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ON MULTIPLICATION fs-MODULES AND DIMENSION SYMMETRY

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Abstract. In this paper, we first study fs-modules, i.e., modules with finitely many small submodules. We show that every fs-module with finite hollow dimension is Noetherian. Also, we prove that an R-module M with finite Goldie dimension, is an fs-module if and only if $M = M_1 \oplus M_2$, where M_1 is semisimple and M_2 is an fs-module with $Soc(M_2) \ll M$. Then, we investigate multiplication fs-modules over commutative rings and we prove that the lattices of R-submodules of M and S-submodules of M are coincide, where $S = End_R(M)$. Consequently, M_R and $_SM$ have the same Krull (Noetherian, Goldie and hollow) dimension. Further, we prove that for any self-generator multiplication module M, the fs-module as a right R-module and as a left S-module are equivalent.

Keywords: Small submodules, fs-modules, Multiplication modules, Dimension symmetry.
2020 MSC: Primary 16P60, 16P20, 16P40.

1. Introduction

In this paper, we focus on modules with finitely many small submodules (briefly, fs-modules). It is well known that M is a semisimple and Noetherian module if and only if it is Artinian with $Rad(M) = 0$ (i.e., 0 is the only small submodule of M). Thus every Artinian module M with $Rad(M) = 0$, is Noetherian. Motivated by this, it is natural to ask: Is any Artinian fs-module, Noetherian? We first try to answer to this question and then investigate multiplication fs-modules over commutative rings . For this, we study some basic properties of modules with finitely many small submodules. For instance, we show that M is an fs-module if and only if $Rad(M)$ has only finitely many submodules. Also, we show that fs-modules are closed under submodules and small quotients (i.e., every factor $\frac{M}{N}$, where N is small in M). We prove that every fs-module with finite hollow dimension is Noetherian. Actually, we extend the latter fact to a larger class of modules (note, every Artinian module has finite hollow dimension). In particular, we show that an R-module M with finite Goldie dimension is an fs-module if and only if $M = M_1 \oplus M_2$, where M_1 is semisimple and M_2 is an fs-module with $Soc(M_2) \ll M$.

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Moreover, we give some examples, to show that for an arbitrary module M, the properties of being an fs-module and to have finite hollow dimension are independent. Then, we focus on multiplication fs-modules over commutative rings. In particular, for any $N \subseteq M$, we prove that N is an R-submodule of M if and only if it is an S-submodule of M, where $S = End_R(M)$. This immediately implies that the lattices of R-submodules of M and S-submodules of M are coincide. Consequently, M_R and ${}_SM$ have the same Krull (Noetherian, Goldie, hollow) dimension. Also, we show that for any self-generator multiplication module M, the concept of an fs-module as a right R-module and as a left S-module are equivalent.

Throughout this article, all rings are associative with non-zero identity and all modules are unital right modules. Let R be a ring and M be an R-module. R is called local, if it has a unique maximal right (equivalently, left) ideal. M is called local if it has exactly one maximal submodule that contains all its proper submodules. The notation $N \subseteq_e M$ (resp., $N \ll M$) will denote N is an essential (resp., a small) submodule of M, that is, $N \cap A \neq 0$ (resp., $N + A \neq M$), for all non-zero (resp., proper) submodules A of M. M has finite Goldie (resp., hollow) dimension if for any ascending (resp., descending) chain $N_1 \subseteq N_2 \subseteq \cdots$ (resp., $N_1 \supseteq N_2 \supseteq \cdots$) of submodules of M there exists an integer $n \geq 1$, such that $N_n \subseteq_e N_k$ (resp., $\frac{N_n}{N_k} \ll \frac{M}{N_k}$) for all $k \geq n$. $Soc(M)$ (resp., $Rad(M)$) will denote the socle (resp., radical) of M, i.e., the sum (resp., intersection) of all minimal (resp., maximal) submodules of M. Also, $Soc(M)$ (resp., $Rad(M)$) is equal to the intersection (resp., sum) of all essential (resp., small) submodules of M. If M fails to have minimal (resp., maximal) submodules, we set $Soc(M) = 0$ (resp., $Rad(M) = M$) and in case, M fails to have proper essential (resp., nonzero small) submodule, then $Soc(M) = M$ (resp., $Rad(M) = 0$). Hence, any module with non-trivial socle (resp., radical) has both minimal and proper essential (resp., maximal and non-zero small) submodules. Also, $J(R)$ denote the Jacobson radical of R, i.e., the intesection of all maximal right ideals of R. Note that $J(R) = Rad(R_R) = Rad({}_RR)$. Finally, k-dim M, n-dim M, G-dim M and h-dim M respectively, denote the Krull dimension, the Noetherian dimension, the Goldie dimension and the hollow dimension of M. The Noetherian dimension is also known as the dual Krull dimension and N-dimension. For more details on these dimensions and undefined terms and notations, we refer to [1, 3, 6, 7, 12, 15, 21].

