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FIXED POINT RESULTS OF FUZZY $(\theta, \mathcal{L})-$ WEAK CONTRACTION IN $\mathbb{G}$-METRIC SPACE

Y. Mahmood AND M.S. Shagari

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Abstract. In this paper, the notion of fuzzy $(\theta, \mathcal{L})$-weak contraction in $\mathbb{G}-$metric space is introduced, and sufficient conditions for the existence of fuzzy fixed points for such mappings are investigated. Relevant illustrative examples are constructed to support the assumptions of our established theorems. It is observed that the principal ideas obtained herein extend and subsume some well-known results in the corresponding literature. A few of these special cases of our results are noted and discussed as corollaries.

Keywords: Fixed point; Fuzzy set; Fuzzy mapping; Contractive type mapping; $(\theta, \mathcal{L})-$weak contraction; $\mathbb{G}-$metric space.
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1. Introduction

Fixed point theory is a renowned and huge field of research in mathematical sciences. This field is known as the combination of analysis which includes topology, geometry and algebra. One of the well-known set-valued fixed point results in spaces with metric structure was announced by Nadler [24]. As a fuzzy extension of the key idea in [24], Heilpern [8] gave the concept of fuzzy mapping and proved the fixed point theorem for fuzzy contractive mapping in metric space. Thereafter, several authors (see, e.g. [1, 5, 14–18, 26, 27]) have modified the ideas of fuzzy sets and examined the existence of fixed point of fuzzy mapping in different directions.

The concept of weak contraction was introduced by Alber and Gurre [1] in 1997. In 2003, Berinde [5] initiated the notion of $(\theta, \mathcal{L})-$weak contraction and studied fixed point theorems for the related contraction. Then Berinde and Berinde [6] extended the concept of $(\theta, \mathcal{L})-$weak contraction from single-valued mapping to multi-valued mapping and presented corresponding fixed point theorems. On the other hand, Samet et al. [26] presented a new mapping namely $\alpha - \psi-$contractive mapping and studied fixed point theorems for the new contraction. With the passage of time, fixed point results of $\alpha-$admissible mappings have been examined in several directions, see, for example [3, 10, 21, 25].

✉ shagaris@ymail.com, ORCID: 0000-0001-6632-8365
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Along the line, Mustafa and Sims [20] presented an idea of a generalized metric space under the name $\mathbb{G}-$metric space, in 2006. In the first paper on $\mathbb{G}-$metric space, Sims and Mustafa [20] introduced some properties of $\mathbb{G}-$metric spaces and also discussed its topology, compactness, completeness, product and the criteria regarding the convergence and continuity of sequences in $\mathbb{G}-$metric space. Some theorems concerning these properties were also proved. Subsequently, Mustafa et al. [22] obtained some new fixed point results for Lipschitzian-type mappings on G-metric space. Later after, more than a handful of authors have investigated the existence of fixed points in G-metric spaces. For example, one can refer [7, 11, 12] and some references therein.

Following the existing literature, we observe that fuzzy fixed point results in G-metric space are not sufficiently investigated. Therefore, motivated by the basic ideas in [1, 5, 20], the aim of this paper is to present new fixed point theorems for fuzzy $(\theta, \mathcal{L})-$weak contraction mappings in $\mathbb{G}-$metric space. Our results generalize and extend a few known results in the comparable literature of fuzzy and classical mathematics.

2. Preliminaries

In this section, we recall some basic concepts that are necessary in the establishment of our main results. Most of these preliminaries are recorded from [4, 9, 19, 20, 23, 24, 26, 28].

Definition 2.1. Let $\mathcal{F}$ be a nonempty set and $\mathbb{G} : \mathcal{F} \times \mathcal{F} \times \mathcal{F} \longrightarrow \mathbb{R}^+$ be a function such that the following are satisfied:

($\mathbb{G}1$) $\mathbb{G}(\tau, \sigma, \upsilon) = 0$ if $\tau = \sigma = \upsilon$,
($\mathbb{G}2$) $\mathbb{G}(\tau, \tau, \sigma) > 0$ for all $\tau, \sigma \in \mathcal{F}$ with $\tau \neq \sigma$,
($\mathbb{G}3$) $\mathbb{G}(\tau, \tau, \sigma) \leq \mathbb{G}(\tau, \sigma, \upsilon)$ for all $\tau, \sigma, \upsilon \in \mathcal{F}$ with $\upsilon \neq \sigma$,
($\mathbb{G}4$) $\mathbb{G}(\tau, \sigma, \upsilon) = \mathbb{G}(\tau, \upsilon, \sigma) = \mathbb{G}(\sigma, \upsilon, \tau) = ...$(symmetric with respect to τ, σ, υ),
($\mathbb{G}5$) $\mathbb{G}(\tau, \sigma, \upsilon) \leq \mathbb{G}(\tau, a, a) + \mathbb{G}(a, \sigma, \upsilon)$ for all $\tau, \sigma, \upsilon, a \in \mathcal{F}$ (rectangular property).

Then $\mathbb{G}$ is called a $\mathbb{G}-$metric function and $(\mathcal{F}, \mathbb{G})$ is said to be a $\mathbb{G}-$metric space.

Example 2.2. *Let $\mathcal{F} = \mathbb{R}$. Then a $\mathbb{G}-$metric on $\mathbb{R}$ is defined as:*

$$\mathbb{G}(\tau, \sigma, \upsilon) = |\tau - \sigma| + |\sigma - \upsilon| + |\tau - \upsilon| \text{ for all } \tau, \sigma, \upsilon \in \mathcal{F}.$$

Definition 2.3. Let $(\mathcal{F}, \mathbb{G})$ be a $\mathbb{G}-$metric space. A sequence $\{\tau_e\}$ in $\mathcal{F}$ is $\mathbb{G}-$convergent if, for any $\delta > 0$, there exists $\tau \in \mathcal{F}$, $O(\delta) \in \mathbb{N}$ such that $\mathbb{G}(\tau, \tau_e, \tau_\rho) < \delta$, for all $e, \rho \geq O(\delta)$. We call τ the limit of the sequence and write $\tau_e \to \tau$ or $\lim_{e\to\infty} \tau_e = \tau$.

Definition 2.4. Let $(\mathcal{F}, \mathbb{G})$ be a $\mathbb{G}-$metric space. A sequence $\{\tau_e\}$ in $\mathcal{F}$ is called $\mathbb{G}-$Cauchy if, for any $\delta > 0$, there exists $O(\delta) \in \mathbb{N}$ such that $\mathbb{G}(\tau_\varsigma, \tau_e, \tau_\rho) < \delta$, for each $e, \rho, \varsigma \geq O(\delta)$, that is, $\mathbb{G}(\tau_\varsigma, \tau_e, \tau_\rho) \to 0$ as $e, \rho, \varsigma \to \infty$.

Definition 2.5. Let $(\mathcal{F}, \mathbb{G})$ be a $\mathbb{G}-$metric space. A sequence $\{\tau_e\}$ in $\mathcal{F}$ is called $\mathbb{G}-$Complete if every $\mathbb{G}-$Cauchy sequence in $(\mathcal{F}, \mathbb{G})$ is convergent in $\mathcal{F}$.

