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MERGING OF UNITS BASED ON INVERSE DATA ENVELOPMENT ANALYSIS

S. Ghobadi ✉ and Kh. Soleimani-Chamkhorami

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Abstract. Inverse data envelopment analysis (InvDEA) is a remarkable and popular management tool. This paper deals with one application of this tool. In fact, the problem of the merging of units is investigated in the presence of negative data. The problem of merging units refers to the fact that a set of units create a new unit based on synergy to improve their performance. We use multiple objective programming for this purpose and suggest new models based on predetermined conditions for new units. The proposed models estimate inputs and outputs simultaneously. Importance advantages of the proposed models are: i) We can follow multiple goals in the problem of merging units because multiple objective programming is applied. ii) Models can simultaneously estimate the inputs and outputs of the combined unit. iii) Unlike the existing methods in the InvDEA-based merging literature, the negative data do not need to be transferred to positive data. Finally, a numerical example is used to explain and validate the model proposed in this paper.

Keywords: Inverse Data Envelopment Analysis (InvDEA), Merging, Efficiency, Multiple-objective programming (MOP), Negative data.

2020 MSC: 90C08, 90C90, 90B50, 90C29, 91B06.

1. Introduction

Evaluating and analyzing the performance of decision-making units has always been important for decision-makers because knowing the performance of units under the manager is an effective factor in appropriate decisions to guide the unit. A proper and efficient technique for evaluating and analyzing performance and providing a model for decision-making units (DMUs) is data envelopment analysis (DEA), which is one of the most essential and practical branches of operations research. DEA is a non-parametric method based on mathematical programming that is widely used to evaluate the performance of a set of decision-making units in production technology with multiple inputs and outputs. In DEA a parameter named "efficiency size" is utilized to evaluate the efficiency of units. The performance index of a unit is a function of different elements such as the number of units, the number of resources

✉ ghobadi@iaukhsh.ac.ir, ORCID: 0000-0002-1884-0767

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and products of units, the kind of production technology, and the employed model. This technique has been studied and utilized in many theoretical and applied publications such as evaluation of the units' performance [3,7,8], ranking units [21,22], and identification of unit congestion [26].

In traditional DEA, it is assumed that all inputs and outputs have non-negative values. Nevertheless, some of the inputs or outputs of the DMUs may be negative [6,23]. For example, financial statements and growth rates could have positive or negative values. In these cases, the traditional DEA models may not be helpful. There are different approaches to handling this challenge, including the Range Directional Measures (RDM)-model provided by Portela et. al. [23]. This model is employed in the current study which can work without transferring negative data. Moreover, this model has two essential features: The translation invariant and the units invariant. These advantages are our main reason for using the RDM-model.

On the other hand, in some scientific works, researchers have studied and evaluated some parameters that affect the size of efficiency, such as the amount of input or output of a unit so that the size of efficiency is maintained or improved to a certain extent. These issues are studied under the heading of inverse data envelopment analysis (InvDEA) in the performance analysis literature. In fact, the traditional DEA technique does not allow characterizing the levels of a unit's production indices for a specific efficiency target, despite the utility of DEA as an analytical tool for evaluating various alternatives. InvDEA considers efficiency as a pre-defined parameter that can be an efficiency level pre-determined as a strategic goal and aims to estimate the quantities of inputs and outputs that are needed to achieve a pre-specified efficiency level, in contrast to traditional DEA, where the goal is to estimate the efficiency index of a specific unit. After introducing inverse DEA by Zhang and Cui [32], this idea has been used in many theoretical and applied publications such as resource allocation [12], sensitivity analysis [17,18]. Also, the InvDEA concept has been employed by Gattoufi et. al. [9] to propose an appropriate approach to solving the problem of merging units. The problem of merging units refers to the fact that a set of units create a new unit based on synergy to improve their performance. Moreover, the same idea has been utilized by Amin and Al-Muharrami [1] to solve the problem of merging units in the presence of negative data. They have used the approach of transferring negative data to positive data to deal with negative data. But it is possible that in some cases the data transfer does not lead to correct results [6]. To solve this challenge, the current paper proposes the Range Directional Measures (RDM) approach [23] to deal with negative data in the problem of merging units. This model estimates the efficiency score for each of unit, similar to radial methods in DEA, while negative data are used without any transformations.

The main contribution of the present study is to prepare a scientific and practicable framework based on the RDM-model for the simultaneous calculation of input and output levels of the merged unit in the presence of negative data. In other words, a new method based on incorporating the RDM-model into InvDEA is proposed to simultaneously estimate the inputs and outputs in the problem of the merging of the units. In the proposed method, due to the use of multi-objective planning tools, the decision-maker can pursue multiple goals in the problem of unit composition. The proposed models have less computational complexity because the number of variables is greatly reduced. Also, the proposed approach can simultaneously estimate the input and output levels of the new unit. At the same time, contrary to the proposed method in the reference [1], there is no need to transfer the negative data between the data.