2. fs-modules and fs-rings

In this section, we give our definition of fs-modules and study their properties.

Definition 2.1. An R-module M with only finitely many non-zero small submodules is said to be an fs-module. A ring R is called a right (left) fs-ring if as a right (left) R-module, it is an fs-module.

Remark 2.2. Let M be an R-module.

(1) If $Rad(M) = 0$, then M is an fs-module.
(2) If $Rad(M) = M$, then M is not an fs-module. For this, note that every finite sum of small submodules is a small submodule, so if M is an fs-module M, then $Rad(M)$ is a small submodule of M, hence $Rad(M) \neq M$.
(3) If M is an fs-module, then M has at least one maximal submodule.

The following easy access results show that the class of fs-modules are closed under submodules and small quotients small quotients (i.e., every factor $\frac{M}{N}$, where N is small in M), see also [11, Propositions 2.5, 2.6].

Proposition 2.3. The following are equivalent for any R-module M.
(1) M is an fs-module.
(2) Every submodule of M is an fs-module.
(3) Every small quotient of M is an fs-module.

Proposition 2.4. Let M be an R-module. Then M is an fs-module if and only if $Rad(M)$ has only finitely many submodules.

Proof. If $Rad(M)$ has only finitely many submodules, then M necessarily is an fs-module, since $Rad(M)$ contains every small submodule of M. Conversely, let M be an fs-module. Since $Rad(M)$ equals to the sum of all small submodules of M, it follows that $Rad(M)$ is small in M, so is every submodule that is contained in $Rad(M)$. Hence, $Rad(M)$ has only finitely many submodules. □

The following result is also in [11, Corollary 2.7].

Corollary 2.5. If M is an fs-module, then
(1) $Rad(M)$ has finite length, so it is both Artinian and Noetherian.
(2) M is Noetherian (Artinian) if and only if $\frac{M}{Rad(M)}$ is Noetherian (Artinian).

Proof. (1) By the previous proposition, $Rad(M)$ has only finitely many submodules, thus it has finite length and so it is both Artinian and Noetherian.
(2) By part (1), it is evident. □

Corollary 2.6. Every local fs-module M is both Noetherian and Artinian. In particular, every local and right fs-ring R is both right Noetherian and right Artinian.

Proof. Let K be the unique maximal submodule of M, then $Rad(M) = K$ and so $\frac{M}{Rad(M)} = \frac{M}{K}$ is simple. Hence, $\frac{M}{Rad(M)}$ is both Noetherian and Artinian and by Corollary 2.5, we are done. □

Remark 2.7. If S is a non-zero small submodule of M, then so is every non-zero submodule of M, contained in S. Hence, every minimal member with respect to inclusion, in the set of non-zero small submodules of M, necessarily is a minimal submodule of M. In other words, every minimal non-zero small submodule is a minimal submodule. From this, it follows that a "minimal non-zero small submodule" or equivalently a "small minimal submodule", is precisely a non-zero submodule which is "both minimal and small submodule" of M.

Proposition 2.8. Let M be an fs-module with $Rad(M) \neq 0$. Then M has a minimal and small submodule, that is, $Soc(Rad(M)) \neq 0$.

Proof. The set of all non-zero small submodules of M is finite, so it has a minimal element, say S, which is also a minimal submodule of M, by the above remark. Hence $0 \neq S \subseteq Rad(M) \cap Soc(M) = Soc(Rad(M))$, by [21, 21.2(2)]. □

Remark 2.9. For every R-module M, it follows from [2, Lemma 2.2(9)], that $Soc(Rad(M))$ is a small submodule of M. Hence, a semisimple submodule S of M is small in M if and only if $S \subseteq Rad(M)$.

Lemma 2.10. Every minimal submodule of an R-module M is either small or a direct summand. For this, let S be a minimal submodule of M. If $S \subseteq Rad(M)$, then $S \subseteq Soc(Rad(M))$, so it is small by previous remark. If $S \nsubseteq Rad(M)$, then $S \nsubseteq K$ for some maximal submodule K of M. Hence, $S \cap K = 0$ and $S \oplus K = M$.

