evaluation of the sum formulas.

Theorem 6.5. *If $r = 1, s = 1, t = 1, u = 1, v = 2, y = 2$, then for $n \geq 1$ we have the following formulas:*

$\sum_{k=1}^{n} k(-1)^k W_{-k} = \frac{1}{54}((-1)^n(-(3n^2-11n-96)W_{-n+5} + 2(3n^2-8n-100)W_{-n+4} - (3n^2+7n-114)W_{-n+3} + 2(3n^2+n-100)W_{-n+2} - (3n^2+25n-96)W_{-n+1} + 2(3n^2+10n-82)W_{-n}) - 96W_5 + 200W_4 - 114W_3 + 200W_2 - 96W_1 + 164W_0)$.

Proof. We use Theorem 5.1. If we set $r = 1, s = 1, t = 1, u = 2$ in Theorem 5.1, then we have

$$\sum_{k=1}^{n} kx^k W_{-k} = \frac{g_2(x)}{(x-2)^2(x+1)^2(x+x^2+1)^2(-x+x^2+1)^2},$$

where

$g_2(x) = x^{n+1}(x^2 - n(-x^6+x^5+x^4+x^3+x^2+x+2) + 2x^3 + 3x^4 + 4x^5 - 5x^6 - 2)W_{5-n} - x^{n+1}(4x+2x^2+2x^3+2x^4+2x^5-8x^6+4x^7+n(x-1)(-x^6+x^5+x^4+x^3+x^2+x+2)-2)W_{4-n} - x^{n+1}(6x^2-4x+4x^3+5x^4+6x^5-3x^6-6x^7+3x^8-n(-x^2+x+1)(-x^6+x^5+x^4+x^3+x^2+x+2)-2)W_{3-n} + x^{n+1}(4x+6x^2-8x^3-6x^4-8x^5+2x^7+4x^8-2x^9+n(-x^3+x^2+x+1)(-x^6+x^5+x^4+x^3+x^2+x+2)+2)W_{2-n} + x^{n+1}(4x+6x^2+8x^3-10x^4-8x^5-x^6+x^8+2x^9-x^{10}+n(-x^4+x^3+x^2+x+1)(-x^6+x^5+x^4+x^3+x^2+x+2)+2)W_{1-n} + x^{n+1}(4x+6x^2+8x^3+10x^4-12x^5+n(-x^5+x^4+x^3+x^2+x+1)(-x^6+x^5+x^4+x^3+x^2+x+2)+2)W_{-n} - x(-5x^6+4x^5+3x^4+2x^3+x^2-2)W_5 + x(4x^7-8x^6+2x^5+2x^4+2x^3+2x^2+4x-2)W_4 + x(3x^8-6x^7-3x^6+6x^5+5x^4+4x^3+6x^2-4x-2)W_3 - x(-2x^9+4x^8+2x^7-8x^5-6x^4-8x^3+6x^2+4x+2)W_2 - x(-x^{10}+2x^9+x^8-x^6-8x^5-10x^4+8x^3+6x^2+4x+2)W_1 - 2x(-6x^5+5x^4+4x^3+3x^2+2x+1)W_0$

For $x = -1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule. Then we get the requiret result using

$$\begin{aligned}\sum_{k=1}^{n} k(-1)^k W_{-k} &= \left.\frac{\frac{d^2}{dx^2}(g_2(x))}{\frac{d^2}{dx^2}((x-2)^2(x+1)^2(x+x^2+1)^2(-x+x^2+1)^2)}\right|_{x=-1} \\ &= \frac{1}{54}((-1)^n(-(3n^2-11n-96)W_{-n+5} \\ &\quad +2(3n^2-8n-100)W_{-n+4} - (3n^2+7n-114)W_{-n+3} \\ &\quad +2(3n^2+n-100)W_{-n+2} - (3n^2+25n-96)W_{-n+1} \\ &\quad +2(3n^2+10n-82)W_{-n}) - 96W_5 + 200W_4 - 114W_3 \\ &\quad +200W_2 - 96W_1 + 164W_0).\end{aligned}$$

□

Taking, respectively,

$W_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 1, J_3 = 1, J_4 = 1, J_5 = 1$ (sixth-order Jacobsthal numbers),

$W_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5, j_3 = 10, j_4 = 20, j_5 = 40$ (sixth order Jacobsthal-Lucas numbers),

$W_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3, K_3 = 10, K_4 = 20, K_5 = 40$ (modified sixth order Jacobsthal numbers),

$W_n = Q_n$ with $Q_0 = 3, Q_1 = 0, Q_2 = 2, Q_3 = 8, Q_4 = 16, Q_5 = 32$ (sixth-order Jacobsthal Perrin numbers),

$W_n = S_n$ with $S_0 = 0, S_1 = 1, S_2 = 1, S_3 = 2, S_4 = 4, S_5 = 8$ (adjusted sixth-order Jacobsthal numbers),

$W_n = R_n$ with $R_0 = 6, R_1 = 1, R_2 = 3, R_3 = 7, R_4 = 15, R_5 = 31$ (modified sixth-order Jacobsthal-Lucas numbers),

in the last theorem, we have the following corollary.

