Journal of Mahani Mathematical Research
Shahid Bahonar University of Kerman
Print ISSN: 2251-7952 Online ISSN: 2645-4505

HIGHER HOMOMORPHISMS AND THEIR APPROXIMATIONS

S.Kh. Ekrami

Article type: Research Article
(Received: 14 April 2022, Received in revised form 01 September 2022)
(Accepted: 08 September 2022, Published Online: 09 September 2022)

Abstract. In this paper, we introduce a class of higher homomorphisms on an algebra $\mathcal{A}$ and we characterize the structure of them as a linear combination of some sequences of homomorphisms. Also, we prove that for any approximate higher ring homomorphism on a Banach algebra $\mathcal{A}$ under some sequences of control funtions, there exists a unique higher ring homomorphism near it. Using special sequences of control functions, we show that the approximate higher ring homomorphism is an exact higher ring homomorphism.

Keywords: Banach algebra, Higher homomorphism, Approximate higher homomorphism, Fixed-point Theorem.
2020 MSC: Primary 47L10, 39B82, 47H10.

1. Introduction

Let $\mathcal{A}$ be an algebra. Suppose that $\{H_n\}_{n=1}^{\infty}$ is a sequence of linear mappings from $\mathcal{A}$ into $\mathcal{A}$. $\{H_n\}_{n=1}^{\infty}$ is called a

(1) *higher homomorphism* (resp., *higher anti-homomorphism*), if it satisfies the equation

$$H_n(xy) = \sum_{i=1}^{n} H_i(x)H_i(y) \quad \big(\text{resp.,} \quad H_n(xy) = \sum_{i=1}^{n} H_i(y)H_i(x)\big)$$

for all $x, y \in \mathcal{A}$ and each positive integer n,

(2) *Jordan higher homomorphism*, if it satisfies the equation

$$H_n(xy + yx) = \sum_{i=1}^{n} H_i(x)H_i(y) + H_i(y)H_i(x)$$

for all $x, y \in \mathcal{A}$ and each positive integer n.

A. K. Faraj et al. [7] proved that every Jordan higher homomorphism of a ring R onto a 2-torsion free prime ring R' is either a higher homomorphism or a higher anti-homomorphism.

There is no more information about higher homomorphisms and their structures. In this paper, we introduce a class of higher homomorphisms and we

✉ khalil.ekrami@gmail.com, ORCID: 0000-0002-6233-5741
DOI: 10.22103/jmmr.2022.19303.1233
Publisher: Shahid Bahonar University of Kerman
How to cite: S.Kh. Ekrami, *Higher Homomorphisms and Their Approximations*, J. Mahani Math. Res. 2023; 12(1): 327-337.
© the Author

characterize the structure of them as a linear combination of some sequences of homomorphisms.

Let $\mathcal{F}$ denote a functional equation. We say that $\mathcal{F}$ is *stable*, if any approximate solution of $\mathcal{F}$ is near to a true solution of $\mathcal{F}$. We say that $\mathcal{F}$ is *super-stable*, if every approximate solution is an exact solution of it. The stability problem of functional equations originated from the following question of Ulam [23]: *Under what condition does there exist an additive mapping near an approximately additive mapping?* Hyers [9] gave a partial affirmative answer to the question of Ulam in the context of Banach spaces. A generalized version of the theorem of Hyers for approximately linear mapping was given by Th. M. Rassias [21]. After that, several functional equations have been extensively investigated by a number of authors (for instances, [1,3,4,6,8,10–17,20,22]). A new method for obtaining the existence of exact solutions and error estimations, based on the fixed point alternative proposed by Radu [19] (see also [2,18]). Radu employed the following result to prove the stability of a Cauchy functional equation (see also [5]).

Proposition 1.1. *(The fixed point alternative principle). Let (X, d) be a generalized complete metric space and $J : X \to X$ be a strictly contractive mapping; that is*

$$d(J(x), J(y)) \leq Ld(x, y) \quad (\forall x, y \in X)$$

for some (Lipschitz) constant $0 < L < 1$. Then, for a given element $x \in X$, exactly one of the following assertions is true: either

- (a) *$d(J^n x, J^{n+1}x) = \infty$ for all $n \geq 0$, or*
- (b) *there exists some integer n_0 such that $d(J^n x, J^{n+1}x) < \infty$ for all $n \geq n_0$.*

Actually, if (b) holds, then

- (b_1) *the sequence $\{J^n x\}$ converges to a fixed point x^* of J,*
- (b_2) *x^* is the unique fixed point of J in $X_0 := \{y \in X; d(J^{n_0}x, y) < \infty\}$;*
- (b_3) *$d(y, x^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in X_0$.*

In this paper, using the fixed point alternative principle, we prove that for any approximate higher ring homomorphism on a Banach algebra $\mathcal{A}$ under some sequences of control funtions, there exists a unique higher ring homomorphism near it. Using special sequences of control functions, we show that the approximate higher ring homomorphism is a higher ring homomorphism.