This paper is organized as follows. A literature review of InvDEA and DEA in the presence of negative data is done in Section 2. Some conventional models in DEA in the presence of negative data are reviewed in section 3. Section 4 deals with the issue of merging decision-making units based on InvDEA and models for answering the raised questions in this field are proposed. A numerical example is presented to illustrate how to use the proposed models in section 5. A brief conclusion is presented in the final section.

2. Literature review

2.1. **A literature review on InvDEA.** Zhang and Cui [32] first developed an evaluation system that led to the emergence of InvDEA. Then, Wei et. al., [29] sought to answer the following questions: "If the efficiency index remains unchanged, but the inputs (outputs) increase, how much should the outputs (inputs) of DMU under evaluation increase?" Jahanshahloo et. al., [19] proposed models to improve the efficiency of inefficient units to answer the questions raised in InvDEA. Hadi-Vencheh et. al. [18], while modifying the sufficient conditions for estimating the inputs in the models proposed in the reference [29], expressed the necessary conditions to answer the question posed in the literature of InvDEA. After introducing InvDEA, this technique was used in various theoretical and practical fields such as preserving (improve) efficiency values [11], setting revenue target [4], resource allocation [12], sensitivity analysis [17,18], temporary interdependence of data [12,15], restructuring of units [2], and random data [16].

Some researchers based on InvDEA, examined the merging of decision-making units to improve the size of performance [9, 30]. These studies, in various DEA frameworks such as dynamic DEA [31], DEA-R [27], and inaccurate DEA [13], have proposed InvDEA-based models to provide the desired level of outputs and inputs for the merged unit with the aim of achieving an efficient and predetermined goal. Moreover, according to the semi-oriented radial measurement (SORM) approach [6], an InvDEA-based model is provided to calculate of the input/output levels in the problem of merging units in the presence of negative data by Amin and Al-Muharrami [1].

2.2. **A literature review on DEA with negative data.** There are different approaches to dealing with the negative data in the literature. Data transformations have been utilized to convert negative data to positive ones [20, 25].As another approach, the absolute values of negative inputs and outputs have been considered by Scheel [24] as outputs and inputs, respectively. Portela et. al., [23] proposed a model for measuring the efficiency of a set of DMUs in the presence of negative data, called RDM (Range Directional Measures). This method could offer efficiency scores for each DMUs, similar to radial methods in DEA, while negative data are used without transformations. A semi-oriented radial measure (SORM) has been developed by Emrouznejad et. al., [6] for the efficiency measurement of a set of DMUs with both negative inputs and outputs. In this paper, by considering the RDM model proposed by Portela et. al., [23] as the basic DEA model, a novel inverse DEA model is presented for adjusting the merging DMUs targets. It is worth noting that the SORM approach is based on converting negative data to positive, which sometimes may not lead to acceptable results. The limitations of the basic DEA models could be the reason for this. To handle this challenge, the Range Directional Measures (RDM)-model provided by Portela et. al., [23] employed in the current study, can work without transferring negative data. Moreover, this model has two essential features: The translation invariant and the units invariant. These advantages are our main reason for using the RDM-model.

As previously mentioned, in traditional DEA, all inputs and outputs are assumed to have non-negative values. Nevertheless, some of the inputs or outputs of the DMUs may be negative such as growth rates. Some researchers have modified the existing traditional models in DEA in such a way that these models can be used in the presence of negative data in the different frameworks such as evaluation of the units' performance [6], ranking of units [28], and InvDEA [10]. Also, some practical applications of InvDEA in the presence of negative data have been studied and developed, such as merging of units [1] and restructuring of units [14]. The current study is also devoted to this topic.

3. DEA with negative data

In the current section, an LP model reviewed for efficiency measurement with negative data. To do this, suppose a set of n units, $\{DMU_j : j \in J = \{1, 2, \ldots, n\}\}$, is under evaluation such that DMU_j produces multiple outputs $y_{rj}(r = 1, 2, \ldots, s)$ to consume multiple inputs $x_{ij}(i = 1, 2, \ldots, m)$. Unlike the traditional DEA, suppose the inputs and outputs of $DMU_j(j \in J)$ can be negative values. Portela et. al. [23] presented an LP model to estimating the efficiency score of DMU_o as follows.

