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MEROMORPHIC FUNCTIONS WITH MISSING COEFFICIENTS DEFINED BY q-DERIVATIVE

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Abstract. By considering a fixed point in the punctured unit disk and using the q–derivative , a new subfamily of meromorphic and univalent functions is defined. Also, the first and second order q–derivative of meromorphic functions are introduced. Coefficient bounds, extreme points, radii of starlikeness and convexity are obtained. Furthermore, the convexity and preserving under convolution with some restrictions on parameters are investigated.

Keywords: Meromophic functions, q–derivative , Convex set, Extreme points, Coefficient estimate.
2020 MSC: 30C45, 30C50.

1. Introduction

Let $\sum$ be the class of meromorphic functions of the type:

$$f(z) = \frac{1}{z} + \sum_{n=k}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the punctured unit disk $\Delta^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$.

Let w be a fixed point in Δ^* and $\sum_w^u$ consist of functions F which are univalent in Δ^* and are of the form:

$$F(z) = f(z^*) = \frac{1}{z^*} + \sum_{n=k}^{\infty} a_n (z^*)^n, \qquad \big(z^* = z - w,\big). \tag{2}$$

Definition 1.1. We say that a function $f(z)$ is meromorphic on a domain Δ^* if $f(z)$ is analytic on Δ^* except possibly singularities, each of which is a pole. Also, $f(z)$ is a meromorphically starlike function of order λ if and only if:

$$-\operatorname{Re}\left\{\frac{zf'}{f}\right\} > \lambda, \qquad (z \in \Delta^*).$$

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Similarly, $f(z)$ is a meromorphically convex function of order λ if and only if:

$$-\operatorname{Re}\left\{1+\frac{zf''}{f'}\right\}>\lambda.$$

Example 1.2. *The functions* $\frac{e^z}{z}$, $\frac{1}{z}$ *and* $-\frac{1}{z}$ *are meromorphic, meromorphically convex and meromorphically starlike functions, respectively.*

Gosper and Rahman [1], defined the q–derivative $(0<q<1)$ of a function f of the form (1) by:

$$D_q f(z)=\frac{f(qz)-f(z)}{(q-1)z}, \qquad (z\in\Delta^*). \tag{3}$$

This definition can be extended for functions $F\in\sum_w^u$ of the form (2) as follow:

$$D_q F(z)=D_q f(z^*)=\frac{f(qz^*)-f(z^*)}{(q-1)z^*}, \qquad (z^*=z-w). \tag{4}$$

From (4), we get:

$$D_q F(z)=\frac{1}{q(z-w)^2}+\sum_{n=k}^{\infty}[n]_q a_n (z-w)^{n-1}, \tag{5}$$

where

$$[n]_q:=\frac{1-q^n}{1-q}=1+q+\cdots+q^{n-1}. \tag{6}$$

If $q\to 1^-$, then $[n]_q\to n$, hence we have:

$$\lim_{q\to 1^-} D_q F(z)=F'(z).$$

A function $F(z)$ belonging to the class $\sum_w^u$ is in the class $\sum_w^u(\alpha,\beta,q)$ if it satisfies the inequality:

$$\left|\frac{\frac{q}{3}\left[(z-w)\frac{(D_qF(z))''}{(D_qF(z))'}+2\right]+\frac{q}{3}}{\beta q\left[(z-w)\frac{(D_qF(z))''}{(D_qF(z))'}+1\right]+\alpha q}\right| 1, \tag{7}$$

where $-1\leqslant 2\beta<\alpha<1$ and $0\leqslant\alpha\leqslant 1$.

2. Main Results

First, we obtain coefficient bounds for functions in $\sum_w^u(\alpha,\beta,q)$. Then we prove the convexity of this class.