2. Higher homomorphisms

Definition 2.1. Let $\mathcal{A}$ be an algebra. A sequence $\{H_n\}_{n=1}^{\infty}$ of linear mappings from $\mathcal{A}$ into $\mathcal{A}$ is called a *higher homomorphism* on $\mathcal{A}$, if it satisfies the equation

$$H_n(xy) = \sum_{i=1}^{n} H_i(x)H_i(y)$$

for each $x, y \in \mathcal{A}$ and each positive integer n. When $\{H_n\}_{n=1}^{\infty}$ is a higher homomorphism, H_1 is a homomorphism.

In this paper, let $\{\lambda_n\}_{n=0}^{\infty}$ be the sequence of complex numbers satisfying the equations

$$\begin{aligned}
\lambda_0 &= 1 \\
\lambda_1 &= \sum_{i=0}^{1} \lambda_i^2 = \lambda_0^2 + \lambda_1^2 \\
\lambda_2 &= \sum_{i=0}^{2} \lambda_i^2 = \lambda_0^2 + \lambda_1^2 + \lambda_2^2 \\
&\vdots \\
\lambda_n &= \sum_{i=0}^{n} \lambda_i^2 = \lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \cdots + \lambda_n^2
\end{aligned} \tag{1}$$

for all $n \in \mathbb{N}$. In the next theorem, we introduce a class of higher homomorphisms on an algebra.

Theorem 2.2. *Let $\mathcal{A}$ be an algebra and $\{h_n\}_{n=1}^{\infty}$ be a sequence of homomorphisms from $\mathcal{A}$ into $\mathcal{A}$ such that $h_i(x)h_j(y) = 0$ for all $i, j \in \mathbb{N}$ with $i \neq j$ and for all $x, y \in \mathcal{A}$. Then the sequence $\{H_n\}_{n=1}^{\infty}$ of mappings from $\mathcal{A}$ into $\mathcal{A}$ defined by*

$$H_n = \sum_{i=1}^{n} \lambda_{n-i} h_i = \lambda_{n-1} h_1 + \lambda_{n-2} h_2 + \lambda_{n-3} h_3 + \cdots + \lambda_0 h_n \tag{2}$$

for all $n \in \mathbb{N}$, is a higher homomorphism on $\mathcal{A}$.

Proof. Let $n \in \mathbb{N}$, trivially each H_n is linear. It follows from (1) and (2) that

$$H_n(xy) = \sum_{i=1}^{n} \lambda_{n-i} h_i(xy) = \sum_{i=1}^{n} \sum_{j=0}^{n-i} \lambda_j^2 h_i(xy)$$

for all $x, y \in \mathcal{A}$. In the above summation, we have $1 \leq i + j \leq n$ and $i \neq 0$. Thus if we put $r = i + j$, then we can write it as the form $\sum_{r=1}^{n} \sum_{i+j=r, i\neq 0}$. Putting $j = r - i$, we indeed have

$$H_n(xy) = \sum_{r=1}^{n} \sum_{i=1}^{r} \lambda_{r-i}^2 h_i(xy) = \sum_{r=1}^{n} \sum_{i=1}^{r} \lambda_{r-i}^2 h_i(x) h_i(y).$$

Also since for all $i, j \in \mathbb{N}$ with $i \neq j$, $h_i(x)h_j(y) = 0$ for all $x, y \in \mathcal{A}$, we have

$$\begin{aligned} H_n(xy) &= \sum_{r=1}^{n}\sum_{i=1}^{r}\sum_{j=1}^{r} \lambda_{r-i}\lambda_{r-j}h_i(x)h_j(y) \\ &= \sum_{r=1}^{n}\Big(\sum_{i=1}^{r}\lambda_{r-i}h_i(x)\Big)\Big(\sum_{j=1}^{r}\lambda_{r-j}h_j(y)\Big) \\ &= \sum_{r=1}^{n} H_r(x)H_r(y) \end{aligned}$$

for all $x, y \in \mathcal{A}$. This completes the proof. □

Example 2.3. *Let $\mathcal{A}$ be the algebra of all bounded complex sequences*

$$\ell^\infty = \big\{x = (x_k)_{k\in\mathbb{N}} \subset \mathbb{C}, \|x\|_\infty = \sup_{k\in\mathbb{N}} |x_k| < \infty\big\}.$$

