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ANALYTIC UNIVALENT FUCNTIONS DEFINED BY GEGENBAUER POLYNOMIALS

S.O. Olatunji ✉

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Abstract. The numerical tools that have outshinning many others in the history of Geometric Function Theory (GFT) are the Chebyshev and Gegenbauer polynomials in the present time. Recently, Gegenbauer polynomials have been used to define several subclasses of an analytic functions and their yielded results are in the public domain. In this work, analytic univalent functions defined by Gegenbauer polynomials is considered using close-to-convex approach of starlike function. Some early few coefficient bounds obtained are used to establish the famous Fekete-Szegö inequalities.

Keywords: Starlike function, Convex function, Coefficient Bounds, Analytic Univalent Function, Gegenbauer polynomials.
2020 MSC: Primary 30C45, 30C50, 33C45.

1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$ with conditions $f(0) = f'(0) - 1 = 0$. Recall that, S is representing a univalent function with some of the above conditions. With simple modificaton and differentiation, various subclasses of A are known such as starlike function, convex function, bounded turning point, close-to-convex just to mention but a few with representations $Re\left(\frac{zf'(z)}{f(z)}\right) > 0, Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, Re\left(f'(z)\right) > 0, Re\left(\frac{zf'(z)}{g(z)}\right) > 0$ where $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ belong to starlike function and so on. All took their sources from the class of Caratheodory functions $\mathcal{P}$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \quad |p_n| \leq 2 \tag{2}$$

which are analytic in U and also satisfy the conditions $p(0) = 1$ and $Rep(z) > 0$ respectively. See [5–9, 11–13].

✉ olatunjiso@futa.edu.ng, ORCID: 0000-0002-4651-6876
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The class S_{μ}^{*} contained in S^{*} which satisfy

$$\left|\frac{zf'(z)}{f(z)}-1\right|<1$$

with $a_n \leq \frac{1}{n-1}, n \geq 2$ was investigated by Singh [15]. He proved some geometric properties for the class defined and the results are flying around. See more detail in [10].
Szynal [16] introduced the class $\mathcal{T}(\lambda)$ as a subclass of A consisting of functions of the form

$$c(z)=\int_{-1}^{1} k(z,m)d\sigma(m) \tag{3}$$

where

$$k(z,m)=\frac{z}{(1-2mz+z^{z})^{\lambda}}, (z \in U) \tag{4}$$

$m \in [-1,1], \lambda \geq 0$ and σ is a probability measure on the interval $[-1,1]$. The collection of such measures on $[s,t]$ is denoted by $P_{[s,t]}$.
The series expansion in (4) gives

$$k(z,m)=z+\sum_{n=2}^{\infty} c_{n-1}^{\lambda}(m)z^{n} \tag{5}$$

where $c_0^{\lambda}(m)=1, c_1^{\lambda}(m)=2\lambda m, c_2^{\lambda}(m)=2\lambda(\lambda+1)m^2-m, c_3^{\lambda}(m)=\frac{4\lambda(\lambda+1)(\lambda+2)}{3}m^2-2\lambda(\lambda+1)m$ and $c_n^{\lambda}(m)$ is representing Gegenbauer polynomials of degree n which are analytic in the unit disk U. It has a wide application in queueing theory, signal analysis, automatic control, scattering theory and many others [1–4, 14]. Let

$$\mathcal{G}_{\lambda,m}f(z)=k(z,m)*f=z+\sum_{n=2}^{\infty} c_{n-1}^{\lambda}(m)a_n z^{n} \tag{6}$$

where $*$ denotes the Hadamard product of (1) and (5) which represents the main interest of the study.
Also, two analytic functions g and h with $g(0)=h(0)$, g is said to be subordinate to h, denoted by $g \prec h$, if there exists an analytic function

$$\omega=\omega_1 z+\omega_2 z^2+\omega_3 z^3+\omega_4 z^4+... \tag{7}$$

such that $\omega(0)=0, |\omega(z)|<1$ and $g(z)-h(\omega(z))$, for all $z \in U$.
For $p \in \mathcal{P}$ given by (2), then

$$p(z)=\frac{1+\omega}{1-\omega}=1+p_1 z+p_2 z^2+p_3 z^3+.... \tag{8}$$

On equating the coefficients in (8), one will obtain

$$p_1=2\omega_1, p_2=2\omega_2+2\omega_1^2, ... \tag{9}$$

The author employed Oladipo [10] and Vasudevarao et al. [17] analogue to establish that Gegenbauer polynomials also promote the growth of GFT. In this work, analytic univalent functions defined by Gegenbauer polynomials is studied. The initial coefficient bounds obtained is used to determine the sharp estimate for the class defined using famous Fekete.Szegö Inequalities.
For the purpose of the main results, the underneath lemmas and definition are the major weapons to use.