Now, we give our structure theorem for fs-modules with finite Goldie dimension, see also [11, Theorem 2.20].

Theorem 2.11. Let M be an R-module with finite Goldie dimension. Then M is an fs-module if and only if $M = M_1 \oplus M_2$, where M_1 is semisimple and M_2 is an fs-module such that $Soc(M_2) \ll M$.

Proof. First of all, note that since M has finite Goldie dimension, it follows that M has finitely many minimal submodules. Let M be an fs-module. If every minimal submodule of M is small in M, then $Soc(M)$ is a finite direct sum of small submodules. It follows that $Soc(M) \ll M$, so in this case $M = 0 \oplus M$ and we are done. In other case, M has some minimal submodules which are non-small, let N_1 be a minimal and non-small submodules of M. By Lemma 2.10 there exists a submodule K_1 of M such that $N_1 \oplus K_1 = M$. If K_1 has no minimal and non-small submodule, then $Soc(K_2) \ll M$. Set $M_1 = N_1$ and $M_2 = K_2$ and we are done. Otherwise $K_1 = N_2 \oplus K_2$, for some minimal and non-small submoule N_2 of K_1 and some submodule K_2 of K_1 and so $M = N_1 \oplus N_2 \oplus K_2$. Since $Soc(M)$ has finite Goldie dimension, so M has finite number of minimal submodules, finally we have $M = N_1 \oplus N_2 \oplus \cdots \oplus N_m \oplus K_m$ such that N_i's are minimal and non-small and all minimal submodules of K_m are small, that is, $Soc(K_m) \ll M$. Let $M_1 = N_1 \oplus N_2 \oplus \cdots \oplus N_m$ and $M_2 = K_m$, then we are done. To prove the converse, since M_1 is semisimple, it follows from [2, Lemma 2.2(6)] that every small submodule of M is in the form of $0 \oplus S_2$ such that $S_2 \ll M_2$, and so M is an fs-module. □

Definition 2.12. An R-module M is called homogeneous, if every non-zero submodule of M has a non-zero small submodule.

The next result is devoted to homogeneous fs-modules.

Proposition 2.13. Let M be a homogeneous fs-module. Then $Soc(M)$ is essential in M.

Proof. Let N be a non-zero submodule of M. By Proposition 2.3, N is an fs-module. So it follows from Proposition 2.8 that $0 \neq Soc(N) \subseteq N \cap Soc(M)$, and hence we are done. □

We recall that an R-module M is called finitely embedded if $Soc(M)$ is a finitely generated and essential submodule of M.

Corollary 2.14. Let M be a homogeneous fs-module. If $Soc(M)$ is finitely generated, then M is finitely embedded.

Proof. This follows from previous proposition and definition of finitely embedded modules. □

Recall that R is called a duo ring, if every one-sided ideal of R is an ideal of R. Also, a commutative ring R is locally Noetherian if the localization R_M is Noetherian for every maximal ideal M of R.

Proposition 2.15. Let R be a locally Noetherian or a Noetherian duo ring and M be a homogeneous fs-module for which $Soc(M)$ is finitely generated. Then M is Artinian.

Proof. If R is locally Noetherian or a Noetherian duo ring, then every finitely embedded module is Artinian, see [19, Theorem 2] and [7, Theorem 2.4]. □

We cite the following important fact from [20, Corollary 1.10].

Theorem 2.16. An R-module M with $Rad(M) = 0$ has finite hollow dimension if and only if it is finitely generated semisimple.

The following result is also in [11, Theorem 2.13].

Theorem 2.17. Let M be an fs-module with finite hollow dimension over a ring R. The following holds.

(1) M is Artinian.
(2) M is Noetherian.
(3) $Rad(M)$ is finitely generated.
(4) M has finite Goldie dimension.
(5) M has a finite composition series.
(6) M has finite length.

Proof. It follows from previous theorem that $\frac{M}{Rad(M)}$ is both Artinian and Noetherian. Then so is M, by the part (2) of Corollary 2.5. It follows that $Rad(M)$

is finitely generated and M has finite Goldie dimension. Moreover, such a module M has a finite composition series, so it has finite length.

□

Corollary 2.18. Let M be an fs-module. Then M is Artinian if and only if it has finite hollow dimension.