For each $n \in \mathbb{N}$ define the mapping $h_n : \ell^\infty \to \ell^\infty$ by

$$h_n(x_1, x_2, x_3, \ldots) = (0, \ldots, 0, \underbrace{x_n}_{n\text{th}}, 0, \ldots)$$

for all $x = (x_1, x_2, x_3, \ldots) \in \ell^\infty$. Then $\{h_n\}_{n=1}^\infty$ is a sequence of homomorphisms on ℓ^∞ such that

$$h_i(x)h_j(y) = (0, \ldots, 0, \underbrace{x_i}_{i\text{th}}, 0, \ldots)(0, \ldots, 0, \underbrace{x_j}_{j\text{th}}, 0, \ldots) = 0$$

for all $i, j \in \mathbb{N}$ with $i \neq j$ and for all $x, y \in \ell^\infty$. It follows from Theorem 2.2 that the sequence $\{H_n\}_{n=1}^\infty$ defined by

$$\begin{aligned} H_1(x_1, x_2, x_3, \ldots) &= (\lambda_0 x_1, 0, 0, 0, \ldots) \\ H_2(x_1, x_2, x_3, \ldots) &= (\lambda_1 x_1, \lambda_0 x_2, 0, 0, 0, \ldots) \\ H_3(x_1, x_2, x_3, \ldots) &= (\lambda_2 x_1, \lambda_1 x_2, \lambda_0 x_3, 0, 0, 0, \ldots) \\ &\vdots \\ H_n(x_1, x_2, x_3, \ldots) &= (\lambda_{n-1} x_1, \lambda_{n-2} x_2, \lambda_{n-3} x_3, \ldots, \lambda_0 x_n, 0, 0, 0, \ldots) \end{aligned}$$

for all $x = (x_1, x_2, x_3, \ldots) \in \ell^\infty$ is a higher homomorphism on ℓ^∞.

Corollary 2.4. *Let $\mathcal{A}$ be an algebra and h be a homomorphism from $\mathcal{A}$ into $\mathcal{A}$. Then the sequence $\{H_n\}_{n=1}^\infty$ of mappings from $\mathcal{A}$ into $\mathcal{A}$ defined by*

(3) $$H_n = \lambda_{n-1} h$$

for all $n \in \mathbb{N}$, is a higher homomorphism on $\mathcal{A}$.

Proof. Let $h : \mathcal{A} \to \mathcal{A}$ be a homomorphism. The sequence of homomorphisms $\{h_n\}_{n=1}^\infty$ with $h_1 = h$ and $h_n = 0$ for $n \geq 2$, satisfies the condition of Theorem 2.2. Thus the sequence of mappings $\{H_n\}_{n=1}^\infty$ defined by $H_n = \lambda_{n-1} h$ for all $n \in \mathbb{N}$, is a higher homomorphism on $\mathcal{A}$. □

3. Approximate higher homomorphism

In this section, first we introduce the concept of higher ring homomorphisms.

Definition 3.1. Let $\mathcal{A}$ be a Banach algebra and $a, b \neq 0, \pm 1$ be real numbers. A sequence of mappings $\{H_n\}_{n=1}^{\infty}$ from $\mathcal{A}$ into $\mathcal{A}$ is called a higher ring homomorphism, if

$$H_n(ax+by) = aH_n(x) + bH_n(y),$$
$$H_n(xy) = \sum_{i=1}^{n} H_i(x)H_i(y)$$

for all $x, y \in \mathcal{A}$ and all $n \in \mathbb{N}$.

Now, using the fixed point alternative principle, we prove that for any approximate higher ring homomorphism on a Banach algebra $\mathcal{A}$ under some sequences of control funtions, there exists a unique higher ring homomorphism near it.