Lemma 1.1. *[12]Let ω given by (7). then*

$$|\omega_{2k-1}| \leq 1 - |\omega_1|^2 - |\omega_2|^2 - |\omega_3|^2 - ... - |\omega_k|^2 \quad for \quad k = 2, 3, ...$$

$$|\omega_{2k}| \leq 1 - |\omega_1|^2 - |\omega_2|^2 - |\omega_3|^2 - ... - |\omega_{k-1}|^2 - |\omega_k|^2 \quad for \quad k = 1, 2, 3, ...$$

Lemma 1.2. *[12]Let ω given by (7). If $\mu \in \mathbb{C}$, then*

$$|\omega_2 - \mu\omega_1^2| \leq max\{1, |\mu|\}$$

By Lemmas 1.1 and 1.2, the following are generated.

Lemma 1.3. *[12]Let $p \in P$ be given by (2). Then for $\mu \in \mathbb{C}$,*

$$|p_2 - \mu p_1^2| \leq 2 - max\{1, |2\mu - 1|\}.$$

Lemma 1.4. *[12]Let $g \in S^*$. Then for any $\mu \in \mathbb{C}$*

$$|b_3 - \mu b_2^2| \leq max\{1, |4\mu - 3|\}$$

$$|b_3 - \mu b_2^2| \leq 1 + (|4\mu - 3| - 1)\frac{|b_2|^2}{4},$$

both inequalities are sharp.

Definition 1.5. One can say $\mathcal{G}_{\lambda,m} \in K_\mu$ if $\mathcal{G}_{\lambda,m} \in A$ and there exist $g \in S^*$

$$\left|\frac{z\left(\mathcal{G}_{\lambda,m}f\right)'(z)}{g(z)} - 1\right| < 1, z \in U$$

where $\lambda \geq 0, m \in [-1, 1]$ and K_μ denotes the natural close-to-convex analogue of S_μ^*.

2. Main Result

Theorem 2.1. *Let $\mathcal{G}_{\lambda,m} \in K_\mu$ where $\lambda \geq 0$ and $m \in [-1, 1]$ be given by (1). Then*

$$|a_2| \leq \frac{3}{2\left|c_1^\lambda(m)\right|}$$

$$|a_3| \leq \frac{5}{3\left|c_2^\lambda(m)\right|}$$

and

$$|a_4| \leq \frac{7.3731}{4|c_3^\lambda(m)|}.$$

Proof. Let

$$z\left(\mathcal{G}_{\lambda,m}f\right)'(z) = g(z)\left(1+\omega(z)\right) \tag{10}$$

for some $g \in S^*$ and ω given in (7), on equating the coefficients in (10) and using (6), one will have

$$2c_1^{\lambda}(m)a_2 = b_2 + \omega_1 \tag{11}$$

$$3c_2^{\lambda}(m)a_3 = b_3 + b_2\omega_1 + \omega_2 \tag{12}$$

$$4c_3^{\lambda}(m)a_4 = b_4 + b_3\omega_1 + b_2\omega_2 + \omega_3 \tag{13}$$

where $|b_k| \leq k$ and $|\omega_k| \leq 1$ for $k \geq 1$. Therefore (11) will give

$$2\left|c_1^{\lambda}(m)a_2\right| \leq |b_2| + |\omega_1| \rightarrow 2\left|c_1^{\lambda}(m)a_2\right| \leq 3.$$

Now, let $x_1 = |\omega_1|, x_2 = |\omega_2|, x_3 = |\omega_3|, x_4 = |\omega_4|$ from (12), to get

$$3\left|c_2^{\lambda}(m)a_3\right| \leq |b_3| + |b_2|\,|\omega_1| + |\omega_2|$$

such that Lemma 1.1 implies

$$3\left|c_2^{\lambda}(m)a_3\right| \leq 3 + 2\,|\omega_1| + \left(1 - \left|\omega_1^2\right|\right) \leq 5$$

Since $0 \leq 4 + 2x_1 - x_1^2 \leq 5$ for $x_1 \in [0,1]$, then $|a_3| \leq \frac{5}{3\left|c_2^{\lambda}(m)\right|}$.