$$
\begin{aligned}
\theta_o = \quad & 1 - \max\ \theta \\
& s.t.\ \sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{io} - \theta F_{io},\ i = 1, 2, \ldots, m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro} + \theta F_{ro},\ r = 1, 2, \ldots, s, \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& \theta \geq 0, \lambda_j \geq 0,\ j = 1, 2, \ldots, n,
\end{aligned} \tag{1}
$$

where

$$F_{io} = x_{io} - \min\{x_{ij} \mid \forall j \in J\},\ i = 1, 2, \ldots, m,$$

$$F_{ro} = \max\{y_{rj} \mid \forall j \in J\} - y_{ro},\ r = 1, 2, \ldots, s.$$

In Model (1), $(\theta,\ \lambda) \in \mathbb{R} \times \mathbb{R}^n$ is the variable vector. (F_{io}, F_{ro}) is the domain of possible improvement of DMU_o. $\theta_o = 1 - \theta^*$ is the relative efficiency of DMU_o and can be easily shown that $\theta_o \leq 1$.

4. Merging of units using the InvDEA concept

A merger is an agreement that two or more existing units into one new unit. There are several types of mergers and also several reasons why units complete mergers. Mergers and acquisitions (M&A) are commonly made to expand a unit's reach, expand into new segments, or gain market share. Generally, the motives of mergers are to enhance the competitiveness of a new merged entity in the form of synergies, growth, etc. Also, a merger can be motivated by a desire to acquire certain assets that cannot be obtained using other methods. In M&A transactions, it is quite common that some units arrange mergers to gain access to assets that are unique or to assets that usually take a long time to develop internally. Unlike the traditional DEA, where the goal is to estimate the efficiency index of a specific unit, InvDEA considers efficiency as a pre-defined parameter that can be an efficiency level pre-determined as a strategic goal and aims to estimate the quantities of inputs and outputs that are needed to attain a pre-specified efficiency level. In the current section, we provide a new InvDEA model to handle the problem of merging units with negative data. To attain this goal, let us to assume that there is a set of selected units $(\{DMU_j :\ j \in \Pi \subset J\})$ to create synergy through merging and generate a new unit (DMU_M) to reach the pre-determined efficiency goal $(\bar{\theta}_M)$.

After merging units, the following model is presented to measure of the relative efficiency of DMU_M.

(2)
$$\begin{aligned}
\theta_M = \quad & 1 - \max\ \theta \\
& s.t. \sum_{j\in\Lambda} \lambda_j x_{ij} + \lambda_M x_{iM} \le x_{iM} - \theta F_{iM},\ i = 1,2,\dots,m, \\
& \sum_{j\in\Lambda} \lambda_j y_{rj} + \lambda_M y_{rM} \ge y_{rM} + \theta F_{rM},\ r = 1,2,\dots,s, \\
& \sum_{j\in\Lambda} \lambda_j + \lambda_M = 1, \\
& \theta \ge 0, \lambda_j \ge 0;\ \forall j \in \Lambda, \lambda_M \ge 0,
\end{aligned}$$

where

$$F_{iM} = x_{iM} - \min\{x_{ij} \mid j \in \Lambda \cup \{M\}\},\ i = 1,2,\dots,m,$$
$$F_{rM} = \max\{y_{rj} \mid j \in \Lambda \cup \{M\}\} - y_{rM},\ r = 1,2,\dots,s.$$

In the above model, $\Lambda = J - \Pi$ is the set of units that have not play a role in this merging process. Also, $(\theta,\ \lambda_j; j \in \Lambda, \lambda_M)$ is the variable vector. If $\theta_M = \bar{\theta}_M$, then it is said that the relative efficiency of DMU_M is $\bar{\theta}_M$. We propose the following mathematical programming to compute of the inputs and outputs of the new merged unit (DMU_M).

(3)

$$\begin{aligned}
& \min\ (\alpha_{ij}^M;\ \forall j \in \Pi,\ i = 1,2,\dots,m), \\
& \max\ (\beta_{rj}^M;\ \forall j \in \Pi,\ r = 1,2,\dots,s), \\
& s.t. \sum_{j\in\Lambda} \lambda_j x_{ij} + \lambda_M \sum_{j\in\Pi} \alpha_{ij}^M \le \sum_{j\in\Pi} \alpha_{ij}^M - \bar{\theta} F_{iM},\ i = 1,2,\dots,m,\ (3.1) \\
& \sum_{j\in\Lambda} \lambda_j y_{rj} + \lambda_M \sum_{j\in\Pi} \beta_{rj}^M \ge \sum_{j\in\Pi} \beta_{rj}^M + \bar{\theta} F_{rM},\ r = 1,2,\dots,s,\ (3.2) \\
& \sum_{j\in\Lambda} \lambda_j + \lambda_M = 1,\ (3.3) \\
& \alpha_{ij}^M \le x_{ij},\ i = 1,2,\dots,m,\ \forall j \in \Pi,\ (3.4) \\
& \beta_{rj}^M \ge y_{rj},\ r = 1,2,\dots,s,\ \forall j \in \Pi,\ (3.5) \\
& \sum_{j\in\Pi} \alpha_{ij}^M \ge x_i^{Ideal},\ i = 1,2,\dots,m,\ (3.6) \\
& \sum_{j\in\Pi} \beta_{rj}^M \le y_r^{Ideal},\ r = 1,2,\dots,s,\ (3.7) \\
& \lambda_M \ge 0,\ \lambda_j \ge 0\ \forall j \in \Lambda,\ (3.8)
\end{aligned}$$