Theorem 3.2. *Let $\mathcal{A}$ be a Banach algebra and $a, b \neq 0, \pm 1$ be real numbers. Suppose that $\{\varphi_n : \mathcal{A} \times \mathcal{A} \to [0, \infty)\}$ and $\{\psi_n : \mathcal{A} \times \mathcal{A} \to [0, \infty)\}$ are sequences of functions for which there exist constants $0 < L, L' < 1$ such that for each $n \in \mathbb{N}$,*

$$\varphi_n(x,y) \leq |a|L\varphi_n\Big(\frac{x}{a}, \frac{y}{a}\Big), \quad \psi_n(x,y) \leq |a|^2 L'\psi_n\Big(\frac{x}{a}, \frac{y}{a}\Big) \tag{4}$$

for all $x, y \in \mathcal{A}$. If $\{f_n\}_{n=1}^{\infty}$ is a sequence of mappings from $\mathcal{A}$ into $\mathcal{A}$ such that for any $n \in \mathbb{N}$, $f_n(0) = 0$ and

$$\|f_n(ax+by) - af_n(x) - bf_n(y)\| \leq \varphi_n(x,y), \tag{5}$$

$$\Big\|f_n(xy) - \sum_{i=1}^{n} f_i(x)f_i(y)\Big\| \leq \psi_n(x,y) \tag{6}$$

for all $x, y \in \mathcal{A}$, then there exists a unique higher ring homomorphism $\{H_n\}_{n=1}^{\infty}$ such that for every $n \in \mathbb{N}$,

$$\|f_n(x) - H_n(x)\| \leq \frac{1}{|a|(1-L)}\varphi_n(x,0) \quad (x \in \mathcal{A}). \tag{7}$$

Proof. It follows from (4) that

$$\lim_{k\to\infty} \frac{\varphi_n(a^k x, a^k y)}{|a|^k} = \lim_{k\to\infty} \frac{\psi_n(a^k x, a^k y)}{|a|^{2k}} = 0 \tag{8}$$

for all $x, y \in \mathcal{A}$. Putting $y = 0$ in (5), we get

$$\|f_n(ax) - af_n(x)\| \leq \varphi_n(x,0) \quad (x \in \mathcal{A}) \tag{9}$$

and so

$$\Big\|f_n(x) - \frac{f_n(ax)}{a}\Big\| \leq \frac{1}{|a|}\varphi_n(x,0) \quad (x \in \mathcal{A}). \tag{10}$$

Let $n \in \mathbb{N}$ be fixed. Let $X = \{g_n : \mathcal{A} \to \mathcal{A}, g_n(0) = 0\}$ and define the generalized metric $d : X \times X \to [0, \infty]$ by

$$d(g_n, h_n) = \inf\{\alpha > 0 : \|g_n(x) - h_n(x)\| \leq \alpha\varphi_n(x, 0), \forall x \in \mathcal{A}\} \quad (g_n, h_n \in X).$$

Then (X, d) is a complete generalized metric space (See the proof of [2, Theorem 2.5]). Define $J : X \to X$ by $J(g_n)(x) = \frac{1}{a} g_n(ax)$ for each $x \in \mathcal{A}$. Then J is a strictly contractive mapping on X with the Lipschitz constant L. From (10) we have

$$\|J(f_n)(x) - f_n(x)\| = \left\| \frac{1}{a} f_n(ax) - f_n(x) \right\| \leq \frac{1}{|a|} \varphi_n(x, 0)$$

for each $x \in \mathcal{A}$. This means that $d(J(f_n), f_n) \leq \frac{1}{|a|}$. Therefore, by Proposition 1.1, J has a unique fixed point in the set $X_0 = \{g_n \in X : d(J(f_n), g_n) < \infty\}$. Let $H_n : \mathcal{A} \to \mathcal{A}$ be the unique fixed point of J. We have $\lim_k (J^k(f_n), H_n) = 0$, so H_n is defined by

$$H_n(x) := \lim_{k \to \infty} \frac{f_n(a^k x)}{a^k} \quad (x \in \mathcal{A}). \tag{11}$$

On the other hand, we have $d(f_n, J(f_n)) \leq \frac{1}{|a|}$ and $J(H_n) = H_n$, then

$$d(f_n, H_n) \leq d(f_n, J(f_n)) + d(J(f_n), J(H_n)) \leq \frac{1}{|a|} + L d(f_n, H_n).$$

So

$$d(f_n, H_n) \leq \frac{1}{|a|(1 - L)},$$

which implies the inequality (7).

Let $x, y \in \mathcal{A}$. It follows from (5) and (8) that for every $n \in \mathbb{N}$,

$$\begin{aligned} &\|H_n(ax + by) - aH_n(x) - bH_n(y)\| \\ &= \lim_{k \to \infty} \left\| \frac{f_n(a^k(ax + by))}{a^k} - a\frac{f_n(a^k x)}{a^k} - b\frac{f_n(a^k y)}{a^k} \right\| \leq \lim_{k \to \infty} \frac{\varphi_n(a^k x, a^k y)}{|a|^k} = 0. \end{aligned}$$

That is, for each $n \in \mathbb{N}$, H_n is additive.