Also from (12) and Lemma 1.1.

$$4\left|c_3^{\lambda}(m)a_4\right| \leq |b_4| + |b_3|\,|\omega_1| + |b_2|\,|\omega_2| + |\omega_3| \leq 4 + 3x_1 + 2x_2 + x_3.$$

For $x_1 \in [0,1], x_2 \leq 1 - x_1^2, x_3 \leq 1 - x_1^2 - x_2^2$, one need to find
$max_H g\left(x_1, x_2, x_3\right)$ where $g\left(x_1, x_2, x_3\right) = 4 + 3x_1 + 2x_2 + x_3$
and

$$H\left\{(x_1, x_2, x_3) : x_1 \leq 1, x_2 \leq 1 - x_1^2, x_3 \leq 1 - x_1^2 - x_2^2\right\}.$$

It is obvious that the maximum over H occurs on the boundary δH which one will now consider. Suppose

$$x_3 = 1 - x_1^2 - x_2^2 \quad and \quad x_2 = 1 - x_1^2, \quad then$$

$g\left(x_1, x_2, x_3\right) = 6 + 3x_1 - x_1^2 - x_1^4, \quad x \in [0,1]$.
Solving the equation above using Wolfram Alpha
$max\left\{6 + 3x_1 - x_1^2 - x_1^4 : 0 \leq x_1 \leq 1\right\} = 7.3731...$ at $x_1 = 0.72808$
where

$$7.3731 = \frac{1}{24}\left\{148 - \frac{968}{(54181)^{\frac{1}{3}} + 2259\sqrt{753}}\right\} + (54181 + 2259\sqrt{753})^{\frac{1}{3}} \tag{14}$$

$$0.72808 = \frac{(27+\sqrt{753})^{\frac{1}{3}}}{2\sqrt{39}} - \frac{1}{\left(3(27+\sqrt{753})\right)^{\frac{1}{3}}}. \tag{15}$$

Hence $|a_4| \leq \frac{7.3731}{4|c_3^{\lambda}(m)|}$. This completes the proof. □

Setting $\lambda = 1$ in Theorem 2.1, to obtain

Corollary 2.2. *Let $\mathcal{G}_{1,m} \in K_\mu$ where $\lambda \geq 0$ and $m \in [-1,1]$ be given by (1). Then*

$$|a_2| \leq \frac{3}{4\,|(m)|} \tag{16}$$

$$|a_3| \leq \frac{5}{3\,|4m^2 - m|} \tag{17}$$

and

$$|a_4| \leq \frac{7.3731}{4|8m^2 - 4m|}. \tag{18}$$

Taking $m = 1$ in Corollary 2.2, one will get

Corollary 2.3. *Let $\mathcal{G}_{1,1} \in K_\mu$ where $\lambda \geq 0$ and $m \in [-1,1]$ be given by (1). Then*

$$|a_2| \leq \frac{3}{4} \tag{19}$$

$$|a_3| \leq \frac{5}{9} \tag{20}$$

and

$$|a_4| \leq \frac{7.3731}{16}. \tag{21}$$

According to corollary 2.3, there is a great difference between this result and the one obtained by [10] and [17].

Theorem 2.4. *Let $\mathcal{G}_{\lambda,m} \in K_\mu$ where $\lambda \geq 0, m \in [-1,1]$ be given by (1) and $\mu \in \mathbb{R}$. For $\mu \leq 0$*

$$|a_3 - a_2^2| \leq \frac{5}{3c_2^\lambda(m))} - \frac{9\mu}{(c_1^\lambda(m))^2}.$$

Proof. Since $\mathcal{G}_{\lambda,m} \in K_\mu$, it is easy to write

$$z\left(\mathcal{G}_{\lambda,m} f\right)'(z) = g(z)\frac{2p(z)}{1+p(z)} \tag{22}$$

