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RELATIVE MODEL OF THE LOGICAL ENTROPY OF SUB-σ_Θ-ALGEBRAS

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Abstract. In the context of observers, any mathematical model according to the viewpoint of an observer Θ is called a relative model. The purpose of the present paper is to study the relative model of logical entropy. Given an observer Θ, we define the relative logical entropy and relative conditional logical entropy of a sub-σ_Θ-algebra having finitely many atoms on the relative probability $\Theta-$measure space and prove the ergodic properties of these measures. Finally, it is shown that the relative logical entropy is invariant under the relation of equivalence modulo zero.

Keywords: Observer, Relative logical entropy, Sub-σ_Θ-algebra, Invariant.
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1. Introduction

Entropy plays a fundamental role in information theory, physics, statistics, biology, sociology, computer sciences, chemistry, general systems theory and many other fields [1]. It measures the amount of uncertainty in random events [12]. The importance of entropy arises from its invariance under isomorphism. Therefore, two systems with different entropies cannot be isomorphic [4]. In [2], Ellerman defined and studied the concept of logical entropy.

Let (X, β) denotes a $\sigma-$finite measure space, i.e., a set equiped with a $\sigma-$algebra β of subsets of X. Further, let p denote a probability measure on (X, β). Then (X, β, p) is called a probability space. When solving some specific problems, instead of Shannon's entropy, it is more appropriate to use an approach based on the concept of logical entropy [7]. The logical entropy of a partition $\xi = \{A_1, ..., A_n\}$ of a probability space is defined by

$$H_l(\xi, p) = -\sum_{i=1}^{n} p(A_i)(1 - p(A_i)).$$

Rao introduced precisely this concept as a quadratic entropy [11]. In [2], historical aspects of the logical entropy formula $H_l(\xi, P)$ are discussed and the relationship between logical entropy and Shannon's entropy is examined. For the latest material on logical entropy one can see [5].

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The notion of one dimensional "observer" is very important in physics and has been applied in ergodic theory [9]. In the paper by Mohammadi [6], the observational modeling of Kolmogorov-Sinai entropy was defined and studied. One must pay attention to this point that an observer is a fuzzy set, but each fuzzy set is not an observer. In fact the relative probability spaces are generalization of the soft fuzzy probability spaces [9]. In [3], the logical entropy and conditional logical entropy of fuzzy $\sigma-$algebras on the F-probability measure spaces is defined. In this paper, we provide analogies of the results on fuzzy probability measure spaces for the case of the relative logical entropy.
To define the concept of the logical entropy on a relative probability $\Theta-$measure space, we use the notion of "observer" [10]. A modeling for an observer of a set X is a fuzzy set $\Theta : X \to [0, 1]$ [9]. First, the notion of the relative logical entropy of a sub-σ_Θ-algebra having finite atoms is defined. Then, the relative conditional logical entropy under the common refinement of sub-σ_Θ-algebras is defined. Finally, we prove the relative logical entropy of sub-σ_Θ-algebras on a relative probability $\Theta-$measure space is invariant under the relation of equivalence modulo zero.

2. Preliminaries

We provide some basic preliminaries in this section.

Definition 2.1. [9] A collection F_Θ of subsets of Θ is said to be a σ_Θ-algebra in Θ if F_Θ satisfies the following conditions ,

(i). $\Theta \in F_\Theta$,

(ii). if $\lambda \in F_\Theta$, then $\lambda' = \Theta - \lambda \in F_\Theta$. Where λ' is the complement of λ with respect to Θ,

(iii). if $\{\lambda_i\}_{i=1}^{\infty}$ is a sequence in F_Θ, then $\vee_{i=1}^{\infty}\lambda_i = \sup_i \lambda_i \in F_\Theta$,

(iv). $\dfrac{\Theta}{2}$ does not belong to F_Θ.

If $\lambda_1, \lambda_2 \subseteq \Theta$, then $\lambda_1 \vee \lambda_2$ and $\lambda_1 \wedge \lambda_2$ are subsets of Θ, we define

$$(\lambda_1 \vee \lambda_2)(x) = \sup\{\lambda_1(x), \lambda_2(x)\},$$

and

$$(\lambda_1 \wedge \lambda_2)(x) = \inf\{\lambda_1(x), \lambda_2(x)\},$$

where $x \in X$. Let Θ be an observer on X. Then we say $\lambda \subseteq \Theta$ if $\lambda(x) \leq \Theta(x)$ for all $x \in X$. The relation $\subseteq$ is a partial ordering on F_Θ. If P_1 and P_2 are σ_Θ-algebras on X, then $P_1 \vee P_2$ is the smallest σ_Θ-algebra that contains $P_1 \cup P_2$, denoted by $[P_1 \cup P_2]$.