Corollary 6.6. *For $n \geq 1$, we have the following properties:*

- **(a):** $\sum_{k=1}^{n} k(-1)^k J_{-k} = \frac{1}{54}((-1)^n(-(3n^2-11n-96)J_{-n+5} + 2(3n^2-8n-100)J_{-n+4} - (3n^2+7n-114)J_{-n+3} + 2(3n^2+n-100)J_{-n+2} - (3n^2+25n-96)J_{-n+1} + 2(3n^2+10n-82)J_{-n}) + 94)$.
- **(b):** $\sum_{k=1}^{n} k(-1)^k j_{-k} = \frac{1}{54}((-1)^n(-(3n^2-11n-96)j_{-n+5} + 2(3n^2-8n-100)j_{-n+4} - (3n^2+7n-114)j_{-n+3} + 2(3n^2+n-100)j_{-n+2} - (3n^2+25n-96)j_{-n+1} + 2(3n^2+10n-82)j_{-n}) + 252)$.
- **(c):** $\sum_{k=1}^{n} k(-1)^k K_{-k} = \frac{1}{54}((-1)^n(-(3n^2-11n-96)K_{-n+5} + 2(3n^2-8n-100)K_{-n+4} - (3n^2+7n-114)K_{-n+3} + 2(3n^2+n-100)K_{-n+2} - (3n^2+25n-96)K_{-n+1} + 2(3n^2+10n-82)K_{-n}) + 16)$.
- **(d):** $\sum_{k=1}^{n} k(-1)^k Q_{-k} = \frac{1}{54}((-1)^n(-(3n^2-11n-96)Q_{-n+5} + 2(3n^2-8n-100)Q_{-n+4} - (3n^2+7n-114)Q_{-n+3} + 2(3n^2+n-100)Q_{-n+2} - (3n^2+25n-96)Q_{-n+1} + 2(3n^2+10n-82)Q_{-n}) + 108)$.
- **(e):** $\sum_{k=1}^{n} k(-1)^k S_{-k} = \frac{1}{54}((-1)^n(-(3n^2-11n-96)S_{-n+5} + 2(3n^2-8n-100)S_{-n+4} - (3n^2+7n-114)S_{-n+3} + 2(3n^2+n-100)S_{-n+2} - (3n^2+25n-96)S_{-n+1} + 2(3n^2+10n-82)S_{-n}) - 92)$.
- **(f):** $\sum_{k=1}^{n} k(-1)^k R_{-k} = \frac{1}{54}((-1)^n(-(3n^2-11n-96)R_{-n+5} + 2(3n^2-8n-100)R_{-n+4} - (3n^2+7n-114)R_{-n+3} + 2(3n^2+n-100)R_{-n+2} - (3n^2+25n-96)R_{-n+1} + 2(3n^2+10n-82)R_{-n}) + 714)$.

Taking $r = 2, s = 3, t = 5, u = 7, v = 11, y = 13$ in Theorem 5.1, we obtain the following proposition.

Proposition 6.7. *If $r = 2, s = 3, t = 5, u = 7, v = 11, y = 13$, then for $n \geq 1$ we have the following formula:*

$\sum_{k=1}^{n} k(-1)^k W_{-k} = \frac{1}{4}((-1)^n((n+5)W_{-n+5} - (3n+16)W_{-n+4} + 4W_{-n+3} - (5n+29)W_{-n+2} - (2n+1)W_{-n+1} - (9n+52)W_{-n}) - 5W_5 + 16W_4 - 4W_3 + 29W_2 + W_1 + 52W_0)$.

From the above proposition, we have the following corollary which gives sum formulas of 6-primes, Lucas 6-primes and modified 6-primes numbers (take $W_n = G_n$ with $G_0 = 0, G_1 = 0, G_2 = 0, G_3 = 0, G_4 = 1, G_5 = 2$ and take $W_n = H_n$ with $H_0 = 6, H_1 = 2, H_2 = 10, H_3 = 41, H_4 = 150, H_5 = 542$ and take $W_n = E_n$ with $E_0 = 0, E_1 = 0, E_2 = 0, E_3 = 0, E_4 = 1, E_5 = 1$, respectively).