Let $p \in P$ be as earlier defined and $g \in S^*$, equating the coefficients in (22) using (2) and (6)

$$\begin{aligned} a_3 - a_2^2 &= \frac{1}{3c_2^\lambda(m)}\left(b_3 - \frac{3\mu c_2^\lambda(m) b_2^2}{4(c_1^\lambda(m))^2}\right) + \frac{b_2 p_1}{12c_2^\lambda(m)}\left(2 - \frac{3\mu c_2^\lambda(m)}{(c_1^\lambda(m))^2}\right) \\ &\quad + \frac{1}{6c_2^\lambda(m)}\left(p_2 - \frac{p_1^2}{2}\right) - \frac{\mu p_1^2}{16(c_1^\lambda(m))^2} \\ a_3 - a_2^2 &= \frac{1}{3c_2^\lambda(m)}\left(b_3 - \frac{3\mu c_2^\lambda(m) b_2^2}{4(c_1^\lambda(m))^2}\right) + \frac{b_2 p_1}{12c_2^\lambda(m)}\left(2 - \frac{3\mu c_2^\lambda(m)}{(c_1^\lambda(m))^2}\right) \\ &\quad + \frac{1}{6}\left(\frac{p_2}{c_2^\lambda(m)} - \frac{4(c_1^\lambda(m))^2 + 3c_2^\lambda(m)}{8(c_1^\lambda(m))^2 c_2^\lambda(m)}\right) \end{aligned}$$

For $\mu \leq 0$ with $x = |p_1|$ and applying Lemmas 1.3 and 1.4 with $|b_2| \leq 2$, one will have

$$\begin{aligned}
\left|a_3 - a_2^2\right| &= \left|\frac{1}{3c_2^\lambda(m)}\left(b_3 - \frac{3\mu c_2^\lambda(m) b_2^2}{4(c_1^\lambda(m))^2}\right) + \frac{b_2 p_1}{12c_2^\lambda(m)}\left(2 - \frac{3\mu c_2^\lambda(m)}{(c_1^\lambda(m))^2}\right)\right. \\
&\left. + \frac{1}{6c_2^\lambda(m)}\left(p_2 - \frac{p_1^2}{2}\right) - \frac{\mu p_1^2}{16(c_1^\lambda(m))^2}\right| \leq \frac{1}{3|c_2^\lambda(m)|}\left|\frac{3\mu c_2^\lambda(m)}{(c_1^\lambda(m))^2} - 3\right| \\
&+ \frac{1}{6|c_2^\lambda(m)|}\left|2 - \frac{3\mu c_2^\lambda(m)}{(c_1^\lambda(m))^2}\right| x + \frac{1}{6|c_2^\lambda(m)|}\left|2 - \frac{x^2}{2}\right| - \frac{\mu x^2}{16|(c_1^\lambda(m))^2|} \\
&= \frac{1}{3c_2^\lambda(m)}\left(\frac{3\mu c_2^\lambda(m)}{(c_1^\lambda(m))^2} - 3\right) + \frac{1}{6c_2^\lambda(m)}\left(2 - \frac{3\mu c_2^\lambda(m)}{(c_1^\lambda(m))^2}\right) x \\
&+ \frac{1}{6c_2^\lambda(m)}\left(2 - \frac{x^2}{2}\right) - \frac{\mu x^2}{16(c_1^\lambda(m))^2}
\end{aligned}$$

where $x \in [0, 2]$. Since the last equation above increases with respect to $x \in [0, 2]$

$$\begin{aligned}
\left|a_3 - a_2^2\right| \leq \left[\frac{1}{3c_2^\lambda(m)}\left(\frac{3\mu c_2^\lambda(m)}{(c_1^\lambda(m))^2} - 3\right) + \frac{1}{6c_2^\lambda(m)}\left(2 - \frac{3\mu c_2^\lambda(m)}{(c_1^\lambda(m))^2}\right) x\right. \\
\left. + \frac{1}{6c_2^\lambda(m)}\left(2 - \frac{x^2}{2}\right) - \frac{\mu x^2}{16(c_1^\lambda(m))^2}\right]_{x=2}
\end{aligned}$$

which establishes the prove.

□

Putting $\lambda = 1$ in Theorem 2.4, to get

Corollary 2.5. *Let $\mathcal{G}_{1,m} \in K_\mu$ where $\lambda \geq 0, m \in [-1, 1]$ be given by (1) and $\mu \in \mathbb{R}$. For $\mu \leq 0$,*

$$\left|a_3 - a_2^2\right| \leq \frac{5}{3(4m^2 - m)} - \frac{9\mu}{(4m)^2}.$$

Taking $m = 1$ in corollary 2.5, to obtain

Corollary 2.6. *Let $\mathcal{G}_{1,1} \in K_\mu$ where $\lambda \geq 0, m \in [-1, 1]$ be given by (1) and $\mu \in \mathbb{R}$. For $\mu \leq 0$,*

$$\left|a_3 - a_2^2\right| \leq \frac{5}{9} - \frac{9\mu}{16}.$$

3. Conclusion

It is clearly observed that Gegenbauer polynomials performed better in this work than generalized discrete probability distribution used by [10].