Definition 2.2. A positive $\Theta-$measure m_Θ over F_Θ is a function $m_\Theta : F_\Theta \to I$ which is countably additive. This means that if λ_i is a disjoint countable collection of members of F_Θ, (i.e., $\lambda_i \subseteq \lambda_j' = \Theta - \lambda_j$ whenever $i \neq j$), then

$$m_\Theta(\vee_{i=1}^{\infty}\lambda_i) = \sum_{i=1}^{\infty} m_\Theta(\lambda_i).$$

The $\Theta-$measure m_Θ has the following properties [9],

(i). $m_\Theta(\chi_\emptyset) = 0$,
(ii). $m_\Theta(\lambda^{'} \vee \lambda) = m_\Theta(\Theta)$ and $m_\Theta(\lambda^{'}) = m_\Theta(\Theta) - m_\Theta(\lambda)$ for all $\lambda \in F_\Theta$,
(iii). $m_\Theta(\lambda \vee \mu) + m_\Theta(\lambda \wedge \mu) = m_\Theta(\lambda) + m_\Theta(\mu)$ for each $\lambda, \mu \in F_\Theta$,
(iv). m_Θ is a nondecreasing function, i.e., if $\lambda, \eta \in F_\Theta$ and $\lambda \subseteq \Theta$, then $m_\Theta(\lambda) \leq m_\Theta(\eta)$.

The triple (X, F_Θ, m_Θ) is called a $\Theta-$measure space and the elements of F_Θ are called relative measurable sets. The $\Theta-$measure space, (X, F_Θ, m_Θ), is called a relative probability $\Theta-$measure space if $m_\Theta(\Theta) = 1$ [6].

Definition 2.3. Let (X, F_Θ, m) be an $\Theta-$measure space and let P be a sub-σ_Θ-algebra of F_Θ. The elements μ and λ of F_Θ are called m_Θ-disjoint if $m_\Theta(\lambda \wedge \mu) = 0$ [6].

A $\Theta-$relation "=(mod m_Θ)" on F_Θ is defined as below

$$\lambda = \mu (\mathrm{mod}\, m_\Theta) \quad \text{if and only if} \quad m_\Theta(\lambda) = m_\Theta(\mu) = m_\Theta(\lambda \wedge \mu) \qquad \lambda, \mu \in F_\Theta.$$

$\Theta-$relation "=(mod m_Θ)" is an equivalence relation [4]. $\tilde{F_\Theta}$ denotes the set of all equivalence classes induced by this relation, and $\tilde{\mu}$ is the equivalence class determined by μ. For $\lambda, \mu \in F_\Theta$, $\lambda \wedge \mu = 0$ (mod m_Θ) if and only if λ and μ are m_Θ-disjoint. We shall identify $\tilde{\mu}$ with μ.

Definition 2.4. Let (X, F_Θ, m_Θ) be a $\Theta-$measure space and let P be a sub-σ_Θ-algebra of F_Θ. An element $\tilde{\lambda} \in \tilde{P}$ is called an atom of P if $m_\Theta(\lambda) > 0$ and for any $\tilde{\mu} \in \tilde{P}$

$$m_\Theta(\lambda \wedge \mu) = m_\Theta(\mu) \neq m_\Theta(\lambda) \Longrightarrow m_\Theta(\mu) = 0.$$

Denote by $\bar{P}$ the set of all atoms of P, and by $R_*(F_\Theta)$ the family of sub-σ_Θ-algebras of F_Θ which include finitely many atoms.
Assume that F_Θ is a σ_Θ-algebra and $P_1, P_2 \in R_*(F_\Theta)$, and $\{\lambda_i; i = 1, 2, ..., n\}$ and $\{\mu_j; j = 1, ..., m\}$ denote the atoms of P_1 and P_2, respectively, then the atoms of $P_1 \vee P_2$ are $\lambda_i \wedge \mu_j$ which $m_\Theta(\lambda_i \wedge \mu_j) > 0$ for each $1 \leq i \leq n$ and $1 \leq j \leq m$.