Lemma 2.6. [20]. *Let $(\mathcal{F}, \mathbb{G})$ be a $\mathbb{G}-$metric space and $\{\tau_e\}$ be a sequence in $\mathcal{F}$. Then the following statements are equivalent:*

(i) $\{\tau_e\}$ *is $\mathbb{G}-$convergent to τ;*
(ii) $\mathbb{G}(\tau_e, \tau_e, \tau) \to 0$, *as e approaches to infinity;*
(iii) $\mathbb{G}(\tau_e, \tau, \tau) \to 0$, *as e approaches to infinity;*
(iv) $\mathbb{G}(\tau_e, \tau_\rho, \tau) \to 0$, *as e, ρ approaches to infinity.*

3. Hausdorff $\mathbb{G}-$distance of fuzzy sets

Kaewcharoen and Kaewkhao [13] introduced the concept of Hausdorff $\mathbb{G}-$distance. Let $\mathcal{F}$ be a $\mathbb{G}-$metric space and $CB(\mathcal{F})$ be the family of all non empty closed and bounded subsets of $\mathcal{F}$. Then, the Hausdorff $\mathbb{G}-$distance is defined as:

$$\mathcal{H}_{\mathbb{G}}(\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3) = \max\left\{ \sup_{\tau \in \mathcal{Z}_1} \mathbb{G}(\tau, \mathcal{Z}_2, \mathcal{Z}_3), \sup_{\tau \in \mathcal{Z}_2} \mathbb{G}(\tau, \mathcal{Z}_1, \mathcal{Z}_3), \sup_{\tau \in \mathcal{Z}_3} \mathbb{G}(\tau, \mathcal{Z}_1, \mathcal{Z}_2) \right\},$$

where

$$\begin{aligned}
\mathbb{G}(\tau, \mathcal{Z}_2, \mathcal{Z}_3) =& d_{\mathbb{G}}(\tau, \mathcal{Z}_2) + d_{\mathbb{G}}(\mathcal{Z}_2, \mathcal{Z}_3) + d_{\mathbb{G}}(\tau, \mathcal{Z}_3),\\
d_{\mathbb{G}}(\tau, \mathcal{Z}_2) =& \inf_{\sigma \in \mathcal{Z}_2} d_{\mathbb{G}}(\tau, \sigma),\\
d_{\mathbb{G}}(\mathcal{Z}_1, \mathcal{Z}_2) =& \inf_{\tau \in \mathcal{Z}_1, \sigma \in \mathcal{Z}_2} d_{\mathbb{G}}(\tau, \sigma),\\
\mathbb{G}(\tau, \sigma, \mathcal{Z}_3) =& \inf_{\tau \in \mathcal{Z}_1, \sigma \in \mathcal{Z}_2, \upsilon \in \mathcal{Z}_3} d_{\mathbb{G}}(\tau, \sigma, \upsilon).
\end{aligned}$$

Remark 3.1. *[13].* Let $(\mathcal{F}, \mathbb{G})$ be a $\mathbb{G}-$M space, $\tau \in \mathcal{F}$ and $\mathcal{Z} \subseteq \mathcal{F}$. For each $\sigma \in \mathcal{Z}$ we have

$$\begin{aligned}
\mathbb{G}(\tau, \mathcal{Z}, \mathcal{Z}) &= d_{\mathbb{G}}(\tau, \mathcal{Z}) + d_{\mathbb{G}}(\mathcal{Z}, \mathcal{Z}) + d_{\mathbb{G}}(\tau, \mathcal{Z})\\
&\leq 2 d_{\mathbb{G}}(\tau, \sigma)\\
&= 2[d_{\mathbb{G}}(\tau, \tau, \sigma) + d_{\mathbb{G}}(\tau, \sigma, \sigma)]\\
&\leq 2[d_{\mathbb{G}}(\tau, \sigma, \sigma) + d_{\mathbb{G}}(\tau, \sigma, \sigma) + d_{\mathbb{G}}(\tau, \sigma, \sigma)]\\
&= 6 d_{\mathbb{G}}(\tau, \sigma, \sigma).
\end{aligned}$$

Consider $\mathcal{S} : \mathcal{F} \to 2^{\mathcal{F}}$, and $\tau \in \mathcal{F}$. Then τ is a fixed point of $\mathcal{F}$ if $\tau \in \mathcal{S}\tau$ (see [24]).

Let $(\mathcal{F}, d_{\mathbb{G}})$ be a metric space, a fuzzy set in $\mathcal{F}$ is a function with domain $\mathcal{F}$ and values in $\mathcal{I} = [0, 1]$. If $\mathcal{Z}$ is a fuzzy set and $\tau \in \mathcal{F}$, then the function value $\eta_{\mathcal{Z}}(\tau)$ is called the degree of membership of τ in $\mathcal{Z}$.

The $\alpha-$level set of $\mathcal{Z}$, denoted by $[\mathcal{Z}]_\alpha$ is defined as

$$\begin{aligned}
[\mathcal{Z}]_\alpha &= \{\tau : \ \eta_{\mathcal{Z}}(\tau) \geq \alpha, \ \alpha \in (0, 1]\}\\
[\mathcal{Z}]_0 &= \overline{\{\tau : \eta_{\mathcal{Z}}(\tau) > 0\}}.
\end{aligned}$$

Where $\overline{B}$ is the closure of the non-fuzzy set B.
Denote by $\mathcal{C}(\mathcal{F})$, the family of all nonempty compact subsets of $\mathcal{F}$. For each $\alpha \in \mathcal{I}$, let $[\mathcal{Z}]_\alpha \in \mathcal{C}(\mathcal{F})$.

Proposition 3.2. [28]. *If $\mathcal{Z}_1, \mathcal{Z}_2 \in \mathcal{C}(\mathcal{F})$ and $\tau \in \mathcal{Z}_1$, then there exists $\sigma \in \mathcal{Z}_2$ such that*

$$2[\mathbb{G}(\tau, \tau, \sigma) + \mathbb{G}(\tau, \sigma, \sigma)] \leq \mathcal{H}_{\mathbb{G}}(\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_2).$$

Lemma 3.3. [23]. *Let $(\mathcal{F}, \mathbb{G})$ be a $\mathbb{G}-$metric space. Denote by $\mathcal{CB}(\mathcal{F})$, the family of nonempty closed and bounded subsets of $\mathcal{F}$. Let $\mathcal{Z}_1, \mathcal{Z}_2 \in \mathcal{CB}(\mathcal{F})$, then for each $\tau \in \mathcal{Z}_1$, we have,*

$$\mathbb{G}(\tau, \mathcal{Z}_2, \mathcal{Z}_2) \leq \mathcal{H}_{\mathbb{G}}(\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_2).$$

Lemma 3.4. [23]. *Let $(\mathcal{F}, \mathbb{G})$ be a $\mathbb{G}-$metric space. If $\mathcal{Z}_1, \mathcal{Z}_2 \in \mathcal{CB}(\mathcal{F})$ and $\tau \in \mathcal{Z}_1$, then for each $\epsilon > 0$ there exists $\sigma \in \mathcal{Z}_2$ such that*

$$\mathbb{G}(\tau, \sigma, \sigma) \leq \mathcal{H}_{\mathbb{G}}(\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_2) + \epsilon.$$

Heilpern [9] gave a fuzzy extension of Banach contraction principle in 1981 and Nadler [24] fixed point theorems by introducing the concept of fuzzy contraction mappings in connection with $d_\infty-$metric for fuzzy sets.

Definition 3.5. [9] Let $\mathcal{E}$ be an arbitrary set, $\mathcal{F}$ be a metric space. A mapping $\mathcal{T} : \mathcal{E} \longrightarrow I^{\mathcal{F}}$ is called fuzzy mapping from $\mathcal{E}$ into $\mathcal{F}$. A fuzzy mapping $\mathcal{T}$ is a fuzzy subset on $\mathcal{E} \times \mathcal{F}$, where $\mathcal{T}(\tau)(\sigma)$ is the grade membership of σ in $T(\tau)$.

Definition 3.6. [9] Let $(\mathcal{F}, d)$ be a metric space and $\mathcal{T} : \mathcal{F} \longrightarrow I^{\mathcal{F}}$ be a fuzzy mapping. A point $\nu \in \mathcal{F}$ is said to be fuzzy fixed point of $\mathcal{T}$ if $\nu \in [\mathcal{T}_\nu]_\alpha$, for some $\alpha \in [0, 1]$.

