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ON THE EXISTENCE OF SUBSPACE-DISKCYCLIC C_0-SEMIGROUPS AND SOME CRITERIA

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Abstract. In this paper, we prove the existance of subspace-diskcyclic C_0-semigroups on any infinite-dimensional separable Banach space. We state that diskcyclic C_0-semigroups are subspace-diskcyclic. Also, we establish some criteria for subspace-diskcyclic C_0-semigroups. Most of these criteria are based on non-empty relatively open sets and some of them are based on dense sets.

Keywords: Subspace-diskcyclicity, Diskcyclicity, C_0-semigroups.
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1. Introduction

Let X be an infinite-dimensional separable Banach space. Let T be a bounded linear operator on X or briefly an operator on X. We denote the set of all bounded linear operators on X by $B(X)$. The orbit of $x \in X$ under T is defined as

$$orb(T, x) = \{x, Tx, T^2x, ...\}.$$

According to orbits properties, there are different categories of operators. For example if $orb(T, x)$ is dense in X for some $x \in X$, then T is called hypercyclic and if for some $x \in X$, $\mathbb{C}orb(T, x) = \{\lambda T^n x : \lambda \in \mathbb{C}, n \in \mathbb{N}_0\}$ is dense in X, then T is called supercyclic [11].

We can construct hypercyclic operators only on infinite-dimensional Banach spaces [10]. Supercyclic operators can appear on Banach spaces with $\dim X \in \{1, 2, \infty\}$ [11]. These types of operators were extensively investigated. For more results, one can see [8] and [10].

A concept between hypercyclicity and supercyclicity is diskcyclicity. This concept was first introduced by Zeana in [15]. An operator $T \in B(X)$ is called diskcyclic if

$$\mathbb{D}orb(T, x) = \{\lambda T^n x : \lambda \in \mathbb{D}, n \in \mathbb{N}_0\}$$

is dense in X [3]. The set $\mathbb{D}$ denotes the closed unit disk, that is, $\mathbb{D} = \{\alpha \in \mathbb{C} : |\alpha| \leq 1\}$. In this case, x is called a diskcyclic vector for T. There are some equivalent criteria for diskcyclicity of an operator [5]. A good review on this topic can be seen in [3].

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In [2] Bamerni and Kilicman defined the concept of subspace-diskcyclic operators. Let M be a closed non-trivial subspace of X. An operator $T \in B(X)$ is said to be subspace-diskcyclic with respect to M or M-diskcyclic if there is an $x \in X$ such that $\mathbb{D}orb(T,x) \cap M$ is dense in M. In fact, they considered the density of $\mathbb{D}orb(T,x)$ in a subspace instead of the whole space. They showed in [2] that there are subspace-diskcyclic operators that are not diskcyclic. They also gave some sufficient conditions for an operator to be subspace-diskcyclic.

C_0-semigroups are exciting structures too. A family $(T_t)_{t\geq 0}$ of operators on a Banach space X is called a C_0-semigroup, if $T_0 = I$ and for all $s, t \geq 0$ and for all $x \in X$,

$$T_{t+s} = T_t T_s \quad \text{and} \quad lim_{s\to t} T_s x = T_t x.$$

Hypercyclicity for C_0-semigroups of operators introduced by Desh, Schappacher and Webb in [9]. A C_0-semigroup $(T_t)_{t\geq 0}$ on a Banach space X is called a hypercyclic C_0-semigroup if for some $x \in X$,

$$\overline{orb((T_t)_{t\geq 0}, x)} = \overline{\{T_t x : t \geq 0\}} = X.$$

Hypercyclicity in C_0-semigroups can be considered as the discrete case instead of the continuous case. Hypercyclic C_0-semigroups exist only on infinite-dimensional spaces [11]. In fact, any infinite-dimensional separable Banach spaces support hypercyclic C_0-semigroups [6, Theorem 3.1]. One can also see [7].