Definition 2.5. Let (X, F_Θ, m_Θ) be a relative probability $\Theta-$measure space and $P_1, P_2 \in R_*(F_\Theta)$. We say that P_2 is an m_Θ-refinement of P_1, denoted by $P_1 \leq_{m_\Theta} P_2$, if for each $\mu \in \bar{P_2}$ there exists $\lambda \in \bar{P_1}$ such that

$$m_\Theta(\lambda \wedge \mu) = m_\Theta(\mu).$$

Definition 2.6. Let (X, F_Θ, m_Θ) be a relative probability $\Theta-$measure space and $P_1, P_2 \in R_*(F_\Theta)$. We say that P_1 and P_2 are m_Θ-equivalent, denoted by $P_1 \approx_{m_\Theta} P_2$, if the following axioms are satisfied:

(i). If $\lambda \in \bar{P_1}$, then $m_\Theta(\lambda \wedge (\vee\{\mu; \mu \in \bar{P_2}\})) = m_\Theta(\lambda)$.
(ii). If $\mu \in \bar{P_2}$, then $m_\Theta(\mu \wedge (\vee\{\lambda; \lambda \in \bar{P_1}\})) = m_\Theta(\mu)$.

Definition 2.7. Let (X, F_Θ, m_Θ) be a relative probability $\Theta-$measure space and $P_1, P_2 \in R_*(F_\Theta)$. We say that P_1 and P_2 are relative equivalent modulo zero, denoted by $P_1 \stackrel{\circ}{=} P_2$, if $P_1 \leq_{m_\Theta} P_2$ and $P_2 \leq_{m_\Theta} P_1$.

Note that the relation of $\stackrel{\circ}{=}$ is an equivalence relation on $R_*(F_\Theta)$.

3. Relative model of the logical entropy of sub-σ_Θ-algebras

In this section, we define the notions of relative logical entropy and relative conditional logical entropy of sub-σ_Θ-algebras on a relative probability $\Theta-$measure space.

Definition 3.1. Let (X, F_Θ, m_Θ) be a relative probability $\Theta-$measure space, and let P be a sub σ_Θ-algebra of F_Θ, where $P \in R_*(F_\Theta)$. The relative entropy of P is defined as

$$H_l(P, m_\Theta) = \sum_{\mu \in \bar{P}} m_\Theta(\mu)(1 - m_\Theta(\mu)).$$

Let (X, β, p) be a classical probability measure space and $\Theta = \chi_X$. Then $F_\Theta = \{\chi_A : A \in \beta\}$ is a σ_Θ-algebra on X. Define $m_\Theta(\chi_A) = p(A)$, $A \in \beta$. Then (X, F_Θ, m_Θ) is a relative probability $\Theta-$ measure space. Let α be a finite sub-$\sigma-$algebra of β and $P = \{\chi_A : A \in \alpha\}$. So, $P \in R_*(F_\Theta)$ and the relative logical entropy of P is given by

$$\begin{aligned} H_l(P, m_\Theta) &= \sum_{A \in \bar{P}} m_\Theta(\chi_A)(1 - m_\Theta(\chi_A)) \\ &= \sum_{A \in \bar{P}} p(A)(1 - p(A)), \end{aligned}$$

which is the logic entropy of the finite classical measurable sub-σ-algebra α of the space (X, β, p). Thus, the relative model of logical entropy of a σ_Θ-algebra is a generalization of the logical entropy of a finite measurable σ-algebra.

Example 3.2. *Let $X = [0,1]$ and $\Theta : X \to [0,1]$ be defined by $\Theta(x) = x$. Suppose that $\lambda \subset \Theta$ is defined by $\lambda(x) = \frac{3}{4}x$. Moreover, let $F_\Theta = \{\Theta, \chi_\emptyset, \lambda, \lambda', \lambda \wedge \lambda', \lambda \vee \lambda'\}$. Then $m_\Theta : F_\Theta \to [0,1]$ defined by $m_\Theta(\Theta) = m_\Theta(\lambda \vee \lambda') = 1, m_\Theta(\chi_\emptyset) = m_\Theta(\lambda \wedge \lambda') = 0$ and $m_\Theta(\lambda) = m_\Theta(\lambda') = \frac{1}{2}$ is a relative probability $\Theta-$measure space. Suppose that $P = \tilde{F_\Theta}$. By a simple calculation, we get $\bar{P} = \{\lambda, \lambda'\}$. From Definition 3.1, we have $H_l(P, m_\Theta) = \frac{1}{2}$.*