4. $\alpha-$admissible and $\alpha - \psi-$Contractive Mappings

The concepts of $\alpha - \psi$ contractive type mappings and $\alpha-$admissible mappings were introduced by Samet [26] in 2012 as follows.

Definition 4.1. [26] Let $\mathcal{S} : \mathcal{F} \longrightarrow \mathcal{F}$ and $\alpha : \mathcal{F} \times \mathcal{F} \longrightarrow [0, +\infty]$, then $\mathcal{S}$ is said to be $\alpha-$admissible if for $\tau, \sigma \in \mathcal{F}$

$$\alpha(\tau, \sigma) \geq 1 \implies \alpha(\mathcal{S}\tau, \mathcal{S}\sigma) \geq 1.$$

Example 4.2. [26]. *Let $\mathcal{F} = (0, +\infty)$. Define $\mathcal{S} : \mathcal{F} \longrightarrow \mathcal{F}$ and $\alpha : \mathcal{F} \times \mathcal{F} \longrightarrow [0, +\infty)$ with $\mathcal{S}\tau = \ln \tau$* for all $\tau \in \mathcal{S}$
and

$$\alpha(\tau, \sigma) = \begin{cases} 2 \text{ if } \tau \geq \sigma, \\ 0 \text{ if } \tau < \sigma. \end{cases}$$

Then, $\mathcal{S}$ is $\alpha-$admissible.

Example 4.3. [26]. *Let* $\mathcal{F} = (0, +\infty)$. *Define* $\mathcal{S} : \mathcal{F} \longrightarrow \mathcal{F}$ *and* $\alpha : \mathcal{F} \times \mathcal{F} \longrightarrow [0, +\infty)$ *with* $\mathcal{S}\tau = \sqrt{\tau}$ for all $\tau \in \mathcal{S}$
and

$$\alpha(\tau, \sigma) = \begin{cases} \exp^{\tau-\sigma} \text{ if } \tau \geq \sigma, \\ 0 \text{ if } \tau < \sigma. \end{cases}$$

Then, $\mathcal{S}$ *is* $\alpha-$*admissible.*

Definition 4.4. [26] Denote by Ψ the family of nondecreasing functions $\Psi : [0, +\infty) \longrightarrow [0, +\infty)$ in such a way that for $\psi \in \Psi, \sum_{e=1}^{\infty} \psi^e(r) < \infty$ and $\psi(r) < r$ for each $r > 0$ and ψ^e is the e-th iterate of ψ. Let for $(\mathcal{F}, \mu)$ be a metric space. Then $\mathcal{S} : \mathcal{F} \longrightarrow \mathcal{F}$ is an $\alpha - \psi-$contractive mapping if for a pair of functions $\alpha : \mathcal{F} \times \mathcal{F} \longrightarrow [0, +\infty)$ and $\psi \in \Psi$, we have

$$\alpha(\tau, \sigma)\mu(\mathcal{S}\tau, \mathcal{S}\sigma) \leq \psi(\mu(\tau, \sigma)) \text{ for all } \tau, \sigma \in \mathcal{F}.$$

5. $\alpha-$admissible and $\alpha - \psi-$Contractive Type Mapping in $\mathbb{G}-$Metric Space:

Alghamdi et al. [2] extended the definitions of $\alpha-$admissible mapping and $\alpha - \psi-$contractive type mapping for $\mathbb{G}-$M space as defined below.

Definition 5.1. [2] Let $(\mathcal{F}, \mathbb{G})$ be a $\mathbb{G}-$metric space, $\mathcal{S} : \mathcal{F} \to \mathcal{F}$ be a single-valued mapping and $\alpha : \mathcal{F} \times \mathcal{F} \times \mathcal{F} \to [0, \infty)$. The mapping $\mathcal{S}$ will be called $\mathbb{G} - \alpha-$admissible if for $\tau, \sigma, \upsilon \in \mathcal{F}$:

$$\alpha(\tau, \sigma, \upsilon) \geq 1 \Rightarrow \alpha(\mathcal{S}\tau, \mathcal{S}\sigma, \mathcal{S}\upsilon) \geq 1.$$

Example 5.2. *Let* $\mathcal{F} = [0, \infty)$ *and* $\mathcal{S} : \mathcal{F} \to \mathcal{F}$. *Define a mapping* $\alpha : \mathcal{F} \times \mathcal{F} \times \mathcal{F} \to [0, \infty)$ *by*

$$\mathcal{S}\tau = \begin{cases} 2\ln\tau \text{ if } \tau \neq 0, \\ e \text{ otherwise}, \end{cases} \quad \text{and } \alpha(\tau, \sigma, \upsilon) = \begin{cases} e \text{ if } \tau \geq \sigma \geq \upsilon, \\ 0 \text{ otherwise}. \end{cases} \quad \text{for all } \tau, \sigma, \upsilon \in \mathcal{F}$$

Then $\mathcal{S}$ *is* $\mathbb{G} - \alpha-$*admissible.*

Definition 5.3. [2] Let $(\mathcal{F}, \mathbb{G})$ be a $\mathbb{G}-$metric space, $\mathcal{S} : \mathcal{F} \to \mathcal{F}$ be a single-valued mapping, then $\mathcal{S}$ is a $\mathbb{G} - \alpha - \psi-$contractive type mapping if there exists a pair of functions $\alpha : \mathcal{F} \times \mathcal{F} \times \mathcal{F} \to [0, \infty)$ and $\psi \in \Psi$, such that $\sum_{e=1}^{\infty}\psi^e(r) < \infty$ and $\psi(r) < r$ for each $r > 0$, for all $\tau, \sigma, \upsilon \in \mathcal{F}$, we have

$$\alpha(\tau, \sigma, \upsilon)\mathbb{G}(\mathcal{S}\tau, \mathcal{S}\sigma, \mathcal{S}\upsilon) \leq \psi(\mathbb{G}(\tau, \sigma, \upsilon)).$$

Definition 5.4. [1] Let $(\mathcal{F}, d)$ be a metric space. A mapping $\mathcal{S} : \mathcal{F} \longrightarrow \mathcal{F}$ is called weak contraction if there exists two constants $\theta \in (0, 1)$ and $\mathcal{L} \geq 0$ such that

$$d(\mathcal{S}\tau, \mathcal{S}\sigma) \leq \theta\mu(\tau, \sigma) + \mathcal{L}d(\sigma, \mathcal{S}\tau), \tag{1}$$

for all $\tau, \sigma \in \mathcal{F}$.

6. **Main Results**

In this section, the idea of $(\theta,\mathcal{L})-$weakly contraction in [3] is extended to $\mathbb{G}-$metric space. Hence, fuzzy $(\theta,\mathcal{L})-$weak contraction in $\mathbb{G}-$metric space and $(\theta,\mathcal{L})-$weak contraction for a pair of fuzzy mappings in $\mathbb{G}-$metric space are defined and fixed point and common fixed point theorems are obtained for fuzzy mappings using the proposed contraction.

We begin with the following definitions.