Proof. Every Artinian module clearly has finite hollow dimension. The converse is true by the previous theorem. □

Proposition 2.19. Let R be a semiprime ring. The following statements are equivalent.

(1) R is a right fs-ring with finite right hollow dimension.
(2) R is a left fs-ring with finite left hollow dimension.
(3) R is a semisimple ring.
Moreover, if R is local, then these are equivalent to
(4) R is a division ring.

Proof. By previous corollary, (1) is equivalent to R is right Artinian and (2) is equivalent to R is left Artinian. Since R is semiprime, these are equivalent to R is semisimple, see [6, Corollary 3.17]. In case R is local, then it is a semisimple local ring. Equivalently, $J(R) = 0$ is the unique maximal right (left) ideal of R, that is, R is a division ring. □

Proposition 2.20. Let R be a ring. Then at the same time, $R[x]$ cannot have finite right (left) hollow dimension and to be a right (left) fs-ring.

Proof. Since $(x) \supsetneq (x^2) \supsetneq (x^3) \supsetneq \cdots$ is an infinite descending chain of ideals in $R[x]$, it follows that $R[x]$ is not right (nor left) Artinian. By Corollary 2.18, we are done. □

Definition 2.21. An R-module M is said to be an $AB5^*$ module, if for every submodule B and inverse system $\{A_i\}_{i\in I}$ of submodules of M,

$$B + \bigcap_{i\in I} A_i = \bigcap_{i\in I}(B + A_i).$$

Artinian modules and linearly compact modules are $AB5^*$, see [21, 29.8]. We also recall that an R-module M is called $q.f.d$ (i.e., quotient finite dimentional) if every factor module of M has finite Goldie dimension.

Lemma 2.22. Let M be an $AB5^*$ fs-module. If M has Krull dimension, then M is both Noetherian and Artinian.

Proof. By [15, Proposition 1.3], M is $q.f.d$ if and only if every submodule of M has finite hollow dimension. Hence, if M is both an fs-module and a $q.f.d$-module, then by Theorem 2.17, it is both Noetherian and Artinian. □

Example 2.23. For an arbitrary R-module M, to be an fs-module and to have finite hollow dimension are independent. For this, we give some examples.

(1) For an fs-module with finite hollow dimension, we refer to every finite R-module M with $Rad(M) = 0$.
(2) For an fs-module with infinite hollow dimension, we refer to $\mathbb{Z}$, the ring of integers as $\mathbb{Z}$-module.
(3) For a non-fs-module with finite hollow dimension, we refer to $\mathbb{Z}_{p^\infty}$ as $\mathbb{Z}$-module, where p is a prime number.
(4) For a non-fs-module with infinite hollow dimension, we refer to $\mathbb{Q}$, the set of rational numbers as $\mathbb{Z}$-module.

Remark 2.24. By the part 4 of above example we infer that a proper essential extension of an fs-module need not to be an fs-module.

Example 2.25. For each $n \in \mathbb{N}$, there exists a non-fs-module M such that $n\text{-dim}\, M = n$. For this, let F be a field and $R = F[x_1, x_2, ..., x_n]$. Then R is a commutative Noetherian ring and every maximal ideal M of R, is exactly of rank n, that is, there exists a chain of length n of prime ideals descending from M, but no longer chain. Now, let S be a simple R-module and $A = E(S)$ be the injective envelope of S. By [19, Theorem 2], A is Artinian and by [3, Poroposion 5], $n\text{-dim}\, A = Rank(M) = n > 1$, where M is a maximal ideal of R such that $S \cong \frac{R}{M}$. This implies that A is not an fs-module, since by Corollary 2.18, for every fs-module with finite hollow dimension, must be Noetherian. This example, also shows that there exist Artinian modules with Noetherian dimension of any natural number.

Definition 2.26. An R-module M is called a us-module, if it has a unique non-zero small submodule. R is called a right us-ring, if as an R-module it is a us-module.

The following well-known fact is due to Brauer.

Theorem 2.27. If A is a minimal right ideal of a ring R, then either $A^2 = 0$ or $A = eR$ for some idempotent $e \in R$.

Theorem 2.28. The following statements are equivalent for any ring R.

(1) R is a right us-ring.
(2) $J = J(R)$ is minimal as a right ideal of R and $J^2 = 0$.
(3) Each right ideal A of R is either minimal or non-small.