Definition 3.3. Let (X, F_Θ, m_Θ) be a relative probability $\Theta-$measure space, and let $P_1, P_2 \in R_*(F_\Theta), \bar{P_1} = \{\lambda_i : i = 1, ..., s\}$ and $\bar{P_2} = \{\mu_j : j = 1, ..., t\}$. The relative conditional logical entropy $H_l(P_1 \mid P_2, m_\Theta)$ is defined as

$$H_l(P_1 \mid P_2, m_\Theta) = \sum_{i=1}^{s} \sum_{j=1}^{t} m_\Theta(\lambda_i \wedge \mu_j)(m_\Theta(\mu_j) - m_\Theta(\lambda_i \wedge \mu_j)).$$

Note that $H_l(P_1 \mid P_2, m_\Theta) \geq 0$.

Theorem 3.4. *Let (X, F_Θ, m_Θ) be a relative probability $\Theta-$measure space, and $P_1, P_2, P_3 \in R_*(F_\Theta)$ where $\bar{P}_1 = \{\lambda_i; 1 \leq i \leq s\}$, $\bar{P}_2 = \{\mu_j; 1 \leq j \leq t\}$ and $\bar{P}_3 = \{\nu_k; 1 \leq k \leq p\}$. If $P_1 \approx_{m_\Theta} P_2$ and $P_2 \approx_{m_\Theta} P_3$, then*

(i). $\sum_{i=1}^{s}\sum_{j=1}^{t}\sum_{k=1}^{p} m_\Theta(\lambda_i \wedge \mu_j \wedge \nu_k) m_\Theta(\nu_k) = \sum_{i=1}^{s}\sum_{k=1}^{p} m_\Theta(\lambda_i \wedge \nu_k)) m_\Theta(\nu_k)$,

(ii). $\sum_{i=1}^{s}\sum_{j=1}^{t} m_\Theta(\lambda_i \wedge \mu_j) m_\Theta(\mu_j) = \sum_{j=1}^{t}(m_\Theta(\mu_j))^2$.

Proof. (i). It has been proved in [4, Theorem 4.5] that $P_1 \vee P_3 \approx_{m_\Theta} P_2$. Hence, we have

$$\begin{aligned}
\sum_{j=1}^{t} m_\Theta(\lambda_i \wedge \mu_j \wedge \nu_k) &= m_\Theta\big(\vee_{j=1}^{s}(\lambda_i \wedge \mu_j \wedge \nu_k)\big) \\
&= m_\Theta\big((\lambda_i \wedge \nu_k) \wedge (\vee_{j=1}^{t} \mu_j)\big) \\
&= m_\Theta(\lambda_i \wedge \nu_k).
\end{aligned}$$

Now the result follows.

(ii). Since $P_1 \approx_{m_\Theta} P_2$, we have for each $1 \leq j \leq t$,

$$\begin{aligned}
\sum_{i=1}^{s} m_\Theta(\lambda_i \wedge \mu_j) &= m_\Theta\Big(\vee_{i=1}^{s}(\lambda_i \wedge \mu_j)\Big) \\
&= m_\Theta(\vee_{i=1}^{s}\Big(\lambda_i \wedge \mu_j)\Big) \\
&= m_\Theta(\mu_i).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{i=1}^{s}\sum_{j=1}^{t} m_\Theta(\lambda_i \wedge \mu_j) m_\Theta(\mu_j) &= \sum_{j=1}^{t} m_\Theta(\mu_j) \sum_{i=1}^{s} m_\Theta(\lambda_i \wedge \mu_j) \\
&= \sum_{j=1}^{t} (m_\Theta(\mu_i))^2.
\end{aligned}$$

□

In the following theorem, we study the relative conditional logical entropy under the common refinement of sub-σ_Θ-algebras that include finitely many atoms on a relative probability $\Theta-$measure space.