where

$$x_i^{Ideal} = \min\{x_{ij} \mid j \in \Lambda \cup \{M\}\},\ i = 1,2,\dots,m,$$
$$y_r^{Ideal} = \max\{y_{rj} \mid j \in \Lambda \cup \{M\}\},\ r = 1,2,\dots,s.$$

In Model (3), $(\lambda_j, \lambda_M, \alpha_{ij}^M, \beta_{rj}^M : \forall j \in \Pi, \forall i, r)$ is the variables vector. Also, $\bar{\theta}_M = 1 - \bar{\theta}$ is the expected relative efficiency for DMU_M.

Equation (3.4) ensures that the amount of the inputs received by DMU_M does not exceed the amount of inputs available to each of the units involved in the merging process. Also, Equation (3.5) guarantees that the amount of outputs produced by DMU_M is not less than the amount of outputs produced for each of the units involved in the integration process. Notice that Model (3) is multiple-objective non-linear programming (MONLP) model. According to the weight-sum method [5], Model (3) can be converted to the following mathematical programming model:

$$
\begin{aligned}
\text{(4)} \qquad \min \quad & \sum_{i=1}^{m} \sum_{j \in \Pi} w_{ij}^M \alpha_{ij}^M - \sum_{r=1}^{s} \sum_{j \in \Pi} w_{rj}^M \beta_{rj}^M, \\
& s.t. \; The \; constraints \; of \; Model \; (3),
\end{aligned}
$$

where $\sum_{i=1}^{m} \sum_{j \in \Pi} w_{ij}^M + \sum_{r=1}^{s} \sum_{j \in \Pi} w_{rj}^M = 1$ and the weights can be proposed by experts. These weights in the above model allows managers to incorporate their preference in targets setting of a merging for producing (saving) specific outputs (inputs) as much as possible. This means that the smaller weight for an output (inputs) implies the less priority for producing (saving) it in the new unit.

The most common combinations happen between entities in the business environment for better performance. Then, we can assume that the units involved in the combining process are inefficient. According to the DEA basic concepts, if the production possibility set (PPS) is identical before and after the merging process, then the new generated unit can be presented by a convex combination of some entities not involved in the merging process. Accordingly, Model (4) is feasible, and also, in each optimal solution of Model (4), we obtain $\lambda_M = 0$. Then, the problem of (4) can be changed to the following mathematical programming problem. It is worth noting that the Model (5) is a linear model.

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{m}\sum_{j\in\Pi} w_{ij}^{M}\alpha_{ij}^{M} - \sum_{r=1}^{s}\sum_{j\in\Pi} w_{rj}^{M}\beta_{rj}^{M}, \\
& s.t. \ \sum_{j\in\Lambda}\lambda_j x_{ij} \leq \sum_{j\in\Pi}\alpha_{ij}^{M} - \bar{\theta}F_{iM},\ i=1,2,\dots,m, \\
& \sum_{j\in\Lambda}\lambda_j y_{rj} \geq \sum_{j\in\Pi}\beta_{rj}^{M} + \bar{\theta}F_{rM},\ r=1,2,\dots,s, \\
& \sum_{j\in\Lambda}\lambda_j = 1, \\
& \alpha_{ij}^{M} \leq x_{ij},\ i=1,2,\dots,m,\ \forall j\in\Pi, \\
& \beta_{rj}^{M} \geq y_{rj},\ r=1,2,\dots,s,\ \forall j\in\Pi, \\
& \sum_{j\in\Pi}\alpha_{ij}^{M} \geq x_i^{Ideal},\ i=1,2,\dots,m, \\
& \sum_{j\in\Pi}\beta_{rj}^{M} \leq y_r^{Ideal},\ r=1,2,\dots,s, \\
& \lambda_j \geq 0\ \forall j\in\Lambda.
\end{aligned} \tag{5}
$$

The next theorem proofs how the problem can be used to estimate of resources and products of DMU_M.