Let $x, y \in \mathcal{A}$ and $n \in \mathbb{N}$. Replacing x by $a^k x$ and y by $a^k y$ in (6) and dividing by $|a|^{2k}$, we have

$$\frac{1}{|a|^{2k}} \left\| f_n((a^k x)(a^k y)) - \sum_{i=1}^{n} f_i(a^k x) f_i(a^k y) \right\| \leq \frac{\psi_n(a^k x, a^k y)}{|a|^{2k}}$$

which tends to zero as $k \to \infty$. Since the sequence $\{a^{-k} f_i(a^k x)\}$ converges for all $x \in \mathcal{A}$, it is bounded. Thus for each $x \in \mathcal{A}$ there is a $C_x > 0$ such that

$\|a^{-k}f_i(a^kx)\| \le C_x$. Therefore

$$\begin{aligned}
&\Big\|H_n(xy) - \sum_{i=1}^{n} H_i(x)H_i(y)\Big\| \\
&\le \Big\|H_n(xy) - \tfrac{f_n(a^{2k}xy)}{a^{2k}}\Big\| + \Big\|\tfrac{f_n(a^{2k}xy)}{a^{2k}} - \sum_{i=1}^{n}(\tfrac{f_i(a^kx)}{a^k})(\tfrac{f_i(a^kx)}{a^k})\Big\| \\
&\quad + \Big\|\sum_{i=1}^{n}(\tfrac{f_i(a^kx)}{a^k} - H_i(x))H_i(y)\Big\| + \Big\|\sum_{i=1}^{n}(\tfrac{f_i(a^kx)}{a^k})(\tfrac{f_i(a^ky)}{a^k} - H_i(y))\Big\| \\
&\le \Big\|H_n(xy) - \tfrac{f_n(a^{2k}xy)}{a^{2k}}\Big\| + |a|^{-2k}\Big\|f_n((a^kx)(a^ky)) - \sum_{i=1}^{n} f_i(a^kx)f_i(a^ky)\Big\| \\
&\quad + \sum_{i=1}^{n}\Big\|\tfrac{f_i(a^kx)}{a^k} - H_i(x)\Big\|\|H_i(y)\| + \sum_{i=1}^{n} C_x\Big\|\tfrac{f_i(a^ky)}{a^k} - H_i(y)\Big\|
\end{aligned}$$

which tends to zero as $k \to \infty$, i.e., the sequence $\{H_n\}_{n=1}^{\infty}$ is a higher ring homomorphism and this completes the proof. □

Theorem 3.3. *Let $\mathcal{A}$ be a Banach algebra and $a, b \neq 0, \pm 1$ be real numbers. Suppose that $\{\varphi_n : \mathcal{A} \times \mathcal{A} \to [0, \infty)\}$ and $\{\psi_n : \mathcal{A} \times \mathcal{A} \to [0, \infty)\}$ are sequences of functions for which there exist constants $0 < L, L' < 1$ such that for each $n \in \mathbb{N}$,*

$$\varphi_n(x, y) \le \frac{L\varphi_n(ax, ay)}{|a|}, \quad \psi_n(x, y) \le \frac{L'\psi_n(ax, ay)}{|a|^2} \tag{12}$$

for all $x, y \in \mathcal{A}$. If $\{f_n\}_{n=1}^{\infty}$ is a sequence of mappings from $\mathcal{A}$ into $\mathcal{A}$ such that for any $n \in \mathbb{N}$, $f_n(0) = 0$ and

$$\|f_n(ax + by) - af_n(x) - bf_n(y)\| \le \varphi_n(x, y), \tag{13}$$

$$\Big\|f_n(xy) - \sum_{i=1}^{n} f_i(x)f_i(y)\Big\| \le \psi_n(x, y) \tag{14}$$

for all $x, y \in \mathcal{A}$, then there exists a unique higher ring homomorphism $\{H_n\}_{n=1}^{\infty}$ such that for every $n \in \mathbb{N}$,

$$\|f_n(x) - H_n(x)\| \le \frac{L}{|a|(1 - L)}\varphi_n(x, 0) \quad (x \in \mathcal{A}). \tag{15}$$