The concept of subspace-hypercyclicity for C_0-semigroups is defined in [14]. Assume $(T_t)_{t\geq 0}$ is a C_0-semigroup on a Banach space X. Presume M is a closed non-trivial subspace of X. If for some $x \in X$,

$$\overline{orb((T_t)_{t\geq 0}, x) \cap M} = M,$$

then we say $(T_t)_{t\geq 0}$ is an M-hypercyclic C_0-semigroup and x is called an M-hypercyclic vector for it. One can also see some criteria for subspace-hypercyclicity of C_0-semigroups in [13].

Like to the concept of subspace-hypercyclicity for C_0-semigroups, the concept of subspace-diskcyclicity for C_0-semigroups has also attracted the attention of mathematicians. The authors in final section of [14], defined subspace-diskcyclic C_0-semigroups as follows.

Definition 1.1. Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on X. Then $(T_t)_{t\geq 0}$ is subspace-diskcyclic C_0-semigroup with respect to M or an M-diskcyclic C_0-semigroup if there is $x \in X$ such that

$$\overline{\mathbb{D}orb((T_t)_{t\geq 0}, x) \cap M} = \overline{\{\lambda T_t x; \lambda \in \mathbb{D}, t \geq 0\}} = M.$$

By definition, it is not hard to see that a subspace-hypercyclic C_0-semigroup is subspace-diskcyclic. Also, any diskcyclic C_0-semigroup is subspace-diskcyclic since it is sufficient to consider $M := X$ [14]. The authors also showed that there are subspace-diskcyclic C_0-semigroup that are not diskcyclic [14, Example 3]. They also proved the following lemma.

Lemma 1.2. *([14]) Assume $(T_t)_{t\geq 0}$ is a C_0-semigroup on X and assume M is a closed non-trivial subspace of X. If one of the following conditions is satisfied, then $(T_t)_{t\geq 0}$ is M-diskcyclic:*

(i) *For any non-empty relatively open sets $U, V \subseteq M$, there are $s > 0$ and $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$ such that $\lambda T_s(U) \cap V$ is non-empty.*
(ii) *For any non-empty relatively open sets $U, V \subseteq M$, there are $s > 0$ and $\lambda \in \mathbb{C}$ with $|\lambda| \geq 1$ such that $\lambda T_s^{-1}(U) \cap V$ is non-empty and relatively-open.*
(iii) *For any non-empty relatively open sets $U, V \subseteq M$, there are $s > 0$ and $\lambda \in \mathbb{C}$ with $|\lambda| \geq 1$ such that $\lambda T_s^{-1}(U) \cap V \neq \phi$ and $T_s(M) \subseteq M$.*

The authors in [1] investigated subspace-diskcyclicity for a sequence of operators. They stated some sufficient conditions that under which, a sequence of operators can be subspace-diskcyclic.

Now, this question arises that if diskcyclicity of a C_0-semigroup implies its subspace-diskcyclicity? Also, we want to know if finite-dimensional or infinite-dimensional Banach spaces support this type of C_0-semigroups or not. Moreover, we interested in discovering new criteria for subspace-diskcyclicity.

In this article, we study the subspace-diskcyclic C_0-semigroups and their properties in more detail. In this article X denotes an infinite-dimensional Banach space and M indicates a closed non-trivial subspace of X.

In Section 2, we prove that if a C_0-semigroup contains a subspace-diskcyclic operator, then the C_0-semigroup is subspace-diskcyclic. Also, we state that diskcyclic C_0-semigroups are subspace-diskcyclic. Moreover, a subspace-diskcyclic C_0-semigroups exists on any infinite-dimensional separable Banach space.

In Section 3, we establish some criteria for subspace-diskcyclic C_0-semigroups. Most of them are based on non-empty relatively open sets and some of them are based on dense sets.

2. Existence of subspace-diskcyclic C_0-semigroups

We start this section by showing the fact that by subspace-diskcyclicity of an operator of a C_0-semigroup we can conclude subspace-diskcyclicity of the C_0-semigroup.