Definition 6.1. Let $(\mathcal{F},\mathbb{G})$ be a $\mathbb{G}-$metric space and $\mathcal{S}:\mathcal{F}\to\mathcal{F}$ be a self mapping. Then $\mathcal{S}$ is called $(\theta,\mathcal{L})$ weak contraction if there exist two constants $\theta\in(0,1)$ and $\mathcal{L}\geq 0$ such that

$$\mathbb{G}(\mathcal{S}\tau,\mathcal{S}\sigma,\mathcal{S}\upsilon)\leq\theta\mathbb{G}(\tau,\sigma,\upsilon)+\mathcal{L}\mu_{\mathbb{G}}(\sigma,\mathcal{S}\tau),\qquad\text{for all }\ \tau,\sigma,\upsilon\in\mathcal{F}.$$

Definition 6.2. Let $(\mathcal{F},\mathbb{G})$ be a $\mathbb{G}-$metric space and $\mathcal{S}:\mathcal{F}\to I^{\mathcal{F}}$ be a fuzzy mapping. Then $\mathcal{S}$ is called fuzzy $(\theta,\mathcal{L})-$weak contraction if there exist two constants $\theta\in(0,1)$ and $\mathcal{L}\geq 0$, such that

$$\mathcal{H}_{\mathbb{G}}\bigg([\mathcal{S}\tau]_\lambda,[\mathcal{S}\sigma]_\lambda,[\mathcal{S}\upsilon]_\lambda\bigg)\leq\theta\mathbb{G}(\tau,\sigma,\upsilon)+\mathcal{L}\mu_{\mathbb{G}}(\sigma,[\mathcal{S}\tau]_\lambda),\qquad\text{for all }\ \tau,\sigma,\upsilon\in\mathcal{F}.$$

Definition 6.3. Let $(\mathcal{F},\mathbb{G})$ be a $\mathbb{G}-$metric space and $\mathcal{S}_1,\mathcal{S}_2:\mathcal{F}\to I^{\mathcal{F}}$ be a pair of fuzzy mappings. The pair $(\mathcal{S}_1,\mathcal{S}_2)$ is said to be fuzzy $(\theta,\mathcal{L})-$weak contraction if there exist constants $\theta\in(0,1)$, $\mathcal{L}_1,\mathcal{L}_2\geq 0$, such that

(i) $\quad\mathcal{H}_{\mathbb{G}}([\mathcal{S}_1\tau]_\lambda,[\mathcal{S}_2\sigma]_\lambda,[\mathcal{S}_2\upsilon]_\lambda)\leq\theta\mathbb{G}(\tau,\sigma,\upsilon)+\mathcal{L}_1\mu_{\mathbb{G}}(\sigma,[\mathcal{S}_1\tau]_\lambda),\ \text{for all }\ \tau,\sigma,\upsilon\in\mathcal{F}.$

(ii) $\quad\mathcal{H}_{\mathbb{G}}([\mathcal{S}_2\tau]_\lambda,[\mathcal{S}_1\sigma]_\lambda,[\mathcal{S}_1\upsilon]_\lambda)\leq\theta\mathbb{G}(\tau,\sigma,\upsilon)+\mathcal{L}_2\mu_{\mathbb{G}}(\sigma,[\mathcal{S}_2\tau]_\lambda),\ \text{for all }\ \tau,\sigma,\upsilon\in\mathcal{F}.$

The following is the principal result of this paper.

Theorem 6.4. *Let $(\mathcal{F},\mathbb{G})$ be a $\mathbb{G}-$metric space and $\mathcal{S}_1,\mathcal{S}_2:\mathcal{F}\to I^{\mathcal{F}}$ be a pair of fuzzy $(\theta,\mathcal{L})$-weak contraction such that for all $\tau,\sigma,\upsilon\in\mathcal{F}$, and $\theta\in(0,1)$, $\mathcal{L}_1,\mathcal{L}_2\geq 0$,*

$$(2)\qquad\mathcal{H}_{\mathbb{G}}([\mathcal{S}_1\tau]_\lambda,[\mathcal{S}_2\sigma]_\lambda,[\mathcal{S}_2\upsilon]_\lambda)\leq\frac{\theta}{6}\mathcal{M}_{i,j}+\mathcal{L}_1\mu_{\mathbb{G}}(\sigma,[\mathcal{S}_1\tau]_\lambda)+\mathcal{L}_2\mathcal{N}_{i,j},$$

where

$$\mathcal{M}_{i,j}=\max\begin{bmatrix}6\mathbb{G}(\tau,\sigma,\upsilon),\mathbb{G}(\tau,[\mathcal{S}_i\tau]_\lambda,[\mathcal{S}_i\tau]_\lambda),\\ \mathbb{G}(\sigma,[\mathcal{S}_j\sigma]_\lambda,[\mathcal{S}_j\sigma]_\lambda),\\ \frac{\mathbb{G}(\sigma,[\mathcal{S}_i\tau]_\lambda,[\mathcal{S}_i\tau]_\lambda)+\mathbb{G}(\tau,[\mathcal{S}_j\sigma]_\lambda,[\mathcal{S}_j\sigma]_\lambda)}{2}\end{bmatrix},$$

$$\mathcal{N}_{i,j}=\min\begin{bmatrix}\mathbb{G}(\sigma,[\mathcal{S}_j\sigma]_\lambda,[\mathcal{S}_j\sigma]_\lambda),\\ \mathbb{G}(\sigma,[\mathcal{S}_i\tau]_\lambda,[\mathcal{S}_i\tau]_\lambda),\\ \mathbb{G}(\tau,[\mathcal{S}_j\sigma]_\lambda,[\mathcal{S}_j\sigma]_\lambda)\end{bmatrix},$$

and $i\neq j,\ i,j=1,2$. Then there exists a common fixed point of fuzzy mappings $\mathcal{S}_i$ and $\mathcal{S}_j$.

Proof. Let $\tau_0 \in \mathcal{F}$. Take $\tau_1 \in [\mathcal{S}_1\tau_0]_\lambda$ and $\tau_2 \in [\mathcal{S}_2\tau_1]_\lambda$ and so on. Generally

$$\tau_{2e+1} \in [\mathcal{S}_1\tau_{2e}]_\lambda,\ \tau_{2e+2} \in [\mathcal{S}_2\tau_{2e+1}]_\lambda \qquad e = 0,1,2,...$$

For $k>0$, let $k\theta = h$. Then, by Condition (2) and Lemma 3.4, we have

$$\begin{aligned}
\mathbb{G}(\tau_1,\tau_2,\tau_2) \leq & kH_{\mathbb{G}}\Big([\mathcal{S}_1\tau_0]_\lambda,[\mathcal{S}_2\tau_1]_\lambda,[\mathcal{S}_2\tau_1]_\lambda\Big)\\
\leq & k\left[\frac{\theta}{6}\max\left(\begin{array}{c} 6\mathbb{G}(\tau_0,\tau_1,\tau_1), G(\tau_0,[\mathcal{S}_1\tau_0]_\lambda,[\mathcal{S}_1\tau_0]_\lambda),\\ \mathbb{G}(\tau_1,[\mathcal{S}_2\tau_1]_\lambda,[\mathcal{S}_2\tau_1]_\lambda),\\ \frac{\mathbb{G}(\tau_1,[\mathcal{S}_1\tau_0]_\lambda,[\mathcal{S}_1\tau_0]_\lambda)+\mathbb{G}(\tau_0,[\mathcal{S}_2\tau_1]_\lambda,[\mathcal{S}_2\tau_1]_\lambda)}{2}\end{array}\right)\right]\\
& + \mathcal{L}_1\mu_{\mathbb{G}}(\tau_1,[\mathcal{S}_1\tau_0]_\lambda) + \mathcal{L}_2\min\left[\begin{array}{c}\mathbb{G}(\tau_1,[\mathcal{S}_2\tau_1]_\lambda,[\mathcal{S}_2\tau_1]_\lambda),\\ \mathbb{G}(\tau_0,[\mathcal{S}_2\tau_1]_\lambda,[\mathcal{S}_2\tau_1]_\lambda),\\ \mathbb{G}(\tau_1,[\mathcal{S}_1\tau_0]_\lambda,[\mathcal{S}_1\tau_0]_\lambda)\end{array}\right]\\
\leq & k\left[\frac{\theta}{6}\max\left(\begin{array}{c}6\mathbb{G}(\tau_0,\tau_1,\tau_1), 6\mathbb{G}(\tau_0,\tau_1,\tau_1), 6\mathbb{G}(\tau_1,\tau_2,\tau_2),\\ \frac{6\mathbb{G}(\tau_1,\tau_1,\tau_1)+6\mathbb{G}(\tau_0,\tau_2,\tau_2)}{2}\end{array}\right)\right]\\
& + \mathcal{L}_1\mu_{\mathbb{G}}(\tau_1,\tau_1) + \mathcal{L}_2\min\Big[6\mathbb{G}(\tau_1,\tau_2,\tau_2), 6\mathbb{G}(\tau_0,\tau_2,\tau_2), 6\mathbb{G}(\tau_1,\tau_1,\tau_1)\Big]\\
\leq & k\left[\frac{\theta}{6}\max\left(6\mathbb{G}(\tau_0,\tau_1,\tau_1), 6\mathbb{G}(\tau_1,\tau_2,\tau_2), \frac{6\mathbb{G}(\tau_0,\tau_2,\tau_2)}{2}\right)\right]\\
& + \mathcal{L}_2\min\Big[6\mathbb{G}(\tau_1,\tau_2,\tau_2), 0, 6\mathbb{G}(\tau_0,\tau_2,\tau_2)\Big]\\
\leq & k\left[\frac{\theta}{6}\left[\max\left(6\mathbb{G}(\tau_0,\tau_1,\tau_1), 6\mathbb{G}(\tau_1,\tau_2,\tau_2), \frac{6\mathbb{G}(\tau_0,\tau_2,\tau_2)}{2}\right)\right]\right].
\end{aligned}$$