Theorem 2.1. *$F(z)$ of the form (2) belongs to $\sum_w^u(\alpha,\beta,q)$ if and only if:*

$$\sum_{n=k}^{\infty} q(n-1)[n]_q\Big(n\big(\beta+\frac{1}{3}\big)+\big(\frac{1}{3}-\beta\big)\Big)a_n\leqslant 2(\alpha-2\beta). \tag{8}$$

Proof. Suppose (8) holds and

$$\mathcal{A} = \left|\frac{q}{3}(z-w)[D_qF(z)]'' + q[D_qF(z)]'\right| - \left|\beta q(z-w)[D_qF(z)]'' + (\alpha+\beta)q[D_qF(z)]'\right|.$$

Replacing $[D_qF(z)]'$ and $[D_qF(z)]''$ by using (5) for $0 < |z-w| = r < 1$, we have:

$$\begin{aligned}\mathcal{A} &= \left|\sum_{n=k}^{\infty} q\big(\frac{n+1}{3}\big)(n-1)[n]_q a_n (z-w)^{n-2}\right| \\ &\quad - \left|\frac{1}{(z-w)^3}(2\alpha-4\beta) + \sum_{n=k}^{\infty} q(n-1)[n]_q\big(\beta(n-1)+\alpha\big)a_n(z-w)^{n-2}\right| \\ &\leqslant \left|\sum_{n=k}^{\infty} q\big(\frac{n+1}{3}\big)(n-1)[n]_q a_n r^{n-2} - (2\alpha-4\beta)\frac{1}{r^3}\right. \\ &\quad \left.+ \sum_{n=k}^{\infty} q(n-1)[n]_q\big(\beta(n-1)+\alpha\big)a_n r^{n-2}\right|.\end{aligned}$$

Since this inequality holds for all r $(0 < r < 1)$, making $r \to 1$, we obtain:

$$\mathcal{A} \leqslant \sum_{n=k}^{\infty} q(n-1)[n]_q\Big(n\big(\beta+\frac{1}{3}\big) + \big(\frac{1}{3}-\beta\big)\Big)a_n - 2(\alpha-2\beta).$$

By (8), we have $\mathcal{A} \leqslant 0$, so we get the required result.

Conversely, let

$$\left|\frac{\sum_{n=k}^{\infty} q\big(\frac{n+1}{3}\big)(n-1)[n]_q a_n(z-w)^{n-2}}{\frac{2(\alpha-2\beta)}{(z-w)^3} + \sum_{n=k}^{\infty} q(n-1)[n]_q\big(\beta(n-1)+\alpha\big)a_n(z-w)^{n-2}}\right| \leqslant 1,$$

hence

$$\left|\frac{\sum_{n=k}^{\infty} q\big(\frac{n+1}{3}\big)(n-1)[n]_q(z-w)^{n+1}}{2(\alpha-2\beta) - \sum_{n=k}^{\infty} q(n-1)[n]_q\big(\beta(n-1)+\alpha\big)a_n(z-w)^{n+1}}\right| \leqslant 1.$$

Since $\mathrm{Re}\{z\} \leqslant |z|$ for all z, then it follows from the above inequality that:

$$\mathrm{Re}\left\{\frac{\sum_{n=k}^{\infty} q\big(\frac{n+1}{3}\big)(n-1)[n]_q(z-w)^{n+1}}{2(\alpha-2\beta) - \sum_{n=k}^{\infty} q(n-1)[n]_q\big[\beta(n-1)+\alpha\big]a_n(z-w)^{n+1}}\right\} \leqslant 1.$$

By putting $z-w=r$ with $0<r<1$ in the above inequality, we get:

$$\text{(9)} \qquad \operatorname{Re}\left\{\frac{\sum\limits_{n=k}^{\infty} q\big(\frac{n+1}{2}\big)(n-1)[n]_q r^{n+1}}{2(\alpha-2\beta)-\sum\limits_{n=k}^{\infty} q(n-1)[n]_q\big[\beta(n-1)+\alpha\big]a_n r^{n+1}}\right\} \leqslant 1.$$

Upon clearing the denominator in (9) and letting $r \to 1$, we have:

$$\sum_{n=k}^{\infty} q\Big(\frac{n+1}{3}\Big)(n-1)[n]_q \leqslant 2(\alpha-2\beta)-\sum_{n=k}^{\infty} q(n-1)[n]_q\big[\beta(n-1)+\alpha\big]a_n.$$

So

$$\sum_{n=k}^{\infty} q(n-1)[n]_q\Big[n\big(\beta+\frac{1}{3}\big)+\big(\frac{1}{3}-\beta\big)\Big]a_n \leqslant 2(\alpha-2\beta).$$