Theorem 3.5. *Let (X, F_Θ, m_Θ) be a relative probability $\Theta-$measure space, and $P_1, P_2, P_3 \in R_*(F_\Theta)$ where $\bar{P}_1 = \{\lambda_i; 1 \leq i \leq s\}$, $\bar{P}_2 = \{\mu_j; 1 \leq j \leq t\}$ and $\bar{P}_3 = \{\nu_k; 1 \leq k \leq p\}$. If $P_1 \approx_{m_\Theta} P_2$ and $P_2 \approx_{m_\Theta} P_3$, then*

$$H_l(P_1 \vee P_2 \mid P_3, m_\Theta) \leq H_l(P_1 \mid P_3, m_\Theta) + H_l(P_2 \mid P_1 \vee P_3, m_\Theta).$$

Proof. From Theorem 3.4, we have

$$H_l(P_1 \vee P_2 \mid P_3, m_\Theta) = \sum_{i=1}^{s}\sum_{j=1}^{t}\sum_{k=1}^{p} m_\Theta(\lambda_i \wedge \mu_j \wedge \nu_k)\big(m_\Theta(\nu_k) - m_\Theta(\lambda_i \wedge \mu_j \wedge \nu_k)\big)$$

$$= \sum_{i=1}^{s}\sum_{j=1}^{t}\sum_{k=1}^{p} m_\Theta(\lambda_i \wedge \mu_j \wedge \nu_k)m_\Theta(\nu_k) - \sum_{i=1}^{s}\sum_{j=1}^{t}\sum_{k=1}^{p} \big(m_\Theta(\lambda_i \wedge \mu_j \wedge \nu_k)\big)^2$$

$$= \sum_{i=1}^{s}\sum_{k=1}^{p} m_\Theta(\lambda_i \wedge \nu_k)m_\Theta(\nu_k) - \sum_{i=1}^{s}\sum_{k=1}^{p} \big(m_\Theta(\lambda_i \wedge \nu_k)\big)^2$$

$$+ \sum_{i=1}^{s}\sum_{k=1}^{p} \big(m_\Theta(\lambda_i \wedge \nu_k)\big)^2 - \sum_{i=1}^{s}\sum_{j=1}^{t}\sum_{k=1}^{p} \big(m_\Theta(\lambda_i \wedge \mu_j \wedge \nu_k)\big)^2.$$

On the other hand we have

$$\begin{aligned} H_l(P_1 \mid P_3, m_\Theta) &= \sum_{i=1}^{s}\sum_{k=1}^{p} m_\Theta(\lambda_i \wedge \nu_k)(m_\Theta(\nu_k) - m_\Theta(\lambda_i \wedge \nu_k)) \\ &= \sum_{i=1}^{s}\sum_{k=1}^{p} m_\Theta(\lambda_i \wedge \nu_k)m_\Theta(\nu_k) \\ &- \sum_{i=1}^{s}\sum_{k=1}^{p} \big(m_\Theta(\lambda_i \wedge \nu_k)\big)^2, \end{aligned}$$

and

$$\begin{aligned} H_l(P_2 \mid P_1 \vee P_3, m_\Theta) &= \sum_{i=1}^{s}\sum_{j=1}^{t}\sum_{k=1}^{p} m_\Theta(\lambda_i \wedge \mu_j \wedge \nu_k)\big(m_\Theta(\lambda_i \wedge \nu_k) - m_\Theta(\lambda_i \wedge \mu_j \wedge \nu_k)\big) \\ &= \sum_{i=1}^{s}\sum_{j=1}^{t}\sum_{k=1}^{p} m_\Theta(\lambda_i \wedge \mu_j \wedge \nu_k)m_\Theta(\lambda_i \wedge \nu_k) \\ &- \sum_{i=1}^{s}\sum_{j=1}^{t}\sum_{k=1}^{p} \big(m_\Theta(\lambda_i \wedge \mu_j \wedge \nu_k)\big)^2 \\ &= \sum_{i=1}^{s}\sum_{k=1}^{p} \big(m_\Theta(\lambda_i \wedge \nu_k)\big)^2 - \sum_{i=1}^{s}\sum_{j=1}^{t}\sum_{k=1}^{p} \big(m_\Theta(\lambda_i \wedge \mu_j \wedge \nu_k)\big)^2. \end{aligned}$$

Combining the above relations, we obtain the assertion. □

Theorem 3.6. *Let* (X, F_Θ, m_Θ) *be a relative probability* $\Theta-$*measure space, and* $P_1, P_2 \in R_*(F_\Theta)$ *where* $\bar{P}_1 = \{\lambda_i; 1 \le i \le s\}$ *and* $\bar{P}_2 = \{\mu_j; 1 \le j \le t\}$*. If* $P_1 \approx_{m_\Theta} P_2$*, then*

(i). $H_l(P_1 \mid \{\Theta, \chi_\emptyset\}, m_\Theta) = H_l(P_1, m_\Theta)$,
(ii). $H_l(P_1 \vee P_2, m_\Theta) = H_l(P_2, m_\Theta) + H_l(P_1 \mid P_2, m_\Theta)$,
(iii). $\max\{H_l(P_1, m_\Theta), H_l(P_2, m_\Theta)\} \le H_l(P_1 \vee P_2, m_\Theta)$.