Since

$$\frac{6\mathbb{G}(\tau_0,\tau_2,\tau_2)}{2} \leq \frac{6\mathbb{G}(\tau_0,\tau_1,\tau_1)+6\mathbb{G}(\tau_1,\tau_2,\tau_2)}{2},$$

so we have

$$\frac{\mathbb{G}(\tau_0,\tau_2,\tau_2)}{2} \leq \max\Big[\mathbb{G}(\tau_0,\tau_1,\tau_1), \mathbb{G}(\tau_1,\tau_2,\tau_2)\Big],$$

which gives,

$$\mathbb{G}(\tau_1,\tau_2,\tau_2) \leq k\Big[\theta\max\Big(\mathbb{G}(\tau_0,\tau_1,\tau_1), \mathbb{G}(\tau_1,\tau_2,\tau_2)\Big)\Big].$$

Suppose $\mathbb{G}(\tau_0,\tau_1,\tau_1) < \mathbb{G}(\tau_1,\tau_2,\tau_2)$, then by property 5.3 of ψ, we get a contradiction. Hence,

$$\mathbb{G}(\tau_1,\tau_2,\tau_2) \leq k\theta[\mathbb{G}(\tau_0,\tau_1,\tau_1)].$$

As $h = k\theta$, then

$$\mathbb{G}(\tau_1,\tau_2,\tau_2) \leq h(\mathbb{G}(\tau_0,\tau_1,\tau_1)).$$

Given that $\tau_2 \in [\mathcal{S}_2\tau_1]_\lambda$ and $\tau_3 \in [\mathcal{S}_1\tau_2]_\lambda$, there exist $k > 0$ and $k\theta = h$. So, again by Lemma 3.4 and Condition (2), we get

$$\begin{aligned}
\mathbb{G}(\tau_2,\tau_3,\tau_3) \leq & kH_{\mathbb{G}}\Bigg([\mathcal{S}_2\tau_1]_\lambda, [\mathcal{S}_1\tau_1]_\lambda, [\mathcal{S}_1\tau_2]_\lambda\Bigg) \\
\leq & k\left[\frac{\theta}{6}\max\left(\begin{array}{c} 6\mathbb{G}(\tau_1,\tau_2,\tau_2), G(\tau_1,[\mathcal{S}_2\tau_1]_\lambda,[\mathcal{S}_2\tau_1]_\lambda), \\ \mathbb{G}(\tau_2,[\mathcal{S}_1\tau_2]_\lambda,[\mathcal{S}_1\tau_2]_\lambda), \\ \frac{\mathbb{G}(\tau_2,[\mathcal{S}_2\tau_1]_\lambda,[\mathcal{S}_2\tau_1]_\lambda)+\mathbb{G}(\tau_1,[\mathcal{S}_1\tau_2]_\lambda,[\mathcal{S}_1\tau_2]_\lambda)}{2} \end{array}\right)\right] \\
& + \mathcal{L}_3\mu_{\mathbb{G}}(\tau_2,[\mathcal{S}_2\tau_1]_\lambda) + \mathcal{L}_4 \min\left[\begin{array}{c} \mathbb{G}(\tau_2,[\mathcal{S}_1\tau_2]_\lambda,[\mathcal{S}_1\tau_2]_\lambda), \\ \mathbb{G}(\tau_1,[\mathcal{S}_1\tau_2]_\lambda,[\mathcal{S}_1\tau_2]_\lambda), \\ \mathbb{G}(\tau_2,[\mathcal{S}_2\tau_1]_\lambda,[\mathcal{S}_2\tau_1]_\lambda) \end{array}\right] \\
\leq & k\left[\frac{\theta}{6}\max\left(\begin{array}{c} 6\mathbb{G}(\tau_1,\tau_2,\tau_2), \\ 6\mathbb{G}(\tau_1,\tau_2,\tau_2), \\ 6\mathbb{G}(\tau_2,\tau_3,\tau_3), \\ \frac{6\mathbb{G}(\tau_2,\tau_2,\tau_2)+6\mathbb{G}(\tau_1,\tau_3,\tau_3)}{2} \end{array}\right)\right] + \mathcal{L}_3\mu_{\mathbb{G}}(\tau_2,\tau_2) \\
& + \mathcal{L}_4 \min\Bigg[6\mathbb{G}(\tau_2,\tau_3,\tau_3), 6\mathbb{G}(\tau_1,\tau_3,\tau_3), 6\mathbb{G}(\tau_2,\tau_2,\tau_2)\Bigg] \\
\leq & k\left[\frac{\theta}{6}\max\left(\begin{array}{c} 6\mathbb{G}(\tau_1,\tau_2,\tau_2), \\ 6\mathbb{G}(\tau_2,\tau_3,\tau_3), \\ \frac{6\mathbb{G}(\tau_1,\tau_3,\tau_3)}{2} \end{array}\right)\right] \\
& + \mathcal{L}_4 \min\left[\begin{array}{c} 6\mathbb{G}(\tau_2,\tau_3,\tau_3), \\ 0, \\ 6\mathbb{G}(\tau_1,\tau_3,\tau_3) \end{array}\right] \\
\leq & k\Bigg[\frac{\theta}{6}\max\Bigg(6\mathbb{G}(\tau_1,\tau_2,\tau_2), 6\mathbb{G}(\tau_2,\tau_3,\tau_3), \frac{6\mathbb{G}(\tau_1,\tau_3,\tau_3)}{2}\Bigg)\Bigg].
\end{aligned}$$

Since

$$\frac{6\mathbb{G}(\tau_1,\tau_3,\tau_3)}{2} \leq \frac{6\mathbb{G}(\tau_1,\tau_2,\tau_2)+6\mathbb{G}(\tau_2,\tau_3,\tau_3)}{2},$$

so we have

$$\frac{\mathbb{G}(\tau_1,\tau_3,\tau_3)}{2} \leq \max[\mathbb{G}(\tau_1,\tau_2,\tau_2), \mathbb{G}(\tau_2,\tau_3,\tau_3)],$$

which gives

$$\mathbb{G}(\tau_2,\tau_3,\tau_3) \leq k\Bigg[\theta\max\Bigg(\mathbb{G}(\tau_1,\tau_2,\tau_2), \mathbb{G}(\tau_2,\tau_3,\tau_3)\Bigg)\Bigg].$$