This completes the proof. □

Corollary 2.2. *If $F(z) \in \sum_w^u(\alpha,\beta,q)$, then the coefficients of $F(z)$ satisfies in the following inequality:*

$$a_n \leqslant \frac{2(\alpha-2\beta)}{q(n-1)[n]_q\Big[n\big(\beta+\frac{1}{3}\big)+\big(\frac{1}{3}-\beta\big)\Big]}, \qquad (n \geqslant k).$$

Remark 2.3. Let $f(z)=z(1-z)^{-1}$ and $g(z)=z(1+iz)^{-1}$ belong to the class of normalized univalent functions. Then $(f+g)'(\frac{1}{2}(1+i))=0$, so $(f+g)(z)$ is not in the same class. But for meromorphic functions we have the following theorem.

Theorem 2.4. *$\sum_w^u(\alpha,\beta,q)$ is a convex set.*

Proof. Let $F_j(z)$ defined by:

$$F_j(z)=\frac{1}{z-w}+\sum_{n=k}^{\infty} a_{n,j}(z-w)^n, \qquad (j=1,2,\ldots),$$

be in the class $\sum_w^u(\alpha,\beta,q)$. It is sufficient to prove that:

$$F(z)=\sum_{j=1}^{m}\delta_j F_j(z),$$

is also in $\sum_w^u(\alpha,\beta,q)$, where $\sum\limits_{j=1}^{m}\delta_j=1$.

Since $F_j(z)\in\sum_w^u(\alpha,\beta,q)$, by (8) we have:

(10)
$$\sum_{n=k}^{\infty} q(n-1)[n]_q\Big[n\big(\beta+\frac{1}{3}\big)+\big(\frac{1}{3}-\beta\big)\Big]a_{n,j} \leqslant 2(\alpha-2\beta), \qquad (j=1,2,\ldots,m),$$

also

$$\begin{aligned} F(z) &= \sum_{j=1}^{m} \delta_j F_j(z) \\ &= \sum_{j=1}^{m} \delta_j \Big[\frac{1}{z-w} + \sum_{n=k}^{\infty} a_{n,j} (z-w)^n \Big] \\ &= \frac{1}{z-w} \sum_{j=1} \delta_j + \sum_{n=k}^{\infty} \Big(\sum_{j=1}^{m} \delta_j a_{n,j} \Big) (z-w)^n. \end{aligned}$$

But

$$\begin{aligned} &\sum_{n=k}^{\infty} q(n-1)[n]_q \Big[n\big(\beta + \frac{1}{3}\big) + \big(\frac{1}{3} - \beta\big)\Big] \Big(\sum_{j=1}^{m} \delta_j a_{n,j} \Big) \\ &= \sum \delta_j \Big(\sum_{n=k}^{\infty} q(n-1)[n]_q \Big[n\big(\beta + \frac{1}{3}\big) + \big(\frac{1}{3} - \beta\big)\Big] a_{n,j} \Big) \end{aligned}$$

by (10), we have:

$$\begin{aligned} &\leqslant \sum_{j=1}^{m} \big[2(\alpha - 2\beta) \big] \\ &= 2(\alpha - 2\beta). \end{aligned}$$

Now the proof by applying Theorem 2.1 is complete. □

3. Extreme points and Radii properties

In the last section, we introduce extreme points of $\sum_w^u(\alpha, \beta, q)$ and obtain radii of starlikeness and convexity for the same class.