Proof. As in the proof of Theorem 3.2, let $n \in \mathbb{N}$ and consider the complete generalized metric space (X, d) and define the strictly contractive mapping J on X by $J(g_n)(x) = ag_n(\frac{x}{a})$ for each $x \in \mathcal{A}$. Replacing x by $\frac{x}{a}$ in (9), it follows from (12) that

$$\|J(f_n)(x) - f_n(x)\| = \Big\|af_n(\frac{x}{a}) - f_n(x)\Big\| \le \varphi_n(\frac{x}{a}, 0) \le \frac{L}{|a|}\varphi_n(x, 0) \quad (x \in \mathcal{A}).$$

This means that $d(J(f_n), f_n) \leq \frac{L}{|a|}$ and then we can similarly find the unique fixed point of J in the set $X_0 = \{g_n \in X : d(J(f_n), g_n) < \infty\}$ as

$$H_n(x) = \lim_{k\to\infty} a^k f_n(\frac{x}{a^k}) \quad (x \in \mathcal{A})$$

such that

$$d(f_n, H_n) \leq \frac{L}{|a|(1-L)}.$$

The rest of proof is similar to the Theorem 3.2. □

Example 3.4. *Let $\{f_n : \mathbb{R} \to \mathbb{R}\}_{n=1}^{\infty}$ be the sequence of mappings defined by*

$$f_n(x) = \lambda_{n-1}\Big(x + \frac{|x|^2}{|x|+1}\Big) \quad (x \in \mathbb{R}).$$

Then for any $n \in \mathbb{N}$, $f_n(0) = 0$ and

$$\begin{aligned}
&|f_n(2x+2y) - 2f_n(x) - 2f_n(y)| \\
&= \Big|\lambda_{n-1}\Big(2x+2y+\tfrac{|2x+2y|^2}{|2x+2y|+1}\Big) - 2\lambda_{n-1}\Big(x+\tfrac{|x|^2}{|x|+1}\Big) - 2\lambda_{n-1}\Big(y+\tfrac{|y|^2}{|y|+1}\Big)\Big| \\
&= |\lambda_{n-1}|\Big|\tfrac{|2x+2y|^2}{|2x+2y|+1} - \tfrac{2|x|^2}{|x|+1} - \tfrac{2|y|^2}{|y|+1}\Big| \\
&\leq |\lambda_{n-1}|\Big(|2x+2y|^2 + 2|x|^2 + 2|y|^2\Big) \\
&= |\lambda_{n-1}|\big(6|x|^2 + 6|y|^2 + 8|xy|\big),
\end{aligned}$$

$$\begin{aligned}
&\Big|f_n(xy) - \sum_{i=1}^{n} f_i(x)f_i(y)\Big| \\
&= \Big|\lambda_{n-1}\Big(xy+\tfrac{|xy|^2}{|xy|+1}\Big) - \sum_{i=1}^{n}\lambda_{i-1}^2\Big(x+\tfrac{|x|^2}{|x|+1}\Big)\Big(y+\tfrac{|y|^2}{|y|+1}\Big)\Big| \\
&= \Big|\lambda_{n-1}\Big(xy+\tfrac{|xy|^2}{|xy|+1}\Big) - \Big(x+\tfrac{|x|^2}{|x|+1}\Big)\Big(y+\tfrac{|y|^2}{|y|+1}\Big)\sum_{i=1}^{n}\lambda_{i-1}^2\Big| \\
&= |\lambda_{n-1}|\Big|\Big(xy+\tfrac{|xy|^2}{|xy|+1}\Big) - \Big(x+\tfrac{|x|^2}{|x|+1}\Big)\Big(y+\tfrac{|y|^2}{|y|+1}\Big)\Big| \quad \Big(\sum_{i=1}^{n}\lambda_{i-1}^2 = \lambda_{n-1}\Big) \\
&= |\lambda_{n-1}|\Big|\tfrac{|xy|^2}{|xy|+1} - \tfrac{|xy|^2}{(|x|+1)(|y|+1)} - \tfrac{y|x|^2}{|x|+1} - \tfrac{x|y|^2}{|y|+1}\Big| \\
&\leq |\lambda_{n-1}|\big(2|xy|^2 + |x||y|^2 + |x|^2|y|\big),
\end{aligned}$$

for all $x, y \in \mathcal{A}$.