Suppose $\mathbb{G}(\tau_1,\tau_2,\tau_2) < \mathbb{G}(\tau_2,\tau_3,\tau_3)$, then by Property 5.3 of ψ, we get a contradiction. Hence,

$$\mathbb{G}(\tau_2,\tau_3,\tau_3) \leq k\theta(\mathbb{G}(\tau_1,\tau_2,\tau_2)).$$

As $h = k\theta$, then

$$\mathbb{G}(\tau_2,\tau_3,\tau_3) \leq h(\mathbb{G}(\tau_1,\tau_2,\tau_2)).$$

Now using the above expression, we can write

$$\mathbb{G}(\tau_2, \tau_3, \tau_3) \leq h(\mathbb{G}(\tau_1, \tau_2, \tau_2)) \leq h(h(\mathbb{G}(\tau_0, \tau_1, \tau_1))) = h^2(\mathbb{G}(\tau_0, \tau_1, \tau_1)).$$

Continuing in this way, we get a sequence τ_e in $\mathcal{F}$ for $\alpha(\tau_e, \tau_{e+1}, \tau_{e+1}) \geq 1$ such that

$$\mathbb{G}(\tau_e, \tau_{e+1}, \tau_{e+1}) \leq h^e(\mathbb{G}(\tau_0, \tau_1, \tau_1)).$$

To see that the sequence τ_e is Cauchy, consider $\rho > e$:

$$\begin{aligned}
\mathbb{G}(\tau_e, \tau_\rho, \tau_\rho) \leq & \mathbb{G}(\tau_e, \tau_{e+1}, \tau_{e+1}) + \mathbb{G}(\tau_{e+1}, \tau_{e+2}, \tau_{e+2}) + ... + \mathbb{G}(\tau_{\rho-1}, \tau_\rho, \tau_\rho) \\
\leq & h^e\mathbb{G}(\tau_0, \tau_1, \tau_1) + h^{e+1}\mathbb{G}(\tau_0, \tau_1, \tau_1) + ... + h^{\rho-1}\mathbb{G}(\tau_0, \tau_1, \tau_1) \\
\leq & (h^e + h^{e+1} + ... + h^{\rho-1})\mathbb{G}(\tau_0, \tau_1, \tau_1) \\
\leq & h^e(1 + h^e + ... + h^{\rho-e-1})\mathbb{G}(\tau_0, \tau_1, \tau_1) \\
\leq & h^e(\frac{1 - h^{\rho-e-1}}{1-h})\mathbb{G}(\tau_0, \tau_1, \tau_1) \\
\leq & h^e\mathbb{G}(\tau_0, \tau_1, \tau_1) \to 0 \ \text{ as } \ e \to \infty.
\end{aligned}$$

This shows that $\{\tau_e\}$ is a Cauchy sequence in $\mathcal{F}$. Since $\mathcal{F}$ is complete so, we can find $\tau^* \in \mathcal{F}$ such that $\tau_e \to \tau^*$ as $e \to \infty$.

Using the fact that $\tau_{2e+1} \in [\mathcal{S}_1\tau_{2e}]_\lambda$ and $\tau_{2e+2} \in [\mathcal{S}_2\tau_{2e+1}]_\lambda$, to show that $\tau^* \in [\mathcal{S}_1\tau^*]_\lambda$ and $\tau^* \in [\mathcal{S}_2\tau^*]_\lambda$, consider

$$\begin{aligned}
&\mathbb{G}(\tau_{2e+1}, [\mathcal{S}_2\tau^*]_\lambda, [\mathcal{S}_2\tau^*]_\lambda) \leq kH_{\mathbb{G}}([\mathcal{S}_1\tau_{2e}]_\lambda, [\mathcal{S}_2\tau^*]_\lambda, [\mathcal{S}_2\tau^*]_\lambda) \\
\leq & k\left[\frac{\theta}{6}\max\begin{pmatrix} 6\mathbb{G}(\tau_{2e}, \tau^*, \tau^*), \mathbb{G}(\tau_{2e}, [\mathcal{S}_1\tau_{2e}]_\lambda, [\mathcal{S}_1\tau_{2e}]_\lambda), \\ \mathbb{G}(\tau^*, [\mathcal{S}_2\tau^*]_\lambda, [\mathcal{S}_2\tau^*]_\lambda), \\ \frac{\mathbb{G}(\tau^*, [\mathcal{S}_1\tau_{2e}]_\lambda, [\mathcal{S}_1\tau_{2e}]_\lambda) + \mathbb{G}(\tau_{2e}, [\mathcal{S}_2\tau^*]_\lambda, [\mathcal{S}_2\tau^*]_\lambda)}{2} \end{pmatrix}\right] \\
& + \mathcal{L}_1\mu_{\mathbb{G}}(\tau^*, [\mathcal{S}_1\tau_{2e}]_\lambda) + \mathcal{L}_2 \min\begin{bmatrix} \mathbb{G}(\tau^*, [\mathcal{S}_2\tau^*]_\lambda, [\mathcal{S}_2\tau^*]_\lambda), \\ \mathbb{G}(\tau^*, [\mathcal{S}_1\tau_{2e}]_\lambda, [\mathcal{S}_1\tau_{2e}]_\lambda), \\ \mathbb{G}(\tau_{2e}, [\mathcal{S}_2\tau^*]_\lambda, [\mathcal{S}_2\tau^*]_\lambda) \end{bmatrix} \\
\leq & k\left[\frac{\theta}{6}\max\begin{pmatrix} 6\mathbb{G}(\tau_{2e}, \tau^*, \tau^*), 6\mathbb{G}(\tau_{2e}, \tau_{2e+1}, \tau_{2e+1}), \\ \mathbb{G}(\tau^*, [\mathcal{S}_2\tau^*]_\lambda, [\mathcal{S}_2\tau^*]_\lambda), \\ \frac{6\mathbb{G}(\tau^*, \tau_{2e+1}, \tau_{2e+1}) + \mathbb{G}(\tau_{2e}, [\mathcal{S}_2\tau^*]_\lambda, [\mathcal{S}_2\tau^*]_\lambda)}{2} \end{pmatrix}\right] \\
& + \mathcal{L}_1\mu_{\mathbb{G}}(\tau^*, \tau_{2e+1}) + \mathcal{L}_2 \min\begin{bmatrix} \mathbb{G}(\tau^*, [\mathcal{S}_2\tau^*]_\lambda, [\mathcal{S}_2\tau^*]_\lambda), \\ \mathbb{G}(\tau_{2e}, [\mathcal{S}_2\tau^*]_\lambda, [\mathcal{S}_2\tau^*]_\lambda), \\ 6\mathbb{G}(\tau^*, \tau_{2e+1}, \tau_{2e+1}) \end{bmatrix}.
\end{aligned}$$

Applying $\lim e \to \infty$ in the above inequality, we get

$$\begin{aligned}
&\lim_{e\to\infty} \mathbb{G}(\tau_{2e+1}, [\mathcal{S}_2\tau^*]_\lambda, [\mathcal{S}_2\tau^*]_\lambda) \leq k \lim_{e\to\infty} \left[\frac{\theta}{6}\max\begin{pmatrix} 6\mathbb{G}(\tau_{2e}, \tau^*, \tau^*), 6\mathbb{G}(\tau_{2e}, \tau_{2e+1}, \tau_{2e+1}), \\ \mathbb{G}(\tau^*, [\mathcal{S}_2\tau^*]_\lambda, [\mathcal{S}_2\tau^*]_\alpha), \\ \frac{6\mathbb{G}(\tau^*, \tau_{2e+1}, \tau_{2e+1}) + \mathbb{G}(\tau_{2e}, [\mathcal{S}_2\tau^*]_\lambda, [\mathcal{S}_2\tau^*]_\lambda)}{2} \end{pmatrix}\right] \\
& + \lim_{e\to\infty} \mathcal{L}_1\mu_{\mathbb{G}}(\tau^*, \tau_{2e+1}) + \lim_{e\to\infty} \mathcal{L}_2 \min\begin{bmatrix} \mathbb{G}(\tau^*, [\mathcal{S}_2\tau^*]_\lambda, [\mathcal{S}_2\tau^*]_\lambda), \\ \mathbb{G}(\tau_{2e}, [\mathcal{S}_2\tau^*]_\lambda, [\mathcal{S}_2\tau^*]_\lambda), \\ 6\mathbb{G}(\tau^*, \tau_{2e+1}, \tau_{2e+1}), \end{bmatrix}.
\end{aligned}$$