Theorem 3.1. *Let*

$$F_0(z) = \frac{1}{z-w}, \tag{11}$$

and for $n \geqslant k$*,*

$$F_n(z) = \frac{1}{z-w} + \frac{2(\alpha - 2\beta)(z-w)^n}{q(n-1)[n]_q \big[n\big(\beta + \frac{1}{3}\big) + \big(\frac{1}{3} - \beta\big)\big]}, \tag{12}$$

then $F(z) \in \sum_w^u(\alpha, \beta, q)$ *if and only if it can be expressed in the form:*

$$F(z) = \sum_{n=0}^{\infty} d_n F_n(z),$$

where $d_n \geqslant 0$*,* $d_i = 0$ *(*$i = 1, 2, \ldots, k-1$*) and* $\sum\limits_{n=0}^{\infty} d_n = 1$*.*

Proof. Let $F(z) = \sum\limits_{n=0}^{\infty} d_n F_n(z)$, so:

$$\begin{aligned}
F(z) &= d_0\Big(\frac{1}{z-w}\Big) \\
&+ \sum d_n \left[\frac{1}{z-w} + \frac{2(\alpha-2\beta)}{q(n-1)[n]_q\big[n\big(\beta+\frac{1}{3}\big)+\big(\frac{1}{3}-\beta\big)\big]}(z-w)^n\right] \\
&+ \sum_{n=k}^{\infty} d_n \left[\frac{1}{z-w} + \frac{2(\alpha-2\beta)}{q(n-1)[n]_q\big[n\big(\beta+\frac{1}{3}\big)+\big(\frac{1}{3}-\beta\big)\big]}(z-w)^n\right] \\
&= \frac{1}{z-w}\sum_{n=0}^{\infty} d_n + \sum_{n=k}^{\infty} \frac{2(\alpha-2\beta)}{q(n-1)[n]_q\big[n\big(\beta+\frac{1}{3}\big)+\big(\frac{1}{3}-\beta\big)\big]} d_n (z-w)^n \\
&= \frac{1}{z-w} + \sum_{n=k}^{\infty} \frac{2(\alpha-2\beta)}{q(n-1)[n]_q\big[n\big(\beta+\frac{1}{3}\big)+\big(\frac{1}{3}-\beta\big)\big]} d_n (z-w)^n.
\end{aligned}$$

Since

$$\begin{aligned}
&\sum_{n=k}^{\infty} \frac{2(\alpha-2\beta)}{q(n-1)[n]_q\big[n\big(\beta+\frac{1}{3}\big)+\big(\frac{1}{3}-\beta\big)\big]} \\
&\times d_n\Big(q(n-1)[n]_q\big[n\big(\beta+\frac{1}{3}\big)+\big(\frac{1}{3}-\beta\big)\big]\Big) \\
&= \sum_{n=k}^{\infty} 2(\alpha-2\beta) d_n \\
&= 2(\alpha-2\beta)\sum_{n=k}^{\infty} d_n \\
&\leqslant 2(\alpha-2\beta)\sum_{n=0}^{\infty} d_n = 2(\alpha-2\beta),
\end{aligned}$$

so by Theorem 2.1, $F(z) \in \sum_w^u(\alpha,\beta,q)$.

Conversely, suppose that $F(z) \in \sum_w^u(\alpha,\beta,q)$. Then by (8), we have:

$$a_n \leqslant \frac{2(\alpha-2\beta)}{q(n-1)[n]_q\big[n\big(\beta+\frac{1}{3}\big)+\big(\frac{1}{3}-\beta\big)\big]}. \tag{13}$$

By setting:

$$\begin{aligned}
d_n &= \frac{q(n-1)[n]_q\big[n\big(\beta+\frac{1}{3}\big)+\big(\frac{1}{3}-\beta\big)\big]}{2(\alpha-2\beta)} a_n, \qquad (n \geqslant 1), \\
d_i &= 0, \qquad (i = 1, 2, \ldots, k-1), \\
d_0 &= 1 - \sum_{n=2}^{\infty} d_n,
\end{aligned}$$

we obtain the required result. □

Theorem 3.2. *Let $F(z) \in \sum_w^u(\alpha, \beta, q)$. Then:*

[(i),noitemsep]$F(z)$ is memorphically univalent starlike of order λ for $0 \leqslant \lambda < 1$ in $|z-w| < R_1$, where:

$$R_1 = \left\{ \frac{q(n-1)[n]_q\big[n\big(\beta + \frac{1}{3}\big) + \big(\frac{1}{3} - \beta\big)\big](1-\lambda)}{2(\alpha - 2\beta)(n+2+\lambda)} \right\}^{\frac{1}{n+1}}.$$