Proof. (i). By the definition of m_Θ we have $m_\Theta(\lambda_i \wedge \Theta) = m_\Theta(\lambda_i)$ for each i. So

$$H_l(P_1 \mid \{\Theta, \chi_\emptyset\}, m_\Theta) = \sum_{i=1}^{s} m_\Theta(\lambda_i \wedge \Theta)(m_\Theta(\Theta) - m_\Theta(\lambda_i \wedge \Theta)) = H_l(P_1, m_\Theta).$$

(ii). From the fact that $P_1 \vee \{\Theta, \chi_\emptyset\} = P_1$ and according to Theorem 3.5 and Theorem 3.6 (i), we get

$$\begin{aligned} H_l(P_1 \vee P_2, m_\Theta) &= H_l(P_1 \vee P_2 \mid \{\Theta, \chi_\emptyset\}, m_\Theta) \\ &= H_l(P_1 \mid \{\Theta, \chi_\emptyset\}, m_\Theta) + H_l(P_2 \mid P_1 \vee \{\Theta, \chi_\emptyset\}, m_\Theta) \\ &= H_l(P_1, m_\Theta) + H_l(P_2 \mid P_1, m_\Theta). \end{aligned}$$

Considering $P_1 \vee P_2 = P_2 \vee P_1$, by changing the role of P_1 and P_2, we obtain

$$H_l(P_1 \vee P_2, m_\Theta) = H_l(P_2, m_\Theta) + H_l(P_1 \mid P_2, m_\Theta).$$

(iii). This follows from Theorem 3.6 (ii) and from the fact that logical entropy is nonnegative. □

Theorem 3.7. *Let (X, F_Θ, m_Θ) be a relative probability $\Theta-$measure space, and $P_1, P_2 \in R_*(F_\Theta)$ where $\bar{P}_1 = \{\lambda_i; 1 \leq i \leq s\}$ and $\bar{P}_2 = \{\mu_j; 1 \leq j \leq t\}$. If $P_1 \approx_{m_\Theta} P_2$, then*

$$H_l(P_1 \mid P_2, m_\Theta) \leq H_l(P_1, m_\Theta).$$

Proof. For each $i = 1, 2, ..., s$, we have

$$\begin{aligned} &\sum_{j=1}^{t} m_\Theta(\lambda_i \wedge \mu_j)\Big(m_\Theta(\mu_j) - m_\Theta(\lambda_i \wedge \mu_j)\Big) \\ \leq\ &\sum_{j=1}^{t} m_\Theta(\lambda_i \wedge \mu_j)\Big(\sum_{j=1}^{t}(m_\Theta(\mu_j) - m_\Theta(\lambda_i \wedge \mu_j)\Big) \\ =\ &m_\Theta(\lambda_i)\Big(\sum_{j=1}^{t}(m_\Theta(\mu_j) - m_\Theta(\lambda_i \wedge \mu_j)\Big) \\ \leq\ &m_\Theta(\lambda_i)(1 - m_\Theta(\lambda_i)). \end{aligned}$$

Therefore,

$$\begin{aligned} H_l(P_1 \mid P_2, m_\Theta) &= \sum_{i=1}^{s}\sum_{j=1}^{t} m_\Theta(\lambda_i \wedge \mu_j)\Big(m_\Theta(\mu_j) - m_\Theta(\lambda_i \wedge \mu_j)\Big) \\ &\leq \sum_{i=1}^{s} m_\Theta(\lambda_i)(1 - m_\Theta(\lambda_i)) = H_l(P_1, m_\Theta). \end{aligned}$$

□

In the following theorem, the subadditivity property of relative logical entropy of sub-σ_Θ-algebras on a relative probability $\Theta-$measure space is proved.