This implies that

$$\begin{aligned}\mathbb{G}(\tau^*,[\mathcal{S}_2\tau^*]_\lambda,[\mathcal{S}_2\tau^*]_\lambda)\leq& k\left[\frac{\theta}{6}\max\begin{pmatrix}6\mathbb{G}(\tau^*,\tau^*,\tau^*),6\mathbb{G}(\tau^*,\tau^*,\tau^*),\\ \mathbb{G}(\tau^*,[\mathcal{S}_2\tau^*]_\lambda,[\mathcal{S}_2\tau^*]_\lambda),\\ \frac{6\mathbb{G}(\tau^*,\tau^*,\tau^*)+\mathbb{G}(\tau^*,[\mathcal{S}_2\tau^*]_\lambda,[\mathcal{S}_2\tau^*]_\lambda)}{2}\end{pmatrix}\right]\\ &+\mathcal{L}_1\mu_{\mathbb{G}}(\tau^*,\tau^*)+\mathcal{L}_2\min\begin{bmatrix}\mathbb{G}(\tau^*,[\mathcal{S}_2\tau^*]_\lambda,[\mathcal{S}_2\tau^*]_\lambda),\\ \mathbb{G}(\tau^*,[\mathcal{S}_2\tau^*]_\lambda,[\mathcal{S}_2\tau^*]_\lambda),\\ 6\mathbb{G}(\tau^*,\tau^*,\tau^*)\end{bmatrix}\\ \leq& k\left[\frac{\theta}{6}\max\left(\mathbb{G}(\tau^*,[\mathcal{S}_2\tau^*]_\lambda,[\mathcal{S}_2\tau^*]_\lambda),\frac{\mathbb{G}(\tau^*,[\mathcal{S}_2\tau^*]_\lambda,[\mathcal{S}_2\tau^*]_\lambda)}{2}\right)\right]\\ &+\mathcal{L}_2\min[\mathbb{G}(\tau^*,[\mathcal{S}_2\tau^*]_\lambda,[\mathcal{S}_2\tau^*]_\lambda),0].\\ \leq& k\frac{\theta}{6}\left[\max\left(\mathbb{G}(\tau^*,[\mathcal{S}_2\tau^*]_\lambda,[\mathcal{S}_2\tau^*]_\lambda),\frac{\mathbb{G}(\tau^*,[\mathcal{S}_2\tau^*]_\lambda,[\mathcal{S}_2\tau^*]_\lambda)}{2}\right)\right]\\ \leq& k\theta\left[\frac{\mathbb{G}(\tau^*,[\mathcal{S}_2\tau^*]_\lambda,[\mathcal{S}_2\tau^*]_\lambda)}{6}\right]\end{aligned}$$

Therefore,

$$\left[1-\frac{k\theta}{6}\right]\mathbb{G}(\tau^*,[\mathcal{S}_2\tau^*]_\lambda,[\mathcal{S}_2\tau^*]_\lambda)=0.$$

Since $[1-\frac{k\theta}{6}]\neq 0$. Hence, $\mathbb{G}(\tau^*,[\mathcal{S}_2\tau^*]_\lambda,[\mathcal{S}_2\tau^*]_\lambda)=0$. This implies that $\tau^*\in[\mathcal{S}_2\tau^*]_\lambda$.
Now,

$$\begin{aligned}&\mathbb{G}(\tau_{2e+2},[\mathcal{S}_1\tau^*]_\lambda,[\mathcal{S}_1\tau^*]_\lambda)\leq kH_{\mathbb{G}}([\mathcal{S}_2\tau_{2e+1}]_\lambda,[\mathcal{S}_1\tau^*]_\lambda,[\mathcal{S}_1\tau^*]_\lambda)\\ &\leq k\left[\frac{\theta}{6}\max\begin{pmatrix}6\mathbb{G}(\tau_{2e+2},\tau^*,\tau^*),\mathbb{G}(\tau_{2e+1},[\mathcal{S}_2\tau_{2e+1}]_\lambda,[\mathcal{S}_2\tau_{2e+1}]_\lambda),\\ \mathbb{G}(\tau^*,[\mathcal{S}_1\tau^*]_\lambda,[\mathcal{S}_1\tau^*]_\lambda),\\ \frac{\mathbb{G}(\tau^*,[\mathcal{S}_2\tau_{2e+1}]_\lambda,[\mathcal{S}_2\tau_{2e+1}]_\lambda)+\mathbb{G}(\tau_{2e+1},[\mathcal{S}_1\tau^*]_\lambda,[\mathcal{S}_1\tau^*]_\lambda)}{2}\end{pmatrix}\right]\\ &\mathcal{L}_3\mu_{\mathbb{G}}(\tau^*,[\mathcal{S}_2\tau_{2e+1}]_\lambda)+\mathcal{L}_4\min\begin{bmatrix}\mathbb{G}(\tau^*,[\mathcal{S}_1\tau^*]_\lambda,[\mathcal{S}_1\tau^*]_\lambda),\\ \mathbb{G}(\tau^*,[\mathcal{S}_2\tau_{2e+1}]_\lambda,[\mathcal{S}_2\tau_{2e+1}]_\lambda),\\ \mathbb{G}(\tau_{2e+1},[\mathcal{S}_1\tau^*]_\lambda,[\mathcal{S}_1\tau^*]_\lambda)\end{bmatrix}\end{aligned}$$

$$\begin{aligned}\mathbb{G}(\tau_{2e+2},[\mathcal{S}_1\tau^*]_\lambda,[\mathcal{S}_1\tau^*]_\lambda)\leq& k\left[\frac{\theta}{6}\max\begin{pmatrix}6\mathbb{G}(\tau_{2e+1},\tau^*,\tau^*),6\mathbb{G}(\tau_{2e+1},\tau_{2e+2},\tau_{2e+2}),\\ \mathbb{G}(\tau^*,[\mathcal{S}_2\tau^*]_\lambda,[\mathcal{S}_1\tau^*]_\lambda),\\ \frac{6\mathbb{G}(\tau^*,\tau_{2e+2},\tau_{2e+2})+\mathbb{G}(\tau_{2e+1},[\mathcal{S}_2\tau^*]_\lambda,[\mathcal{S}_2\tau^*]_\lambda)}{2}\end{pmatrix}\right]\\ &+\mathcal{L}_3\mu_{\mathbb{G}}(\tau^*,\tau_{2e+2})+\mathcal{L}_4\min\begin{bmatrix}\mathbb{G}(\tau^*,[\mathcal{S}_1\tau^*]_\lambda,[\mathcal{S}_1\tau^*]_\lambda),\\ \mathbb{G}(\tau_{2e+1},[\mathcal{S}_1\tau^*]_\lambda,[\mathcal{S}_1\tau^*]_\lambda),\\ 6\mathbb{G}(\tau^*,\tau_{2e+2},\tau_{2e+2})\end{bmatrix}.\end{aligned}$$

Similarly as above applying $\lim e\to\infty$, we get $\mathbb{G}(\tau^*,[\mathcal{S}_1\tau^*]_\lambda,[\mathcal{S}_1\tau^*]_\lambda)=0$, which implies $\tau^*\in[\mathcal{S}_1\tau^*]_\lambda$. So, τ^* is the common fixed point of the pair of mappings $\mathcal{S}_1$ and $\mathcal{S}_2$. □

Remark 6.5. Theorem 6.4 is a fuzzy extension of the main results of [1, 6, 9], even in the case of metric space.

