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# MONTE CARLO COMPARISON OF GOODNESS-OF-FIT TESTS FOR THE INVERSE GAUSSIAN DISTRIBUTION BASED ON EMPIRICAL DISTRIBUTION FUNCTION

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Abstract. The Inverse Gaussian (IG) distribution is widely used to model positively skewed data. In this article, we examine goodness of fit tests for the Inverse Gaussian distribution based on the empirical distribution function. In order to compute the test statistics, parameters of the Inverse Gaussian distribution are estimated by maximum likelihood estimators (MLEs), which are simple explicit estimators. Critical points and the actual sizes of the tests are obtained by Monte Carlo simulation. Through a simulation study, power values of the tests are compared with each other. Finally, an illustrative example is presented and analyzed.

*Keywords*: Empirical distribution function, Inverse Gaussian distribution, Maximum likelihood estimates, Goodness-of-fit test, Monte Carlo simulation, Test power.
*2020 MSC*: Primary 62G10, 62G30.

## 1. **Introduction**

The Inverse Gaussian (IG) distribution is an important statistical model for analyzing right skewed data with positive support. Its density function is

$$f(x;\mu,\lambda)=\left(\frac{\lambda}{2\pi x^3}\right)^{1/2}\exp\left\{-\frac{\lambda}{2\mu^2 x}(x-\mu)^2\right\},\qquad x>0,$$

where $\mu\in R^+$ and $\lambda\in R^+$ are parameters. The mean and variance of this distribution are $\mu$ and $\mu^3/\lambda$, respectively.

Various applications based on IG distribution assumption are widely addressed by the literature in different fields of science as electrical networks, cardiology, hydrology, meteorology, ecology, physiology, demography, employment service, and etc., (e.g., Folks and Chhikara [9,10]; Bardsley [4]; Seshadri [19]; Johnson et al. [14]; Barndorff-Nielsen [5]). Therefore, finding powerful goodness of fit tests for the IG distribution is an important issue. In this article, we investigate different goodness of fit tests for the IG distribution using a simulation study.

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Assuming that $X_1, ..., X_n$ is the sample from a distribution $F$, we wish to assess whether the unknown $F(x)$ can be satisfactorily approximated by a IG model $G(x)$. Goodness-of-fit (GOF) tests are designed to measure how well the observed sample data fits some proposed model. One class of GOF tests that can be used consists of tests based on the distance between the empirical and hypothesized distribution functions. Five of the known tests in this class are Cramer-von Mises ($W^2$), Kolmogorov-Smirnov ($D$), Kuiper ($V$), Watson ($U^2$), and Anderson-Darling ($A^2$). For more details about these tests, see D'Agostino and Stephens [7].

Many researchers have been interested in goodness of fit tests for different distributions and then different tests are developed in the literature. For example, see D'Agostino and Stephen [7] and Huber-Carol et al. [13].

Goodness of fit tests for the IG distribution are investigated by some authors including O'Reilly and Rueda [18], Gunes et al. [11], Henze and Klar [12], Al-Omari and Haq [2], Ofosuhene [17], Allison et al. [1].

Zhang [24] introduced three goodness of fit test statistics based on the empirical distribution function and applied them for testing normality and showed that the new tests have higher power than the competing tests. In the present paper, we will apply these test statistics to test the hypothesis of the IG distribution and compare the power of these tests with the other tests.

Recently, Torabi et al. [20] proposed a new test statistic based on the empirical distribution function and then constructed a test of fit for the normal distribution and show their test is powerful against some alternatives. Also, Torabi el al. [21] again used their test statistic and suggested a test for the exponential distribution. Here, we investigate the behavior of Torabi el al. [20]'s test for the IG distribution and propose some test statistics for test of fit for IG model.

The main contribution of the paper can express as follows. In this paper, we apply EDF-tests for the IG distribution. Moreover, the methods of Zhang [24] and Torabi el al. [20] are stated and based on these methods, we propose some goodness of fit tests for the IG distribution. Table of critical values and properties of the tests are presented. Through extensive simulation studies, we find the powerful tests for different choices of sample sizes and alternatives. We also investigate the behavior of the tests for the IG model with real data.

In Section 2, we consider goodness of fit test statistics based on the empirical distribution function and apply them for the IG distribution. In Section 3, the critical points and the actual sizes of the test are obtained by Monte Carlo simulations. Then power values of the tests are computed and then compared with each other. All simulations were carried out by using R 4.1.1 and with 100,000 replications. Section 4 contains applications of the tests in real examples. The following section contains a brief conclusion.

## 2. The Test Statistics

The GOF test checks whether our sample data is likely to be from a specific theoretical distribution. We have a set of data values, and an idea about how the data values are distributed. The test gives us a way to decide if the data values have a "good enough" fit to our idea, or if our idea is questionable. GOF tests are designed to measure how well the observed sample data fits some proposed model.