Proposition 2.1. *Suppose $(T_t)_{t\geq 0}$ is a C_0-semigroup on X. If $(T_t)_{t\geq 0}$ contains a subspace-diskcyclic operator, then $(T_t)_{t\geq 0}$ is subspace-diskcyclic.*

Proof. Let $s > 0$ exist such that T_s is M-diskcyclic. Let x be an M-diskcyclic vector for T_s. So,

$$\overline{\mathbb{D}orb(T_s, x) \cap M} = M.$$

That means

$$\{\lambda T_s^n x : \lambda \in \mathbb{D}, n \in \mathbb{N}_0\} \cap M$$

is dense in M. But $T_s^n = T_{sn}$ and

$$\{\lambda T_{sn}x : \lambda \in \mathbb{D}, n \in \mathbb{N}_0\} \subseteq \{\lambda T_t x : \lambda \in \mathbb{D}, t \geq 0\}.$$

Hence, $(T_t)_{t\geq 0}$ is an M-diskcyclic C_0-semigroup.

□

Example 2.2. *Consider $(T_t)_{t\geq 0}$ is a C_0-semigroup on $\mathbb{C}$ that is defined with $T_t x = 3^t x$ for any $t \geq 0$. If $t = 1$, then $T_1 x = 3x$. We claim that T_1 is diskcyclic on $\mathbb{C}$ and 1 is a diskcyclic vector for it. For this, note that*

$$\mathbb{D}orb(T_1, 1) = \{\lambda 3^n : \lambda \in \mathbb{D}, n \in \mathbb{N}_0\}. \tag{1}$$

Let $z = a + ib \in \mathbb{C}$ be arbitrary. There is $m \in \mathbb{N}$ such that $\sqrt{a^2 + b^2} \leq 3^m$. We can write $z = 3^m(\frac{a}{3^m} + i\frac{b}{3^m})$. Then $z \in \mathbb{D}orb(T_1, 1)$ since we can write it of the form $3^m\alpha$, where $|\alpha| \leq 1$. Therefore, 1 is a diskcyclic vector for T_1.

Similarly, $1 \oplus \{0\}$ is a subspace-diskcyclic vector for $T_1 \oplus I$ on $\mathbb{C} \oplus \mathbb{C}$ with respect to $M := \mathbb{C} \oplus \{0\}$. Hence, $(T_t \oplus I)_{t\geq 0}$ is an M-diskcyclic C_0-semigroup by Proposition 2.1.

Example 2.2 shows that we can construct subspace-diskcyclic C_0-semigroups on finite-dimensional Banach spaces.

An operator with a dense range that commutes with operators of C_0-semigroups can lead us to a sufficient condition as follows.

Proposition 2.3. *Suppose $(T_t)_{t\geq 0}$ and $(S_t)_{t\geq 0}$ are C_0-semigroups on X. Consider there is $\Phi \in B(X)$ such that $\overline{\Phi(X)} = M$. If $(S_t)_{t\geq 0}$ is M-diskcyclic and $T_t \circ \Phi = \Phi \circ S_t$ for any $t \geq 0$, then $(T_t)_{t\geq 0}$ is M-diskcyclic.*

Proof. Let $U \subseteq M$ be a non-empty relatively open set. Let x be an M-diskcyclic vector for $(S_t)_{t\geq 0}$. Hence, there is $\alpha \in \mathbb{D}$ and $t_0 > 0$ such that $\alpha S_{t_0} x \in \Phi^{-1}(U)$. So, $\alpha\Phi(S_{t_0} x) \in U$.

Therefore, $\alpha T_{t_0}(\Phi x) \in U$. So, for any non-empty open set U in relative topology, there is $t_0 > 0$ and $\alpha \in \mathbb{D}$ such that $\alpha T_{t_0}(\Phi x) \in U$. That means Φx is an M-diskcyclic vector for $(T_t)_{t\geq 0}$.

□

To prove the next theorem, we need to recall a theorem from [4] as follows.

Theorem 2.4. *Let $A \subseteq X$ be a dense subset in X. Then there is a closed non-trivial subspace M of X such that $A \cap M$ is dense in M.*

As we mentioned in the introduction, there are subspace-diskcyclic C_0-semigroups that are not diskcyclic. But in the following theorem, we prove that any diskcyclic C_0-semigroup is subspace-diskcyclic.