$F(z)$ is memorphically univalent convex of order λ for $0 \leqslant \lambda < 1$ in $|z-w| < R_2$, where:

$$R_2 = \left\{ \frac{q(n-1)[n]_q\big[n\big(\beta + \frac{1}{3}\big) + \big(\frac{1}{3} - \beta\big)\big](1-\lambda)}{2n(\alpha - 2\beta)(n+2+\lambda)} \right\}^{\frac{1}{n+1}}.$$

(2) *Proof.*

[(i),noitemsep] It is enough to show that:

$$\left| \frac{(z-w)F'(z)}{F(z)} + 1 \right| \leqslant .$$

But

$$\left| \frac{(z-w)F'(z)}{F(z)} + 1 \right| = \left| \frac{-\frac{1}{z-w} + \sum\limits_{n=k}^{\infty} n a_n (z-w)^n + \frac{1}{z-w} + \sum\limits_{n=k}^{\infty} a_n (z-w)^n}{\frac{1}{z-w}\Big(1 + \sum\limits_{n=k}^{\infty} a_n (z-w)^{n+1}\Big)} \right|$$
$$\leqslant \frac{\sum\limits_{n=k}^{\infty} (n+1) a_n (z-w)^{n+1}}{1 - \sum\limits_{n=k}^{\infty} a_n (z-w)^{n+1}}$$
$$\leqslant 1 - \lambda,$$

or

$$\sum_{n=k}^{\infty} (n+1) a_n |z-w|^{n+1} \leqslant (1-\lambda) - (1-\lambda) \sum_{n=k}^{\infty} a_n |z-w|^{n+1},$$

or

$$\sum_{n=k}^{\infty} \frac{(n+2+\lambda)}{1-\lambda} a_n |z-w|^{n+1} \leqslant 1.$$

Now, by using (13), we have:

$$\sum_{n=k}^{\infty} \frac{n+2+\lambda}{1-\lambda} a_n |z-w|^{n+1}$$
$$\leqslant \sum_{n=k}^{\infty} \frac{2(\alpha - 2\beta)}{q(n-1)[n]_q\big[n\big(\beta + \frac{1}{3}\big) + \big(\frac{1}{3} - \beta\big)\big]} |z-w|^{n+1} \Big(\frac{n+2+\lambda}{1-\lambda} \Big) \leqslant 1.$$

So it is enough to suppose:

$$|z-w|^{n+1} \leqslant \frac{q(n-1)[n]_q\big[n\big(\beta+\frac{1}{3}\big)+\big(\frac{1}{3}-\beta\big)\big](1-\lambda)}{2(\alpha-2\beta)(n+2+\lambda)},$$

and so the proof of 1 is complete. Since "f is convex if and only if zf' is starlike",

$$\frac{(z-w)\big[(z-w)F'(z)\big]'}{(z-w)F'(z)} = \frac{(z-w)\big[F'(z)+(z-w)F''(z)\big]}{(z-w)F'(z)} = 1+\frac{(z-w)F''(z)}{F'(z)}.$$

□

In the last theorem, we show that $\sum_w^u(\alpha,\beta,q)$ is closed under convolution.

Theorem 3.3. *If*

$$F(z)=\frac{1}{z-w}+\sum_{n=k}^{\infty}a_n(z-w)^n,$$

$$G(z)=\frac{1}{z-w}+\sum_{n=k}^{\infty}b_n(z-w)^n,$$

are in the class $\sum_w^u(\alpha,\beta,q)$, *then the Hadamard product (or convolution) of* F *and* G *defined by:*

$$(F*G)(z)=\frac{1}{z-w}+\sum_{n=k}^{\infty}a_nb_n(z-w)^n, \tag{14}$$

is in the class $\sum_w^u(\alpha,\beta^*,q)$, *where:*

$$\beta^* \geqslant \frac{\frac{2}{3}(\alpha-2\beta)^2(n+1)-3\alpha q(n-1)[n]_q\big[n\big(\beta+\frac{1}{3}\big)+\big(\frac{1}{3}-\beta\big)\big]^2}{2(\alpha-2\beta)^2(1-n)-2(n-1)q[n]_q\big[n\big(\beta+\frac{1}{3}\big)+\big(\frac{1}{3}-\beta\big)\big]^2}. \tag{15}$$