Theorem 4.1. *Suppose $\bar{\theta}_M = 1-\bar{\theta}$ is considered as the efficiency goal for the combined unit (DMU_M). Let DMU_M belong to the PPS of before the combining process. Also, suppose $\Psi = (\lambda_j^*, \alpha_{ij}^{M*}, \beta_{rj}^{M*}; \forall j\in\Lambda, \forall i, r)$ is an optimal solution of problem (5). If*

$$
\begin{aligned}
x_{iM} &= \sum_{j\in\Pi}\alpha_{ij}^{M*},\ i=1,2,\dots,m, \\
y_{rM} &= \sum_{j\in\Pi}\beta_{rj}^{M*},\ r=1,2,\dots,s,
\end{aligned} \tag{6}
$$

in which $x_{iM} \neq x_i^{Ideal}$ or $y_{rM} \neq y_r^{Ideal}$ for some of $i,\ r$; then $\theta_M = \bar{\theta}_M$.

Proof. According to feasibility of Ψ to problem (5), we have

$$
\sum_{j\in\Lambda}\lambda_j^* x_{ij} \leq \sum_{j\in\Pi}\alpha_{ij}^{M*} - \bar{\theta}_M F_{iM} = x_{iM} - \bar{\theta}F_{iM},\ i=1,2,\dots,m, \tag{7}
$$

$$
\sum_{j\in\Lambda}\lambda_j^* y_{rj} \geq \sum_{j\in\Pi}\beta_{rj}^{M*} + \bar{\theta}_M F_{rM} = y_{rM} + \bar{\theta}F_{rM},\ r=1,2,\dots,s, \tag{8}
$$

$$
\sum_{j\in\Lambda}\lambda_j^* = 1, \tag{9}
$$

(10) $$\alpha_{ij}^{M*} \leq x_{ij},\ i = 1, 2, \ldots, m,\ \forall j \in \Pi,$$

(11) $$\beta_{rj}^{M*} \geq y_{rj},\ r = 1, 2, \ldots, s,\ \forall j \in \Pi,$$

(12) $$x_{iM} = \sum_{j \in \Pi} \alpha_{ij}^{M*} \geq x_i^{Ideal},\ i = 1, 2, \ldots, m,$$

(13) $$y_{rM} = \sum_{j \in \Pi} \beta_{rj}^{M*} \leq y_r^{Ideal},\ r = 1, 2, \ldots, s,$$

(14) $$\lambda_j^* \geq 0,\ \forall j \in \Lambda.$$

According to relations (7)-(9) and (14), it is not difficult to see that $(\lambda_j^*; \forall j \in \Lambda,\ \lambda_M = 0,\ \bar{\theta}, x_{iM}, y_{rM})$ is a feasible answer to model (2), and so $\theta_M \leq 1 - \bar{\theta} = \bar{\theta}_M$.

With regard to relations (7), (8), and because $\bar{\theta} F_{iM} (\forall i)$ and $\bar{\theta} F_{rM} (\forall r)$ are negative values, therefore

(15) $$\sum_{j \in \Lambda} \lambda_j^* x_{ij} \leq x_{iM},\ i = 1, 2, \ldots, m,$$

(16) $$\sum_{j \in \Lambda} \lambda_j^* y_{rj} \geq y_{rM},\ r = 1, 2, \ldots, s.$$

Now, suppose $\Phi = (\tilde{\lambda}_j; \forall j \in \Lambda, \tilde{\lambda}_M,\ \tilde{\theta})$ is an optimal answer to problem (2). According to feasibility of Φ to model (2), and using relations (15) and (16), we obtain

(17) $$\sum_{j \in \Lambda} \bar{\lambda}_j x_{ij} \leq x_{iM} - \tilde{\theta} F_{iM},\ i = 1, 2, \ldots, m,$$

(18) $$\sum_{j \in \Lambda} \bar{\lambda}_j y_{rj} \geq y_{rM} + \tilde{\theta} F_{rM},\ r = 1, 2, \ldots, s,$$

where $\bar{\lambda}_j := \tilde{\lambda}_j + \tilde{\lambda}_M \lambda_j^*$ for each $j \in \Lambda$. Moreover,

(19) $$\sum_{j \in \Lambda} \bar{\lambda}_j = 1,\ \bar{\lambda}_j \geq 0,\ \forall j \in \Lambda.$$

Assume $\theta_M = 1 - \tilde{\theta} < \bar{\theta}_M = 1 - \bar{\theta}$ (contradiction assumption). Then, $\tilde{\theta} > \bar{\theta}$. According to Equations (17) and (18), we get

(20) $$\sum_{j \in \Lambda} \bar{\lambda}_j x_{ij} \leq x_{iM} - \tilde{\theta} F_{iM} < x_{iM} - \tilde{\theta} F_{iM},\ i = 1, 2, \ldots, m,$$

(21) $$\sum_{j \in \Lambda} \bar{\lambda}_j y_{rj} \geq y_{rM} + \bar{\theta} F_{rM} > y_{rM} + \tilde{\theta} F_{rM},\ r = 1, 2, \ldots, s.$$

With regard to conditions of the theorem, we have $x_{iM} \neq x_i^{Ideal}$ or $y_{rM} \neq y_r^{Ideal}$ for some $i,\ r$. Without losing generality, it is assumed that $x_{tM} \neq x_t^{Ideal}$.