Proof. (1) ⇒ (2). Clearly J is the unique small right ideal of R, hence it is minimal as a right ideal (for, if S is a non-zero right ideal of R contained in J, then S is small by [2, Lemma 2.2(1)], so $S = J$). On the other hand, J have no any idempotent element, so $J^2 = 0$ by Theorem 2.27.
(2) ⇒ (3). Let A be a non-zero right ideal of R. If A is not minimal, then $A \neq J$, hence it is non-small.
(3) ⇒ (1). Since $J(R)$ is small as a right (and left) ideal of R, it is minimal right ideal of R by (3). Hence, R is an us-ring. □

Definition 2.29. An R-module M is said to be dual-local if it has a unique minimal submodule, that is, M has a non-zero submodule N that contained in every non-zero submodule of M.

In [13], a ring with a unique essential proper right ideal is called a right *ue*-ring. Similarly, an R-module with a unique proper essential submodule is called a *ue*-module. The following theorem seems to be interesting.

Theorem 2.30. The following statements are equivalent for an R-module M.

(1) M is a local *us*-module.
(2) M is a dual-local *ue*-module.
(3) M has a unique non-trivial submodule.

Moreover, for such a module $Soc(M) = Rad(M)$.

Proof. (3) $\Rightarrow$ (1) and (3) $\Rightarrow$ (2) are clear.
(1) $\Rightarrow$ (3) Since M is local, $Rad(M)$ is the unique maximal submodule of M. Since M is a *us*-module, $Rad(M)$ and it the unique non-zero small submodule of M. If $0 \neq A$ is a submodule of $Rad(M)$, then A is small and so $A = Rad(M)$. Hence $Rad(M)$ is also a minimal submodule of M and so $Rad(M)$ is both a maximal and a minimal submodule of M. Again, let A be a non-trivial submodule of M. Then $A \subseteq Rad(M)$ and by minimality of $Rad(M)$ we have $A = Rad(M)$. This shows that $Rad(M)$ is the unique non-trivial submodule of M and we are done.
(2) $\Rightarrow$ (3) Since M is dual-local, $Soc(M)$ is the unique minimal submodule of M. Since M is a *ue*-module, $Soc(M)$ is also the unique essential submodule of M. If A is a proper submodule of M such that $Soc(M) \subseteq A$, then A is essential and so $A = Soc(M)$. Hence, $Soc(M)$ is also a maximal submodule of M and so $Soc(M)$ is both a minimal and a maximal submodule of M. Again, let A be a non-trivial submodule of M. Then $Soc(M) \subseteq A$ and by maximality of $Soc(M)$, we have $A = Soc(M)$. This shows that $Soc(M)$ is the unique non-trivial submodule of M and we are done.
Moreover (3) implies that $Soc(M) = Rad(M)$ is just the unique non-trivial submodule of M. □

Corollary 2.31. Let R be a local and right (left) *us*-ring. Then as a right (left) ideal, $J(R)$ is both minimal and maximal, hence it is the unique non-trivial right (2-sided) ideal of R.

Definition 2.32. An R-module M with only finitely many small and minimal (resp., unique small and minimal) submodules is called an fsm-module (resp., usm-module). A ring R with only finitely many small and minimal right ideals is called a right fsm-ring (resp., usm-ring).

Remark 2.33. Note that every fs-module is an fsm-module, but the converse is not true ingeneral. For example, $\mathbb{Z}_{p^\infty}$ as a $\mathbb{Z}$-module is an fsm-module which is not an fs-module. More generally, for every Artinian module M which is not semisimple, we have $Rad(M) \neq 0$, so M has at least one small and minimal submodule. Since $Soc(M)$ is Artinian, so it is finitely generated, hence M is an fsm-module.

Proposition 2.34. Let M be an R-module. Then M is an fsm-module if and only if $Rad(M)$ has only finitely many minimal submodules.

Proof. In case that $Rad(M) = 0$, we need no explanation. For the other case, note that from Remark 2.9, it follows that M has only finitely many small and minimal submodules if and only if $Rad(M)$ has only finitely many minimal submodules. □

3. Multiplication modules and dimension symmetry

Troughout this section, R is a commutative ring. The concept of multiplication modules has been studied in many articles, see for example [4, 17]. An R-module M is called multiplication if for every submodule N of M, there exists an ideal I of R such that $N = MI$. In this case, we can take $I = (N : M) = ann(\frac{M}{N}) = \{r \in R : Mr \subseteq N\}$. The class of multiplication modules contains all projective ideals, all cyclic modules, all finitely generated distributive modules and all ideals eR, where e is an idempotent. In this section, we focus on multiplication fs-modules. We recall that if M is an R-module and $S = End_R(M)$, then ${}_SM_R$, that is, M is an $(S - R)$-bimodule. Also, an R-submodule X of M is called fully invariant provided it is also an S-submodule of M, or equivalently, $f(X) \subseteq X$, for every $f \in S = End_R(M)$.