Theorem 2.5. *Let $(T_t)_{t\geq 0}$ be a diskcyclic C_0-semigroup on X. Then $(T_t)_{t\geq 0}$ is subspace-diskcyclic with respect to a closed non-trivial subspace N of X.*

Proof. Since $(T_t)_{t\geq 0}$ is diskcyclic, there is $x \in X$ such that $\overline{\mathbb{D}orb((T_t)_{t\geq 0}, x)} = X$. By Theorem 2.4, there is a closed non-trivial subspace N of X such that $\overline{\mathbb{D}orb((T_t)_{t\geq 0}, x) \cap N} = N$. So $(T_t)_{t\geq 0}$ is an N-diskcyclic C_0-semigroup. □

As it mentioned in the introduction, the authors in [14, Example 3] constructed an example of a subspace-diskcyclic C_0-semigroup that is not diskcyclic. So, the converse of Theorem 2.5 is not true.

By [12, Proposition 1.4], on any infinite-dimensional Banach space we can find a diskcyclic C_0-semigroup. So we can state the following corollary.

Corollary 2.6. *Subspace-diskcyclic C_0-semigroups can be constructed on any Banach space with infinite-dimension.*

We can also conclude another useful corollary as follows.

Corollary 2.7. *Let $(T_t)_{t\geq 0}$ be a diskcyclic C_0-semigroup on X. If T_s is a diskcyclic operator for some $s > 0$, then $(T_t)_{t\geq 0}$ is subspace-diskcyclic.*

Proof. Suppose there is $s > 0$ such that T_s is diskcyclic. So there is $x \in X$ such that $\overline{\mathbb{D}orb(T_s, x)} = X$. By Theorem 2.4, there is a closed non-trivial subspace N of X such that $\overline{\mathbb{D}orb(T_s, x) \cap N} = N$. Hence, T_s is N-diskcyclic and by Proposition 2.1, $(T_t)_{t\geq 0}$ is N-diskcyclic. □

3. Some sufficient conditions for subspace-diskcyclicity of C_0-semigroups

By dense sets and special sequences we can state sufficient conditions for subspace-diskcyclicity as follows.

Theorem 3.1. *Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on X. Suppose Y and Z are dense subsets of M. Assume $(t_n)_{n=1}^{\infty}$ is an increasing sequence of positive real numbers such that:*

(i) *For any $z \in Z$, $T_{t_n} z \to 0$,*
(ii) *For any $y \in Y$, there is $(u_n) \subseteq M$ and $\beta_y \in \mathbb{C}$ with $|\beta_y| \leq 1$ such that $u_n \to 0$ and $T_{t_n} \beta_y u_n \to y$,*
(iii) *For any $n \in \mathbb{N}$, $T_{t_n}(M) \subseteq M$.*

Then $(T_t)_{t\geq 0}$ is M-diskcyclic.

Proof. Let $U, V \subseteq M$ be non-empty relatively open sets. The subsets Z and Y are dense in M by hypothesis. So there are $z \in V \cap Z$ and $y \in U \cap Y$. Hence, by condition(i),

$$T_{t_n} z \to 0, \tag{2}$$

and by condition(ii), there is $(u_n) \subseteq M$ and $\beta_y \in \mathbb{C}$ with $|\beta_y| \leq 1$ such that

$$u_n \to 0 \quad \text{and} \quad T_{t_n} \beta_y u_n \to y. \tag{3}$$

Consider $x_n := z + u_n$. Hence, $x_n \to z$. Also, when $n \to \infty$,

$$T_{t_n}\beta_y x_n = T_{t_n}\beta_y z + T_{t_n}\beta_y u_n = \beta_y T_{t_n} z + T_{t_n}\beta_y u_n \to y. \tag{4}$$

So for N large enough, $x_N \in V$ and $T_{t_N}\beta_y x_N \in U$. Therefore

$$\beta_y^{-1} T_{t_N}^{-1}(U) \cap V \neq \phi. \tag{5}$$

If we consider $\lambda := \beta_y^{-1}$, then it is concluded from (5) that

$$\lambda T_{t_N}^{-1}(U) \cap V \neq \phi. \tag{6}$$

This completes the proof.

□

By using dense subsets, we can state the following theorem.