Hence, $x_{tM} = \sum_{j\in\Pi} \alpha_{tj}^{M*} > x_t^{Ideal}$. Now, we define $\bar{\beta}_{rj}^{M} = \beta_{rj}^{M*}$ for all $j \in \Pi$, $r = 1, 2, \ldots, s$; and

$$\bar{\alpha}_{ij}^{M} = \begin{cases} \alpha_{ij}^{M*} - \kappa_j & if\ i = t,\ j \in \Pi, \\ \alpha_{ij}^{M*} & Otherwise, \end{cases}$$

in which $\sum_{j\in\Pi} \kappa_j = \kappa$ and

(22)

$$\kappa = \min \left\{ \frac{\sum_{j\in\Lambda} \bar{\lambda}_j x_{tj} - (1-\bar{\theta}) \sum_{j\in\Pi} \alpha_{tj}^{M*} - \bar{\theta} x_t^{Ideal}}{\bar{\theta} - 1},\ \sum_{j\in\Pi} \alpha_{tj}^{M*} - x_t^{Ideal} \right\}.$$

It is obvious that $\kappa > 0$. According to relation (22), we get

$$\sum_{j\in\Lambda} \bar{\lambda}_j x_{tj} \leq \sum_{j\in\Pi} \bar{\alpha}_{tj}^{M} - \bar{\theta} F_{tM}, \tag{23}$$

$$\sum_{j\in\Pi} \bar{\alpha}_{tj}^{M} = \sum_{j\in\Pi} \alpha_{tj}^{M*} - \kappa \geq x_t^{Ideal}. \tag{24}$$

With regard to relations (11), (13), (17), (18), (23), and (24), it is obvious that $(\bar{\lambda}_j, \bar{\alpha}_{ij}^{M}, \bar{\beta}_{rj}^{M})$ is a feasible answer to Model (5) such that, the amount of the goal function of problem (5) at this feasible answer is equal:

$$\sum_{i=1}^{n} \sum_{j\in\Pi} w_{ij}^{M} \bar{\alpha}_{ij}^{M} - \sum_{r=1}^{s} \sum_{j\in\Pi} w_{rj}^{M} \bar{\beta}_{rj}^{M} = \sum_{i=1\&i\neq t}^{n} \sum_{j\in\Pi} w_{ij}^{M} \bar{\alpha}_{ij}^{M} + \sum_{j\in\Pi} w_{tj}^{M} \bar{\alpha}_{ij}^{M}$$

$$- \sum_{r=1}^{s} \sum_{j\in\Pi} w_{rj}^{M} \bar{\beta}_{rj}^{M} = \sum_{i=1\&i\neq t}^{n} \sum_{j\in\Pi} w_{ij}^{M} \alpha_{ij}^{M*} + \sum_{j\in\Pi} w_{tj}^{M} (\alpha_{lj}^{M*} - \kappa_j)$$

$$- \sum_{r=1}^{s} \sum_{j\in\Pi} w_{rj}^{M} \beta_{rj}^{M*} < \sum_{i=1\&i\neq t}^{n} \sum_{j\in\Pi} w_{ij}^{M} \alpha_{ij}^{M*} + \sum_{j\in\Pi} w_{tj}^{M} \alpha_{tj}^{M*} - \sum_{r=1}^{s} \sum_{j\in\Pi} w_{rj}^{M} \beta_{rj}^{M*}$$

$$= \sum_{i=1}^{n} \sum_{j\in\Pi} w_{ij}^{M} \alpha_{ij}^{M*} - \sum_{r=1}^{s} \sum_{j\in\Pi} w_{rj}^{M} \beta_{rj}^{M*}.$$

Therefore, the contradiction assumption is false and so completes the proof. □

We conclude this section by discussing the case where Model (4), and thus Model (5), is not feasible. It is worth noting that if Model (4) is infeasible, than it means that the process of merging units, changed the pre-merging efficiency frontier. In fact, the merged unit falls outside the pre-merging frontier. In this case, there is an optimal solution such that $\lambda_M^* = 1$. Then, it can be assumed that in the Model (4), we have $\lambda_M \in \{0,\ 1\}$. This gives the result that the model (4) can be linearized as follows:

(25)

$$\begin{aligned}
\min \quad & \sum_{i=1}^{m}\sum_{j\in\Pi} w_{ij}^{M}\alpha_{ij}^{M} - \sum_{r=1}^{s}\sum_{j\in\Pi} w_{rj}^{M}\beta_{rj}^{M}, \\
& s.t.\ \sum_{j\in\Lambda}\lambda_j x_{ij} + \sum_{j\in\Pi}\overline{\alpha}_{ij}^{M} \le \sum_{j\in\Pi}\alpha_{ij}^{M} - \bar{\theta}F_{iM},\ i=1,2,\ldots,m, \\
& \sum_{j\in\Lambda}\lambda_j y_{rj} + \sum_{j\in\Pi}\overline{\beta}_{rj}^{M} \ge \sum_{j\in\Pi}\beta_{rj}^{M} + \bar{\theta}F_{rM},\ r=1,2,\ldots,s, \\
& \sum_{j\in\Lambda}\lambda_j + \lambda_M = 1, \\
& \alpha_{ij}^{M} \le x_{ij},\ i=1,2,\ldots,m,\ \forall j\in\Pi, \\
& \beta_{rj}^{M} \ge y_{rj},\ r=1,2,\ldots,s,\ \forall j\in\Pi, \\
& \sum_{j\in\Pi}\alpha_{ij}^{M} \ge x_i^{Ideal},\ i=1,2,\ldots,m, \\
& \sum_{j\in\Pi}\beta_{rj}^{M} \le y_r^{Ideal},\ r=1,2,\ldots,s, \\
& \overline{\alpha}_{ij} \le x_{ij}\alpha_{ij},\ i=1,2,\ldots,m,\forall j\in\Pi, \\
& \overline{\beta}_{rj} \ge y_{rj}\beta_{rj},\ r=1,2,\ldots,s,\ \forall j\in\Pi, \\
& \alpha_{ij} - (1-\lambda_M)x_{ij} \le \overline{\alpha}_{ij} \le \alpha_{ij},\ i=1,2,\ldots,m,\forall j\in\Pi, \\
& \beta_{rj} - (1-\lambda_M)y_{rj} \ge \overline{\beta}_{rj} \ge \beta_{rj},\ r=1,2,\ldots,s,\ \forall j\in\Pi, \\
& \lambda_M \in \{0,1\},\ \lambda_j \ge 0\ \forall j\in\Lambda,
\end{aligned}$$

where $\overline{\alpha}_{ij} = \lambda_M\alpha_{ij}$ and $\overline{\beta}_{rj} = \lambda_M\beta_{rj}$ for each $i=1,2,\ldots,m$, $r=1,2,\ldots,s$, and $j\in\Pi$.

5. An Illustrative Example

In the current section, the performance of the extended theory is demonstrated through a numerical example. The dataset is adapted from those provided by Emrouznejad et. al., [6] and presented in Table 1. We assess 10 units with a resource (x) and two products (y_1 and y_2). The efficiency score for all 10 units is obtained by Model (1) and the outcomes are provided in Table 1.

According to Table 1, DMU_7, DMU_8, and DMU_9 units are inefficient. Suppose these units, to improve their performance, combine and establish a new unit with a pre-determined efficiency level, $\bar{\theta}_M = 0.950$. Suppose these units are decided to merge and establish a new unit (DMU_M) with a pre-determined level of efficiency $\bar{\theta}_M = 0.950$ to improve their performance. To estimate the inputs and outputs of DMU_M, Model (5) corresponding to DMU_M is written as follows:

Table 1. Inputs, outputs, and the performance index according to Model (1).

DMUs	DMU_1	DMU_2	DMU_3	DMU_4	DMU_5	DMU_6	DMU_7	DMU_8	DMU_9	DMU_{10}
x	12	35	25	22	40	50	35	40	25	16
y_1	15	18	20	12	-10	-8	-18	-10	-7	26
y_2	11	6	13	20	25	27	6	22	19	8
θ^*	1.000	0.682	0.992	1.000	1.000	1.000	0.384	0.767	0.842	1.000