We cite the following facts from [17] and [4, Lemma 1].

Proposition 3.1. Let M be a multiplication R-module. Then $S = End_R(M)$ is a commutative ring.

Proposition 3.2. Let M be a multiplication module. Then every submodule of M is fully invariant.

Previous proposition says that every R-submodule of a multiplication module is an S-submodule, where $S = End_R(M)$. In the next theorem we show that not only for multiplication R-modules, but also for every R-module M, every S-submodule is an R-submodule.

Proposition 3.3. Let M be an R-module and $S = End_R(M)$. Then every S-submodule of M is an R-submodule.

Proof. Let N be an S-submodule of M. It suffices to show that $Nr \subseteq N$, for any $r \in R$. For this, let $r \in R$ and define $f : M_R \to M_R$ by $f(m) = mr$, for each $m \in M$. It is easy to see that $f \in S = End_R(M)$. Hence, $Nr = f(N) \subseteq N$, that is, N is an R-submodule of M. □

The next theorem can be one of the major theorems in the theory of commutative rings.

Theorem 3.4. Let M be a multiplication R-module, $N \subseteq M$ and $S = End_R(M)$. Then

(1) N is an R-submodule of M if and only if it is an S-submodule of M.
(2) The lattices of R-submodules of M and S-submodules of M are coincide.

Proof. (1) It comes out from Propositions 3.2 and 3.3.
(2) It follows by the part (1). □

Corollary 3.5. Let M be a multiplication R-module, $N \subseteq M$ and $S = End_R(M)$. Then N is an essential (resp., a small) R-submodule of M if and only if it is an essential (resp., a small) S-submodule of M.

Proof. It follows from of Theorem 3.4(1). □

In [10], as an special case of the Krull symmetry property, see also the appendix of [6], we study R-modules M for which $k\text{-dim}\, M_R = k\text{-dim}\, {}_SM$, where $S = End_R(M)$. It is proved that if R is an FBN ring and M a fully bounded NPG module (i.e., Noetherian, projective and generator), then $k\text{-dim}\, M_R = k\text{-dim}\, {}_SM$. The next theorem shows that for multiplication modules, this symmetry also holds, for the Noetherian, Goldie and hollow dimensions and it also holds for the properties of being α-DICC, α-short and α-Krull. For more details on these latter concepts, we refer the reader, respectively to [14], [5] and [9].

Theorem 3.6. Let M be a multiplication R-module and $S = End_R(M)$. Then

(1) $G\text{-dim}\, M_R = G\text{-dim}\, {}_SM$.
(2) $h\text{-dim}\, M_R = h\text{-dim}_S M$.
(3) $k\text{-dim}\, M_R = k\text{-dim}\, {}_SM$.
(4) $n\text{-dim}\, M_R = n\text{-dim}\, {}_SM$.
Moreover, for every ordinal α,
(5) M_R is α-DICC if and only if ${}_SM$ is α-DICC.
(6) M_R is α-short if and only if ${}_SM$ is α-short.
(7) M_R is α-Krull if and only if ${}_SM$ is α-Krull.

Proof. All these naturally come out from of Theorem 3.4(2). □

We recall that an R-module M is called self-generator if for each submodule X of M, there exists $\Delta \subseteq S = End_R(M)$, such that $N = \sum_{f\in\Delta} f(M)$. For any $X \subseteq M$, we set $I_X = \{f \in S : f(M) \subseteq X\}$.

Proposition 3.7. Let M be a self-generator multiplication R-module and $S = End_R(M)$. Then M is a multiplication S-module.

Proof. Let $X \subseteq M$ be an S-submodule of M. Then X is also an R-submodule of M. Also, S is a commutative ring, by Proposition 3.1. Now, we may invoke [8, Lemma 3.4(1)] to see that $I_X M = X$ and this completes the proof. □

We conclude this paper with the next result that seems to be interesting.

Theorem 3.8. Let M be a self-generator multiplication R-module and $S = End_R(M)$. Then M_R is an fs-module if and only if ${}_SM$ is an fs-module

Proof. It follows from Theorem 3.4(2).

□

4. **Author Contributions**

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