For each $n \in \mathbb{N}$, the functions $\varphi_n(x,y) = |\lambda_{n-1}|\big(6|x|^2+6|y|^2+8|xy|\big)$ and $\psi_n(x,y) = |\lambda_{n-1}|\big(2|xy|^2+|x||y|^2+|x|^2|y|\big)$ satisfy in (12) for $L = L' = \frac{1}{2}$. Thus it follows from Theorem 3.3 that the sequence of mappings $\{H_n : \mathbb{R} \to \mathbb{R}\}_{n=1}^{\infty}$ defined by

$$H_n(x) = \lim_{k\to\infty} a^k f_n(\tfrac{x}{a^k}) = \lambda_{n-1}x \quad (x \in \mathbb{R})$$

is the unique higher ring homomorphism such that for every $n \in \mathbb{N}$,

$$|f_n(x) - H_n(x)| \leq 3|x|^2 \quad (x \in \mathbb{R}).$$

Corollary 3.5. *Let* $\mathcal{A}$ *be a Banach algebra and* $a, b \neq 0, \pm 1$ *be real numbers. Let* $p, \theta_n, \theta'_n > 0$ *(*$n \in \mathbb{N}$*) be real numbers such that* $p > 1$*, whenever* $|a| > 1$ *and* $0 < p < 1$*, whenever* $0 < |a| < 1$*. If* $\{f_n\}_{n=1}^{\infty}$ *is a sequence of mappings from* $\mathcal{A}$ *into* $\mathcal{A}$ *such that for any* $n \in \mathbb{N}$*,* $f_n(0) = 0$ *and*

$$\|f_n(ax + by) - af_n(x) - bf_n(y)\| \leq \theta_n(\|x\|^p + \|y\|^p),$$

$$\left\| f_n(xy) - \sum_{i=1}^{n} f_i(x) f_i(y) \right\| \leq \theta'_n \|x\|^p \|y\|^p$$

for all $x, y \in \mathcal{A}$*, then there exists a unique higher ring homomorphism* $\{H_n\}_{n=1}^{\infty}$ *such that for every* $n \in \mathbb{N}$*,*

$$\|f_n(x) - H_n(x)\| \leq \frac{\theta_n}{|a|^p - |a|} \|x\|^p \quad (x \in \mathcal{A}).$$

Proof. The proof follows from Theorem 3.3 by taking $\varphi_n(x, y) = \theta_n(\|x\|^p + \|y\|^p)$ and $\psi_n(x, y) = \theta'_n \|x\|^p \|y\|^p$ for all $x, y \in \mathcal{A}$. Note that for each $n \in \mathbb{N}$, $\varphi_n(x, y)$ satisfies (12) for $0 < L = |a|^{1-p} < 1$ and $\psi_n(x, y)$ satisfies (12) for $0 < L' = |a|^{2(1-p)} < 1$. □

As a corollary of Theorem 3.3, we prove that for special sequences of control functions $\{\varphi_n\}_{n=1}^{\infty}$ and $\{\psi_n\}_{n=1}^{\infty}$, any approximate higher ring homomorphism on a Banach algebra is a higher ring homomorphism.

Corollary 3.6. *Let* $\mathcal{A}$ *be a Banach algebra and* $a, b \neq 0, \pm 1$ *be real numbers. Let* $p, q, \theta_n, \theta'_n > 0$ *(*$n \in \mathbb{N}$*) be real numbers such that* $p + q > 2$*, whenever* $|a| > 1$ *and* $0 < p + q < 2$*, whenever* $0 < |a| < 1$*. If* $\{f_n\}_{n=1}^{\infty}$ *is a sequence of mappings from* $\mathcal{A}$ *into* $\mathcal{A}$ *such that for any* $n \in \mathbb{N}$*,* $f_n(0) = 0$ *and*

$$\|f_n(ax + by) - af_n(x) - bf_n(y)\| \leq \theta_n \|x\|^{\frac{p}{2}} \|y\|^{\frac{q}{2}},$$

$$\left\| f_n(xy) - \sum_{i=1}^{n} f_i(x) f_i(y) \right\| \leq \theta'_n \|x\|^p \|y\|^q$$

for all $x, y \in \mathcal{A}$*, then* $\{f_n\}_{n=1}^{\infty}$ *is a higher ring homomorphism.*

Proof. The proof follows from Theorem 3.3 by taking $\varphi_n(x, y) = \theta_n \|x\|^{\frac{p}{2}} \|y\|^{\frac{q}{2}}$ and $\psi_n(x, y) = \theta'_n \|x\|^p \|y\|^q$ for all $x, y \in \mathcal{A}$. Note that for each $n \in \mathbb{N}$, $\varphi_n(x, y)$ satisfies (12) for $0 < L = |a|^{1-\frac{p+q}{2}} < 1$ and $\psi_n(x, y)$ satisfies (12) for $0 < L' = |a|^{2-(p+q)} < 1$. It follows from (15) that $\{f_n\}_{n=1}^{\infty}$ is a higher ring homomorphism. □

4. **Conclusion**

In this paper, we introduce a class of higher homomorphisms on an algebra $\mathcal{A}$ and we characterize the structure of them as a linear combination of some sequences of homomorphisms. Also we prove that for any approximate higher ring homomorphism on a Banach algebra $\mathcal{A}$ under some sequences of control funtions, there exists a unique higher ring homomorphism near it. Using special sequences of control functions, we show that the approximate higher ring homomorphism is a higher ring homomorphism.