$$
\begin{aligned}
\text{(26)} \quad \min \quad & \sum_{j\in\Pi} w_{1j}^M \alpha_{1j}^M - \sum_{j\in\Pi} w_{1j}^M \beta_{1j}^M - \sum_{j\in\Pi} w_{2j}^q \beta_{2j}^M, \\
& s.t. \sum_{j\in\Lambda} \lambda_j x_{1j} \le \alpha_{17}^M + \alpha_{18}^M + \alpha_{19}^M - \bar{\theta} F_{1M}, \\
& \sum_{j\in\Lambda} \lambda_j y_{1j} \le \beta_{r7}^M + \beta_{r8}^M + \beta_{r9}^M + \bar{\theta} F_{r1}, \ r = 1, 2, \\
& \sum_{j\in\Lambda} \lambda_j = 1, \\
& \alpha_{17}^M \le 35, \ \alpha_{18}^M \le 40, \ \alpha_{19}^M \le 25, \\
& \beta_{17}^M \ge -18, \ \beta_{18}^M \ge -10, \ \beta_{19}^M \ge -7, \\
& \beta_{27}^M \ge 6, \ \beta_{28}^M \ge 22, \ \beta_{29}^M \ge 19, \\
& \alpha_{17}^M + \alpha_{18}^M + \alpha_{19}^M \ge 12 = x^{Ideal}, \\
& \beta_{17}^M + \beta_{18}^M + \beta_{19}^M \le 26 = y_1^{Ideal}, \\
& \beta_{27}^M + \beta_{28}^M + \beta_{29}^M \le 27 = y_2^{Ideal}, \\
& \alpha_{1j}^M \in \mathbb{R}, \ \forall j \in \Pi \\
& \beta_{rj}^M \in \mathbb{R}, \ \forall j \in \Pi, r = 1, 2, \\
& \lambda_j \ge 0, \ \forall j \in \Lambda,
\end{aligned}
$$

where $\Pi = \{7, 8, 9\}$, $\Lambda = \{1, 2, 3, 4, 5, 6, 10\}$, and $\bar{\theta} = 0.050$. Using different weights, we generated two optimal solution that the outcomes are provided in Table 2.

Table 2. Inputs and outputs inherited of DMU_M from DMU_7, DMU_8, and DMU_9 units.

Optimal solutions	α_{17}^{M*}	α_{18}^{M*}	α_{19}^{M*}	β_{17}^{M*}	β_{18}^{M*}	β_{19}^{M*}	β_{27}^{M*}	β_{28}^{M*}	β_{29}^{M*}
The first optimal solution	24.00	0.00	0.00	10.21	0.00	0.00	-11.87	0.00	0.00
The second optimal solution	35.00	0.00	10.78	-5.35	0.00	0.00	11.44	14.00	0.00

According to Table 1, two solutions are proposed to generate new unit (DMU_M). In other words, the resources and products of the new unit must be as Table 3.

Table 3. Inputs and outputs of DMU_M.

DMU	x_{1M}	y_{1M}	y_{2M}
DMU_M	24.00	10.21	-11.87
DMU_M	45.78	-5.35	25.44

According to Tables 2 and 3, it is clear that if the second optimal solution is selected to generate DMU_M, then: i) The first output of DMU_M is fully supplied by DMU_7 while other units have no contribution. Also, the contributions of DMU_7 and DMU_8 in the second output of DMU_M are approximately 45% and 55%, respectively, while DMU_9 has no contribution. ii) The contributions of DMU_7 and DMU_9 in the input of DMU_M are approximately 76% and 24%, respectively while DMU_9 has no contribution.

6. Conclusion

In the present paper, the problem of merging units with negative data is investigated. Sufficient conditions are proposed for estimating the resources and products of the combined unit with a pre-determined certain level of efficiency. For this purpose, a new InvDEA based model is proposed that can work well in the presence of negative data. In the proposed method, due to the use of multi-objective planning tools, it is possible for the decision maker to pursue multiple goals in the issue of unit composition, such as saving more inputs from a particular unit. As another advantage, this model can simultaneously estimate the inputs and outputs of the combined unit. Moreover, unlike other proposed methods [1], negative data does not need to be transferred to positive data.

The current paper proposed the RDM-model to deal with negative data in the unit merging problem. This model estimates inputs and outputs based on a radial approach. In some cases, it is impossible to have a proportional contraction in the inputs and a proportional expansion in the outputs simultaneously. The proposed method faces challenges in these cases. To handling these challenges, new InvDEA models should be developed based on non-radial approaches (such as reference [28]) in facing negative data. In fact, obtaining the required models in the InvDEA framework based on the non-radial approach [27,28] can be a suitable research path in the future. Moreover, extending the topic discussed in this paper in the frameworks of dynamic DEA and network DEA can also be worth studying.

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Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Saeid Ghobadi
Orcid number: 0000-0002-1884-0767
Department of Mathematics
Khomeinishahr Branch, Islamic Azad University
Isfahan, Iran
Email address: `Email: ghobadi@iaukhsh.ac.ir`

Khosro Soleimani-Chamkhorami
Orcid number: 0000-0002-2738-4708
Department of Mathematics
Najafabad Branch, Islamic Azad University
Najafabad, Iran
Email address: `Soleimani@iaun.ac.ir`