Theorem 3.8. *Let* (X, F_Θ, m_Θ) *be a relative probability* $\Theta-$*measure space, and* $P_1, P_2 \in R_*(F_\Theta)$ *where* $\bar{P}_1 = \{\lambda_i; 1 \leq i \leq s\}$ *and* $\bar{P}_2 = \{\mu_j; 1 \leq j \leq t\}$*. If* $P_1 \approx_{m_\Theta} P_2$*, then*

$$H_l(P_1 \vee P_2, m_\Theta) \leq H_l(P_1, m_\Theta) + H_l(P_2, m_\Theta).$$

Proof. Using Theorems 3.6 and 3.7, we have

$$\begin{aligned} H_l(P_1 \vee P_2, m_\Theta) &= H_l(P_2, m_\Theta) + H_l(P_1 \mid P_2, m_\Theta) \\ &\leq H_l(P_1, m_\Theta) + H_l(P_2, m_\Theta). \end{aligned}$$

□

Theorem 3.9. *Let* (X, F_Θ, m_Θ) *be a relative probability* $\Theta-$*measure space, and* $P_1, P_2 \in R_*(F_\Theta)$ *where* $\bar{P}_1 = \{\lambda_i; 1 \leq i \leq s\}$ *and* $\bar{P}_2 = \{\mu_j; 1 \leq j \leq t\}$*. If* $P_1 \approx_{m_\Theta} P_2$*, then*

$$H_l(P_1 \mid P_2, m_\Theta) = 0 \Longleftrightarrow P_1 \leq_{m_\Theta} P_2.$$

Proof. Let $P_1 \leq_{m_\Theta} P_2$, then for any $\mu_j \in P_2$, there exists $\lambda_i \in P_1$ such that $m_\Theta(\lambda_i \wedge \mu_j) = m_\Theta(\mu_j)$. This means that $m_\Theta(\lambda_i \wedge \mu_j) - m_\Theta(\mu_j) = 0$. Hence, $H_l(P_1 \mid P_2, m_\Theta) = 0$. Conversely, let $H_l(P_1 \mid P_2, m_\Theta) = 0$. Since $m_\Theta(\lambda_i \wedge \mu_j) \geq 0$ and $m_\Theta(\lambda_i \wedge \mu_j) \leq m_\Theta(\mu_j)$, we get for each i, j, $m_\Theta(\lambda_i \wedge \mu_j) = m_\Theta(\mu_j)$ or $m_\Theta(\lambda_i \wedge \mu_j) = 0$ for each i and j. Hence we get $P_1 \leq_{m_\Theta} P_2$. □

The next theorem shows that the relative logical entropy of sub-σ_Θ-algebras is invariant under the relation of equivalence modulo zero.

Theorem 3.10. *Let* (X, F_Θ, m_Θ) *be a relative probability* $\Theta-$*measure space, and* $P_1, P_2 \in R_*(F_\Theta)$ *where* $\bar{P}_1 = \{\lambda_i; 1 \leq i \leq s\}$ *and* $\bar{P}_2 = \{\mu_j; 1 \leq j \leq t\}$*. If* $P_1 \approx_{m_\Theta} P_2$ *and* $P_1 \stackrel{\circ}{=} P_2$*, then*

$$H_l(P_1, m_\Theta) = H_l(P_2, m_\Theta).$$

Proof. Considering $P_1 \leq_{m_\Theta} P_2$ and using Theorem 3.6 (ii), we have $H_l(P_1 \mid P_2, m_\Theta) = H_l(P_1 \vee P_2, m_\Theta) - H_l(P_2, m_\Theta) = 0$. Therefore $H_l(P_1 \vee P_2, m_\Theta) = H_l(P_2, m_\Theta)$. Similarly, $P_2 \leq_{m_\Theta} P_1$, implies $H_l(P_1 \vee P_2, m_\Theta) = H_l(P_1, m_\Theta)$. Since $P_1 \vee P_2 = P_2 \vee P_1$, we get $H_l(P_1, m_\Theta) = H_l(P_2, m_\Theta)$. □

4. Conclusion

In this paper, we presented the notions of relative logical entropy and relative conditional logical entropy for a sub-σ_Θ-algebra with finite atoms . We proved some ergodic properties of the suggested measures. We also studied the relative conditional logical entropy under the common refinement of sub-σ_Θ-algebras and proved the subadditivity property of relative logical entropy of sub-σ_Θ-algebras on a probability $\Theta-$measure space. By Theorem 3.10, this nonnegative quantity is invariant under the relation of equivalence modulo zero.

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