In what follows, we provide an illustrative example to support the hypotheses of Theorem 6.4.

Example 6.6. *Let $\mathcal{F} = [0,1]$, $\mathbb{G}(\tau,\sigma,\upsilon) = |\tau-\sigma| + |\sigma-\upsilon| + |\tau-\upsilon|$ for all $\tau,\sigma,\upsilon \in \mathcal{F}$, then $(\mathcal{F},\mathbb{G})$ is a complete $\mathbb{G}$-metric space. Let $\mathcal{S}_1, \mathcal{S}_2 : \mathcal{F} \longrightarrow I^{\mathcal{F}}$ be two fuzzy mappings defined as:*

$$\mathcal{S}_1(\tau)(t) = \begin{cases} 1 \text{ if } 0 \leq t \leq \frac{\tau}{5}, \\ 0 \text{ if } \frac{\tau}{5} < t \leq 1. \end{cases} \quad \text{and } \mathcal{S}_2(\tau)(t) = \begin{cases} 1 \text{ if } 0 \leq t \leq \frac{\tau}{3}, \\ 0 \text{ if } \frac{\tau}{3} < t \leq 1. \end{cases} \quad \text{for all } \tau \in \mathcal{F},$$

For $\lambda = 1$, we have

$$[\mathcal{S}_1(\tau)]_\lambda = [\mathcal{S}_2(\tau)]_1 = \left[0, \frac{\tau}{5}\right] \text{ and } [\mathcal{S}_2(\tau)]_\lambda = [\mathcal{S}_2(\tau)]_1 = \left[0, \frac{\tau}{3}\right] \text{ for all } \tau \in \mathcal{F}$$

Then for $\theta = \frac{1}{2}$ and $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4 \geq 0$, all the conditions of Theorem 6.4 are fulfilled and 0 is a common fixed point of $\mathcal{S}_1$ and $\mathcal{S}_2$.

Corollary 6.7. *Let $(\mathcal{F},\mathbb{G})$ be a $\mathbb{G}-$metric space and $\mathcal{S} : \mathcal{F} \to I^{\mathcal{F}}$ be a fuzzy $(\theta,\mathcal{L})$-weak contraction such that for all $\tau,\sigma,\upsilon \in \mathcal{F}$, and $\theta \in (0,1)$, $\mathcal{L}_1, \mathcal{L}_2 \geq 0$,*

$$\mathcal{H}_{\mathbb{G}}([\mathcal{S}\tau]_\lambda, [\mathcal{S}\sigma]_\lambda, [\mathcal{S}\upsilon]_\lambda) \leq \frac{\theta}{6}\mathcal{M} + \mathcal{L}_1 \mu_{\mathbb{G}}(\sigma, [\mathcal{S}\tau]_\lambda) + \mathcal{L}_2 \mathcal{N}, \tag{3}$$

where

$$\mathcal{M} = \max \begin{bmatrix} 6\mathbb{G}(\tau,\sigma,\upsilon), \mathbb{G}(\tau, [\mathcal{S}\tau]_\lambda, [\mathcal{S}\tau]_\lambda), \\ \mathbb{G}(\sigma, [\mathcal{S}\sigma]_\lambda, [\mathcal{S}\sigma]_\lambda), \\ \frac{\mathbb{G}(\sigma,[\mathcal{S}\tau]_\lambda,[\mathcal{S}\tau]_\lambda) + \mathbb{G}(\tau,[\mathcal{S}\sigma]_\lambda,[\mathcal{S}\sigma]_\lambda)}{2} \end{bmatrix}$$

$$\mathcal{N} = \min \begin{bmatrix} \mathbb{G}(\sigma, [\mathcal{S}\sigma]_\lambda, [\mathcal{S}\sigma]_\lambda), \\ \mathbb{G}(\sigma, [\mathcal{S}\tau]_\lambda, [\mathcal{S}\tau]_\lambda), \\ \mathbb{G}(\tau, [\mathcal{S}\sigma]_\lambda, [\mathcal{S}\sigma]_\lambda) \end{bmatrix}.$$

Then there exists a fixed point of mapping $\mathcal{S}$.

Proof. Put $\mathcal{S}_1 = \mathcal{S}_2$, in the proof of Theorem 6.4, we get the required result. □

Corollary 6.8. *Let $(\mathcal{F},\mathbb{G})$ be a $\mathbb{G}-$metric space and $\mathcal{S} : \mathcal{F} \to I^{\mathcal{F}}$ be a fuzzy $(\theta,\mathcal{L})-$weak contraction. That is, for all $\tau,\sigma,\upsilon \in \mathcal{F}$ there exist two constants $\theta \in (0,1)$, $\mathcal{L} \geq 0$, such that*

$$\mathcal{H}_{\mathbb{G}}\bigg([\mathcal{S}\tau]_\lambda, [\mathcal{S}\sigma]_\lambda, [\mathcal{S}\upsilon]_\lambda\bigg) \leq \theta\mathbb{G}(\tau,\sigma,\upsilon) + \mathcal{L}\mu_{\mathbb{G}}(\sigma, [\mathcal{S}\tau]_\lambda). \tag{4}$$

Then there exists $\tau^* \in \mathcal{F}$ such that $\tau^* \in [\mathcal{S}\tau^*]_\lambda$, that is , τ^* is a fixed point of $\mathcal{S}$.

Example 6.9. *Let $\mathcal{F} = [0,1]$, $\mathbb{G}-$ be as defined in Example 6.6 and $\mathcal{S} : \mathcal{F} \longrightarrow I^{\mathcal{F}}$ be a fuzzy mapping defined as:*

$$\mathcal{S}(\tau)(t) = \begin{cases} 1 \text{ if } 0 \leq t \leq \frac{\tau}{7}, \\ 0 \text{ if } \frac{\tau}{7} < t \leq 1. \end{cases} \quad \text{for all } \tau \in \mathcal{F},$$

For $\lambda = 1$, we have

$$[\mathcal{S}\tau]_\lambda = [\mathcal{S}\tau]_1 = \left[0, \frac{\tau}{7}\right].$$

Then, for $\theta = \frac{1}{7}$ and $\mathcal{L} \geq 0$, all the conditions of Corollary 6.8 are fulfilled and $\tau = 0$ is a fixed point of $\mathcal{S}$.

7. Conclusion

In this paper, we studied fixed point theorems and common fixed point results for fuzzy $\alpha-$admissible and fuzzy $\alpha - \psi-$admissible mappings and fixed point results for fuzzy $(\theta, \mathcal{L})-$weakly contractive mappings in $\mathbb{G}-$metric space. Starting from the notion of $\mathbb{G}-$metric space, our results complement several significant fixed point theorems of $\mathbb{G}-$metric space in the frame of fuzzy $(\theta, \mathcal{L})-$weak contraction mappings. The main idea of this paper, being discussed in fuzzy setting, is fundamental. Hence, as some future assignments, the concepts of this article can be examined in the context of some generalized fuzzy sets such as L-fuzzy, intuitionistic fuzzy, rough and soft sets. We hope that our presented idea herein will be a source of motivation for interested researchers to extend and improve these results suitable for areas of applications such as in the investigation of existence of solutions of differential and integral equations of different types and related problems.

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Yasir Mahmood
Orcid number: 0000-0002-9077-5947
Department of Mathematics
COMSATS University, Chak Shahzad
Islamabad,44000, Pakistan
Email address: yasirmalik0030@gmail.com

Mohammed Shehu Shagari
Orcid number: 0000-0001-6632-8365
Department of Mathematics
Faculty of Physical Sciences
Ahmadu Bello University
Zaria, Nigeria
Email address: shagaris@ymail.com