Theorem 3.2. *Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on X. Suppose there is an increasing sequence $(t_n) \subseteq \mathbb{R}^+$ and a sequence $(\alpha_n) \subseteq \mathbb{C}$ with $|\alpha_n| \leq 1$ for any $n \in \mathbb{N}$. Let Y be a dense subset of M. Consider for all $y \in Y$, there is $X_y \subseteq M$ such that $\overline{X_y} = M$ and there is $S_{y,t_n} : X_y \to M$ such that:*

(i) *For any $z \in X_y$, $S_{t_n} z \to 0$,*
(ii) *For any $z \in X_y$, $\alpha_n T_{t_n} S_{t_n} z \to z$,*
(iii) *For any $z \in X_y$, $\alpha_n T_{t_n} z \to 0$,*
(iv) *For any $n \in \mathbb{N}$, $T_{t_n}(M) \subseteq M$.*

Then $(T_t)_{t\geq 0}$ is M-diskcyclic.

Proof. Let $U, V \subseteq M$ be non-empty relatively open sets. Suppose $y \in U \cap X_y$. Relevant to y, there exists a dense subset X_y of M.

By density of X_y, there is $z \in V \cap X_y$. Consider $x_n := S_{t_n} y$. Hence,

$$x_n \to 0, \quad z + x_n \to z, \quad \alpha_n T_{t_n} S_{t_n} y \to y, \quad \text{and} \quad \alpha_n T_{t_n} z \to 0. \tag{7}$$

Therefore,

$$\alpha_n T_{t_n}(z + x_n) \to y. \tag{8}$$

Hence, for sufficiently large N, $z + x_N \in \alpha_N^{-1} T_{t_N}^{-1}(U) \cap V$. It follows that $(T_t)_{t\geq 0}$ is M-diskcyclic.

□

Theorem 3.3. *Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on X. Suppose Y is a dense subset of M with this property that for any $x, y \in Y$, there are $(x_n) \subseteq M$, an increasing sequence $(t_n) \subseteq \mathbb{R}^+$ and $(\alpha_n) \subseteq \mathbb{C}$ with $|\alpha_n| \leq 1$ for all $n \in \mathbb{N}$ such that $T_{t_n}(M) \subseteq M$, $x_n \to x$ and $\alpha_n T_{t_n} x_n \to y$. Then $(T_t)_{t\geq 0}$ is subspace-diskcyclic with respect to M.*

Proof. Let $U, V \subseteq M$ be non-empty relatively open sets. By density of Y, there are $x \in V \cap M$ and $y \in U \cap M$. By hypothesis, $(x_n) \subseteq M$ and $(\alpha_n) \subseteq \mathbb{C}$ exist with $|\alpha_n| \leq 1$ for any n such that

$$x_n \to x \quad \text{and} \quad \alpha_n T_{t_n} x_n \to y.$$

Hence, there is n_0 such that $x_{n_0} \in V$ and $\alpha_{n_0} T_{t_{n_0}} x_{n_0} \in U$. Therefore, $x_{n_0} \in \alpha_{n_0}^{-1} T_{t_{n_0}}^{-1}(U) \cap V$.

□

Neighborhoods of zero are good instruments beside open sets to state some sufficient conditions as follows.

Theorem 3.4. *Assume $(T_t)_{t\geq 0}$ is a C_0-semigroup on X. If for any non-empty relatively open sets $U, V \subseteq M$ and any neighborhood W of zero in M, there is $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$ and there is $t > 0$ with $T_t(M) \subseteq M$ such that*

$$\alpha T_t(V) \cap W \neq \phi \quad \text{and} \quad \alpha T_t(W) \cap U \neq \phi.$$

Then $(T_t)_{t\geq 0}$ is M-diskcyclic.