(2) *Proof.* Since $F,G\in\sum_w^u(\alpha,\beta,q)$, so by (8), we have:

$$\sum_{n=k}^{\infty}\frac{q(n-1)[n]_q\big[n\big(\beta+\frac{1}{3}\big)+\big(\frac{1}{3}-\beta\big)\big]a_n}{2(\alpha-2\beta)} \leqslant 1, \tag{16}$$

$$\sum_{n=k}^{\infty}\frac{q(n-1)[n]_q\big[n\big(\beta+\frac{1}{3}\big)+\big(\frac{1}{3}-\beta\big)\big]b_n}{2(\alpha-2\beta)} \leqslant 1. \tag{17}$$

We must find the smallest β^* such that:

$$\sum_{n=k}^{\infty}\frac{q(n-1)[n]_q\big[n\big(\beta^*+\frac{1}{3}\big)+\big(\frac{1}{3}-\beta^*\big)\big]}{2(\alpha-2\beta^*)}a_nb_n \leqslant 1,$$

by using the Cauchy-Schwarz inequality, we have:

$$\sum_{n=k}^{\infty} \frac{q(n-1)[n]_q\big[n\big(\beta+\frac{1}{3}\big)+\big(\frac{1}{3}-\beta\big)\big]}{2(\alpha-2\beta)}\sqrt{a_n b_n} \leqslant 1. \tag{18}$$

Now, it is enough to show that:

$$\frac{q(n-1)[n]_q\big[n\big(\beta^*+\frac{1}{3}\big)+\big(\frac{1}{3}-\beta^*\big)\big]}{2(\alpha-2\beta^*)}a_n b_n \leqslant \frac{q(n-1)[n]_q\big[n\big(\beta+\frac{1}{3}\big)+\big(\frac{1}{3}-\beta\big)\big]}{2(\alpha-2\beta)}\sqrt{a_n b_n},$$

or equivalently:

$$\sqrt{a_n b_n} \leqslant \frac{(\alpha-2\beta^*)\big[n\big(\beta+\frac{1}{3}\big)+\big(\frac{1}{3}-\beta\big)\big]}{(\alpha-2\beta)\big[n\big(\beta^*+\frac{1}{3}\big)+\big(\frac{1}{3}-\beta^*\big)\big]}.$$

But from (17), we have:

$$\sqrt{a_n b_n} \leqslant \frac{2(\alpha-2\beta)}{q(n-1)[n]_q\big[n\big(\beta+\frac{1}{3}\big)+\big(\frac{1}{3}-\beta\big)\big]}.$$

So it is enough that

$$\frac{2(\alpha-2\beta)}{q(n-1)[n]_q\big[n\big(\beta+\frac{1}{3}\big)+\big(\frac{1}{3}-\beta\big)\big]} \leqslant \frac{(\alpha-2\beta^*)\big[n\big(\beta+\frac{1}{3}\big)+\big(\frac{1}{3}-\beta\big)\big]}{(\alpha-2\beta)\big[n\big(\beta^*+\frac{1}{3}\big)+\big(\frac{1}{3}-\beta^*\big)\big]},$$

or

$$\frac{2(\alpha-2\beta)^2}{q(n-1)[n]_q\big[n\big(\beta+\frac{1}{3}\big)+\big(\frac{1}{3}-\beta\big)\big]^2} \leqslant \frac{\alpha-2\beta^*}{n\big(\beta^*+\frac{1}{3}\big)+\big(\frac{1}{3}-\beta^*\big)}.$$

After a simple calculation, we get:

$$\beta^* \geqslant \frac{\frac{2}{3}(\alpha-2\beta)^2(n+1)-3\alpha q(n-1)[n]_q\big[n\big(\beta+\frac{1}{3}\big)+\big(\frac{1}{3}-\beta\big)\big]^2}{2(\alpha-2\beta)^2(1-n)-2(n-1)q[n]_q\big[n\big(\beta+\frac{1}{3}\big)+\big(\frac{1}{3}-\beta\big)\big]^2},$$

so the proof is complete. □

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The authors declare that they have no competing interests.

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