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References

[1] A. Amourah, A. G. Al Amoush, M. Al-Kaseasbeh, *Gegenbauer polynomials and bi-univalent functions*. Palestine Journal of Mathematics, 10 (2), 625-632, (2021) .

[2] A. Amourah, B.A. Frasin, T. Abdeljawad, *Fekete-Szegö Inequality for Analytic and Biunivalent Functions Subordinate to Gegenbauer Polynomials*. Journal of Function Spaces, Volume 2021, Article ID 5574673, 7 pages, (2021).

[3] A. Amourah, M. Alomari, F. Yousef, A. Alsoboh, *Consolidation of a Certain Discrete Probability Distribution with a Subclass of Bi-Univalent Functions Involving Gegenbauer Polynomials*. *Mathematical Problems in Engineering*, Volume 2022, Article ID 6354994, 6 pages, (2022).

[4] H. Bavinck, G. Hooghiemstra, E. De Waard, *An application of Gegenbauer polynomials in queueing theory*. Journal of Computational and Applied Mathematics Mathematics, 49, 1-10, (1993).

[5] O.A. Fadipe-Joseph, A.T. Oladipo, U.A. Ezeafulukwe, *Modified sigmoid function in univalent function theory*. International Journal of Mathematical Sciences and Engineering application, 7, 313-317, (2013).

[6] A.W. Goodman, *Univalent Functions*. Vols 1-2, (1983).

[7] W. Ma, D. Minda, *A unified treatment of some special classes of univalent factors. In Proceedings of the conference on Complex Analysis, Tianjin, China, 19-23, June, 1992, Conference on Proceedings Lecture Notes for Analysis*. International Press; Cambridge, MA, USA, pp 157-169, (1994).

[8] G. Murugusundaramoorthy, T. Janani, *Sigmoid function in the space of univalent $\lambda-$pseudo starlike functions*. International Journal of Pure and Applied Mathematics, 101, 33-41, (2015) .

[9] G. Murugusundaramoorthy, S.O. Olatunji, O.A. Fadipe-Joseph, *Fekete-Szegö problems for analytic functions in the space of logistic sigmoid functions based on quasi-subordination*. Int. J. Nonlinear Anal. Appl., 9, No. 1, 55-68, (2018).

[10] A.T. Oladipo, *Analytic univalent function defined by generalized discrete probability distribution*. Earthline Journal of Mathematical Sciences, vol.5(1), 169-178, (2021).

[11] S.O. Olatunji, *Fekete-Szegö Inequalities on certain subclasses of analytic functions defined by $\lambda-$pseudo-$q-$difference operator associated with $s-$sigmoid function*. Boletín de la Sociedad Matemática Mexicana, to appear.

[12] S.O Olatunji, E.J. Dansu, *Coefficient Estimates for Bazilevič Ma-Minda Functions in the Space of Sigmoid Function*. *Malaya Journal of Matematik*, Mat. 4(3), 505-512, (2016).

[13] S.O. Olatunji, A.T. Oladipo, *On a new subfamilies of analytic and univalent functions with negative coefficient with respect to other points*. Bulletin of Mathematical Analysis and Applications, Vol.3 No. 2, 159-166, (2011).

[14] S.O. Olatunji, A.M. Gbolagade, *On certain subclass of analytic functions associated with gegenbauer polynomials*. Journal of Fractional Calculus and Applications, 9(2), 127-132, (2018).

[15] R. Singh, *On a class of star-like functions*. Compos. Math., 19(1), 78-82, (1968).

[16] J. Szynal, *An extension of typically real functions*. Ann. Univ. Mariae Curie-Sklodowska, Sect A. 48, 193-201 (1994).

[17] A. Vasudevarao, J.Sokol, D.K. Thomas, *On a close-to-convex analogue of certain starlike functions*. Bulletin Aust. Math. Soc., 102 (2), 268-281 (2020).

Sunday Oluwafemi Olatunji
Orcid number: 0000-0002-4651-6876
Department of Mathematical Sciences
Federal University of Technology, P.M.B.704,
Akure, Nigeria
Email address: olatunjiso@futa.edu.ng