Proof. Let $U, V \subseteq M$ be non-empty relatively open sets. There are relatively open sets $U_1, V_1 \subseteq M$ and a neighborhood W_1 of zero in M such that

$$U_1 + W_1 \subseteq U \quad \text{and} \quad V_1 + W_1 \subseteq V. \tag{9}$$

By hypothesis, there is $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$ and there is $t > 0$ such that

$$\alpha T_t(V_1) \cap W_1 \neq \phi \quad \text{and} \quad \alpha T_t(W_1) \cap U_1 \neq \phi. \tag{10}$$

There is $v_1 \in V_1$ such that $\alpha T_t v_1 \in W_1$. Also, there is $w_1 \in W_1$ such that $\alpha T_t w_1 \in U_1$. Therefore,

$$v_1 + w_1 \in V_1 + W_1 \subseteq V \quad \text{and} \quad \alpha T_t(v_1 + w_1) = \alpha T_t v_1 + \alpha T_t w_1 \in W_1 + U_1 \subseteq U.$$

So, $(T_t)_{t\geq 0}$ is M-diskcyclic.

□

By a partial change in conditions of Theorem 3.4, we gain the following theorem.

Theorem 3.5. *Assume $(T_t)_{t\geq 0}$ is a C_0-semigroup on X. If for any non-empty relatively open sets $U, V \subseteq M$ and any neighborhood W of zero in M, there is $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$ and there are $t > 0$ and $p > 0$ with $T_t(M) \subseteq M$ and $T_p(M) \subseteq M$ such that*

$$\alpha T_t(V) \cap W \neq \phi \quad \text{and} \quad \alpha T_{t+p}(W) \cap U \neq \phi.$$

Then $(T_t)_{t\geq 0}$ is M-diskcyclic.

Proof. Let $U, V \subseteq M$ be non-empty relatively open sets and let W be a neighborhood of zero in M. Consider $W' := W \cap T_p^{-1}(W)$. Then W' is a neighborhood of zero. Hence, by hypothesis, there is $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$ such that

$$\alpha T_t(V) \cap (W \cap T_p^{-1}(W)) \neq \phi \quad \text{and} \quad \alpha T_{t+p}(W \cap T_p^{-1}(W)) \cap U \neq \phi. \tag{11}$$

So

$$\alpha T_t(V) \cap W \neq \phi \quad \text{and} \quad \alpha T_{t+p}(T_p^{-1}(W)) \cap U \neq \phi. \tag{12}$$

Therefore,

$$\alpha T_t(V) \cap W \neq \phi \quad \text{and} \quad \alpha T_t(W) \cap U \neq \phi. \tag{13}$$

Now, by Theorem 3.4, $(T_t)_{t\geq 0}$ is M-diskcyclic.

□

Corollary 3.6. *Assume $(T_t)_{t\geq 0}$ is a C_0-semigroup on X. If for any non-empty relatively open sets $U, V \subseteq M$ and any neighborhood W of zero in M, there is $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$ and there is $t > 0$ with $T_t(M) \subseteq M$ and $T_1(M) \subseteq M$ such that*

$$\alpha T_t(V) \cap W \neq \phi \quad \textit{and} \quad \alpha T_{t+1}(W) \cap U \neq \phi.$$

then $(T_t)_{t\geq 0}$ is M-diskcyclic.

Proof. It is sufficient to consider $p := 1$ in Theorem 3.5.

□

4. Conclousion

Investigating properties such as subspace-hypercyclicity, subspace-supercyclicity and subspace-diskcyclicity for C_0-semigroups have attracted the attention and interest of mathematicians.

In this article, we took a closer look at the subspace-diskcyclic C_0-semigroups. We proved that all diskcyclic C_0-semigroups are subspace-diskcyclic but the converse is not true. We showed that subspace-diskcyclic C_0-semigroups exist on both infinite-dimensional and finite-dimensional Banach spaces. Also, we proved that while a C_0-semigroup contains a subspace-diskcyclic operator, then it is subspace-diskcyclic.

By the idea of the criteria that are stated in [3] and [12], we stated some criteria for subspace-diskcyclicity for C_0-semigroups that were based on non-empty relatively open sets, and some of them are based on dense sets. In the stated criteria in this paper for M-diskcyclicity we have the condition $T_t(M) \subseteq M$ for some $t > 0$. This question arises can we state some criteria for subspace-diskcyclicity without this condition?

5. Data Availability Statement

Not applicable.

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7. Conflict of interest

The author declares no conflict of interest.

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