Given a random sample $X_1, \ldots, X_n$ from a continuous probability distribution $F$ with a density $f(x)$ over a non-negative support, the hypothesis of interest is

$$\mathrm{H}_0 : f(x) = f_0(x; \mu, \lambda) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} \exp\left\{-\frac{\lambda}{2\mu^2 x}(x-\mu)^2\right\}, \quad x > 0,\ for\ some\ (\mu, \lambda) \in \Theta,$$

where $\mu$ and $\lambda$ are unspecified and $\Theta = R^+ \times R^+$. The alternative to $H_0$ is

$$H_1 :\ f(x) \neq f_0(x; \mu, \lambda), \qquad for\ any\ (\mu, \lambda) \in \Theta\,.$$

Here, we consider the popular and common tests which are used in practice and statistical software. The test statistics of these tests are briefly described as follows. For more details about these tests, see D'Agostino and Stephens [7].

Let $X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)}$ are the order statistics based on the random sample $X_1, ..., X_n$.

1. The Cramer-von Mises statistic [22]:

$$W^2 = \frac{1}{12n} + \sum_{i=1}^{n} \left(\frac{2i-1}{2n} - F_0(X_{(i)}; \hat{\mu}, \hat{\lambda})\right)^2.$$

2. The Watson statistic [23]:

$$U^2 = W^2 - n\big(\bar{P} - 0.5\big)^2,$$

where $\bar{P}$ is the mean of $F_0(X_{(i)}; \hat{\mu}, \hat{\lambda}),\ i = 1, ..., n$.

3. The Kolmogorov-Smirnov statistic [15]:

$$D = \max(D^+, D^-),$$

where

$$D^+ = \max_{1 \leq i \leq n} \left\{\frac{i}{n} - F_0(X_{(i)}; \hat{\mu}, \hat{\lambda})\right\}; \quad D^- = \max_{1 \leq i \leq n} \left\{F_0(X_{(i)}; \hat{\mu}, \hat{\lambda}) - \frac{i-1}{n}\right\}.$$

4. The Kuiper statistic [16]:

$$V = D^+ + D^-.$$

5. The Anderson-Darling statistic [3]:

$$A^2 = -n - \frac{1}{n} \sum_{i=1}^{n} (2i-1) \left\{\log F_0(X_{(i)}; \hat{\mu}, \hat{\lambda}) + \log\left[1 - F_0(X_{(n-i+1)}; \hat{\mu}, \hat{\lambda})\right]\right\}.$$

In the above test statistics, $F_0(x)$ is the cumulative distribution function of the IG distribution and $(\hat{\mu}, \hat{\lambda})$ are the maximum likelihood estimates of the parameter $(\mu, \lambda)$.

Moreover, we consider the EDF-based tests proposed by Zhang [24]. Briefly, the approach of Zhang [24] for the IG distribution is described as follows. Let

$$H_t: \; F(t) = F_0(t) = \frac{1}{1 + \exp\{-(t-\mu)/\sigma\}},$$

and $\bar{H}_t: \; F(t) \neq F_0(t)$. According Zhang [24], testing $H$ vs $\bar{H}$ is equivalent to testing $H_t$ vs $\bar{H}_t$ for every $t \in (0, \infty)$ in the sense that

$$H = \bigcap_{t \in (0,\infty)} H_t \text{ and } \bar{H} = \bigcup_{t \in (0,\infty)} \bar{H}_t \; .$$

Now, define a binary random sample to test $H_t$ vs $\bar{H}_t$ for each $t$;

$$X_{it} = I\left(X_i \leq t\right) \qquad i = 1, 2, ..., n,$$

where $P\left(X_{it} = 1\right) = F(t)$ and $P\left(X_{it} = 0\right) = 1 - F(t)$.

Let $Z_t$ denotes a statistic based on $X_it$ for testing $H_t$ vs $\bar{H}_t$ where large values of $Z_t$ reject $H_t$. For testing $H$ vs $\bar{H}$, Zhang [24] proposed two test statistics given by

$$Z = \int Z_t dw(t) \text{ and } Z_{\max} = \sup_{t \in (0,\infty)} \left[Z_t w(t)\right],$$

where $w(t)$ is some weight function. Also, large values of these statistics reject $H$.

Zhang [24] for $Z_t$ considered Pearson's Chi squared statistic

$$X_t^2 = \frac{n[F_n(t) - F_0(t)]^2}{F_0(t)\left[1 - F_0(t)\right]},$$

and the likelihood ratio statistic

$$G_t^2 = 2n\left\{F_n(t) \log \frac{F_n(t)}{F_0(t)} + [1 - F_n(t)] \log \frac{1 - F_n(t)}{1 - F_0(t)}\right\},$$

where $F_n(t)$ is the empirical distribution function. If we set $Z_t = X_t^2$ with

$$w(t) = n^{-1} F_0(t)\left[1 - F_0(t)\right], \quad dw(t) = n^{-1} F_0(t)\left[1 - F_0(t)\right] dF_0(t),$$

and $w(t) = F_0(t)$, then the traditional Kolmogorov–Smirnov, Cramer–von Mises and Anderson–Darling statistics are obtained.