5. **Aknowledgement**

I would like to thank the reviewers for their thoughtful comments and efforts towards improving my manuscript.

References

[1] A. Bahraini, G. Askari and M. Eshaghi Gordji, *On approximate orthogonally ring homomorphisms and orthogonally ring derivations in Banach algebras with the new type fixed point*, J. Mahani Math. Res. Cent. 10(2) (2021) 115–123.

[2] L. Cădariu, V. Radu, *On the stability of the Cauchy functional equation: a fixed point approach*, Grazer Math. Ber. 346 (2004) 43–52 .

[3] S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific, 2002.

[4] S. Czerwik, *On the stability of the quadratic mapping in normed spaces*, Abh. Math. Sem. Univ. Hamburg, 62 (1992) 59–64.

[5] J. B. Diaz, B. Margolis, *A fixed point theorem of the alternative for the contractions on generalized complete metric space*, Bull. Amer. Math. Soc. 74 (1968) 305–309.

[6] E. Elqorachi, M. Th. Rassias, *Generalized Hyers-Ulam Stability of Trigonometric Functional Equations*, Mathematics, 6, 83 (2018) 1–11.

[7] A. K. Faraj, A. H. Majeed, C. Haetinger, N. Rehman, *On Generalized Jordan higher homomorphisms*, Palestine Journal of Mathematics, 3 (2014) 406 – 421.

[8] Z. Gajda, *On stability of additive mappings*, Internat. J. Math. Math. Sci., 14 (1991) 431–434.

[9] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. USA, 27 (1941) 222–224.

[10] D. H. Hyers, G. Isac, Th. M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhauser, 1998.

[11] D. H. Hyers, T. Rassias, *Approximate homomorphisms*, Aeq. Math., 44 (1992) 125–153.

[12] Y. F. Jin, C. Park, M. Th. Rassias, *Hom-derivations in C*-ternary algebras*, Acta Mathematica Sinica, English Series, 36, 9 (2020) 1025–1038.

[13] S. -M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis*, Springer, 2011.

[14] S. -M. Jung, D. Popa, M. Th. Rassias, *On the stability of the linear functional equation in a single variable on complete metric groups*, Journal of Global Optimization, 59 (2014) 165–171.

[15] S. -M. Jung, M. Th. Rassias, C. Mortici, *On a functional equation of trigonometric type*, Applied Mathematics and Computation, 252 (2015) 294–303.

[16] Y. -H. Lee, S. -M. Jung, M. Th. Rassias, *On an n-dimensional mixed type additive and quadratic functional equation*, Applied Mathematics and Computation, 228 (2014) 13–16.

[17] Y. -H. Lee, S. -M. Jung, M. Th. Rassias, *Uniqueness theorems on functional inequalities concerning cubic-quadratic-additive equation*, Journal of Mathematical Inequalities, 12, 1 (2018) 43–61.

[18] H. Mazaheri, M. Zarenejhad, *Some New Results on Remotest Points in Normed Spaces*, J. Mahani Math. Res. Cent. 3(1) (2014) 37–50.

[19] V. Radu, *The fixed point alternative and stability of functional equations*, Fixed Point Theory, 4 (2003) 91–96.

[20] Th. M. Rassias, *On the stability of functional equations in Banach spaces*, J. Math. Anal. Appl., 251 (2000) 264–284.

[21] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., 72 (1978) 297–300.

[22] Th. M. Rassias, *The problem of S.M. Ulam for approximately multiplicative mappings*, J. Math. Anal. Appl., 246 (2000) 352–378.

[23] S. M. Ulam, *Problems in Modern Mathematics*, Wiley, New York, 1964.

Sayed khalil Ekrami
Orcid number: 0000-0002-6233-5741
Department of Mathematics
Payame Noor University, P. O. Box 19395-3697
Tehran, Iran.
Email address: khalil.ekrami@gmail.com, ekrami@pnu.ac.ir