Corollary 6.8. *For $n \geq 1$, we have the following properties:*

(a): $\sum_{k=1}^{n} k(-1)^k G_{-k} = \frac{1}{4}((-1)^n((n+5)G_{-n+5} - (3n+16)G_{-n+4} + 4G_{-n+3} - (5n+29)G_{-n+2} - (2n+1)G_{-n+1} - (9n+52)G_{-n}) + 6)$.

(b): $\sum_{k=1}^{n} k(-1)^k H_{-k} = \frac{1}{4}((-1)^n((n+5)H_{-n+5} - (3n+16)H_{-n+4} + 4H_{-n+3} - (5n+29)H_{-n+2} - (2n+1)H_{-n+1} - (9n+52)H_{-n}) + 130)$.

(c): $\sum_{k=1}^{n} k(-1)^k E_{-k} = \frac{1}{4}((-1)^n((n+5)E_{-n+5} - (3n+16)E_{-n+4} + 4E_{-n+3} - (5n+29)E_{-n+2} - (2n+1)E_{-n+1} - (9n+52)E_{-n}) + 11)$.

6.3. **The case $x = i$.** In this subsection, we consider the special case $x = i$.

Taking $r = s = t = u = v = y = 1$ in Theorem 5.1, we obtain the following proposition.

Proposition 6.9. *If $r = s = t = u = v = y = 1$, then for $n \geq 1$ we have the following formula:*

$\sum_{k=1}^{n} ki^k W_{-k} = \frac{1}{3+4i}(i^n(((1-2i)n - (2-6i))W_{5-n} + ((1+3i)n - (2+7i))W_{4-n} + ((6-7i) - (4-3i)n)W_{3-n} + ((6-4i) - (4+2i)n)W_{2-n} + ((8-4i) + (1-2i)n)W_{1-n} + ((8+3i) + (1+3i)n)W_{-n}) + (2-6i)W_5 + (2+7i)W_4 - (6-7i) W_3 - (6-4i)W_2 - (8-4i)W_1 - (8+3i)W_0)$.

From the above proposition, we have the following corollary which gives sum formulas of Hexanacci and Hexanacci-Lucas numbers (take $W_n = H_n$ with $H_0 = 0, H_1 = 1, H_2 = 1, H_3 = 2, H_4 = 4, H_5 = 8$ and take $H_n = E_n$ with $E_0 = 6, E_1 = 1, E_2 = 3, E_3 = 7, E_4 = 15, E_5 = 31$, respectively).

Corollary 6.10. *For $n \geq 1$, we have the following properties:*

(a): $\sum_{k=1}^{n} ki^k H_{-k} = \frac{1}{3+4i}(i^n(((1-2i)n - (2-6i))H_{5-n} + ((1+3i)n - (2+7i))H_{4-n} + ((6-7i) - (4-3i)n)H_{3-n} + ((6-4i) - (4+2i)n)H_{2-n} + ((8-4i) + (1-2i)n)H_{1-n} + ((8+3i) + (1+3i)n)H_{-n}) + (-2+2i))$.

(b): $\sum_{k=1}^{n} ki^k E_{-k} = \frac{1}{3+4i}(i^n(((1-2i)n - (2-6i))E_{5-n} + ((1+3i)n - (2+7i))E_{4-n} + ((6-7i) - (4-3i)n)E_{3-n} + ((6-4i) - (4+2i)n)E_{2-n} + ((8-4i) + (1-2i)n)E_{1-n} + ((8+3i) + (1+3i)n)E_{-n}) - (24+34i))$.

Corresponding sums of the other sixth order generalized Hexanacci numbers can be calculated similarly.

7. Conclusion

Recently, there have been so many studies of the sequences of numbers in the literature and the sequences of numbers were widely used in many research areas, such as architecture, nature, art, physics and engineering, see for example [11], [12]. In this work, sum identities were proved. The method used in this paper can be used for the other linear recurrence sequences, too. We have written sum identities in terms of the generalized Hexanacci sequence, and then we have presented the formulas as special cases the corresponding identity for the Hexanacci, Hexanacci-Lucas, and other sixth-order recurrence sequences. All the listed identities in the corollaries may be proved by induction, but that method of proof gives no clue about their discovery. We give the proofs to indicate how these identities, in general, were discovered.

Computations of the Frobenius norm, spectral norm, maximum column length norm and maximum row length norm of circulant (r-circulant, geometric circulant, semicirculant) matrices with the generalized m-step Fibonacci sequences require the sum of the numbers of the sequences. So, our results can be used to study circulant (r-circulant, geometric circulant, semicirculant) matrices with second-order linear recurrence sequences.

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