Moreover, if we consider $Z_t = G_t^2$ with $w(t) = 1$, $dw(t) = F_0(t)^{-1}[1 - F_0(t)]^{-1} dF_0(t)$ and $dw(t) = F_n(t)^{-1}[1 - F_n(t)]^{-1} dF_n(t)$, respectively, and further, $F_n(X_{(i)}) = \frac{i-0.5}{n}$ , then the test statistics proposed by Zhang [24] are obtained. These test statistics for the IG distribution are as

$$Z_A = -\sum_{i=1}^{n} \left( \frac{\log F_0(X_{(i)}; \hat{\theta})}{n - i + 0.5} + \frac{\log\left[1 - F_0(X_{(i)}; \hat{\theta})\right]}{i - 0.5} \right),$$

$$Z_C=\sum_{i=1}^{n}\left(\log\left\{\frac{F_0(X_{(i)};\hat{\theta})^{-1}-1}{(n-0.5)/(i-0.75)-1}\right\}\right)^2,$$

$$Z_K=\max_{1\le i\le n}\left((i-0.5)\log\left\{\frac{i-0.5}{nF_0(X_{(i)};\hat{\theta})}\right\}+(n-i+0.5)\log\left\{\frac{n-i+0.5}{n(1-F_0(X_{(i)};\hat{\theta}))}\right\}\right).$$

It is obvious that for large values of the above test statistics the null hypothesis $H_0$ will be rejected.

Given $X$ and $Y$ two absolutely continuous random variables with cdfs $F_0$ and $F$, respectively, Torabi et al. [20] defined the following discrepancy measure:

$$D(F_0,F)=\int_{-\infty}^{\infty}h(\frac{1+F_0(x)}{1+F(x)})dF(x)=E_F\left[h(\frac{1+F_0(x)}{1+F(x)})\right],$$

where $E_F[.]$ is the expectation under $F$ and $h:(0,\infty)\to R^+$ is a continuous function, decreasing on (0,1) and increasing on $(1,\infty)$ with an absolute minimum at $x=1$ such that $h(1)=0$. For this measure, $D(F,F_0)=0$ if and only if $F=F_0$ , almost everywhere.

Torabi et al. [20] proposed to use this measure as a criterion of goodness of fit of an iid sample $X_1,...,X_n$ with empirical distribution function $F_n$, to a given distribution $F_0$. It is clear that $D(F,F_0)=0$ can be estimated by

$$H_n=D(F_0,F_n)=\frac{1}{n}\sum_{i=1}^{n}h\left(\frac{1+F_0(X_{(i)})}{1+F_n(X_{(i)})}\right)=\frac{1}{n}\sum_{i=1}^{n}h\left(\frac{1+F_0(X_{(i)})}{1+i/n}\right),$$

and we can consider it as a test statistic. Here, we construct tests for the IG distribution based on $H_n$ as follows.

Let $X_{(1)}\le X_{(2)}\le ...\le X_{(n)}$ are the order statistics based on the random sample $X_1,...,X_n$. Applying the Torabi et al. [20] distance we have

$$H_n=\frac{1}{n}\sum_{i=1}^{n}h\left(\frac{1+F_0(X_{(i)})}{1+F_n(X_{(i)})}\right)=\frac{1}{n}\sum_{i=1}^{n}h\left(\frac{1+F_0(X_{(i)};\hat{\mu},\hat{\lambda})}{1+i/n}\right),$$

where $F_0(x)$ is the cumulative distribution function of the IG distribution. The test statistic $H_n$ is expected to take values close to zero when $H_0$ is true. Hence, the null hypothesis is rejected for large values of $H_n$. Here, we consider the following functions for $h$.

$$h_1(x)=x\log(x)-x+1,$$

$$h_2(x)=\left(\frac{x-1}{x+1}\right)^2.$$

These functions are suggested by Torabi et al. [20].

Note that $h_k:[0,\infty)\to R^+$ is a non-negative function with the absolute minimum at $x=1$, such that $h_k(1)=0$, $k=1,2$. Under $H_0$, we expect that $F_n(x)\approx F_0(x)$. Hence $h_k\left(\frac{1+F_0(X)}{1+F_n(X)}\right)\approx 0$, since $h_k(1)=0$. Thus the value of test statistic is expected to be near zero when $H_0$ is true. Therefore, it is justifiable to reject $H_0$ for large values of $H_n$. Finally, we can write the test statistic as follow.

$$H_n^{(k)} = \frac{1}{n}\sum_{i=1}^{n} h_k\left(\frac{1+F_0(X_{(i)};\hat{\mu},\hat{\lambda})}{1+i/n}\right),$$

where $k = 1, 2$.

**Corollary 2.1.** *From Proposition 2.3 of Torabi et al. [20], we have that for all $x \in R$,*

$$0 \le h_k\left(\frac{1+F_0(X)}{1+F_n(X)}\right) \le \max\left(h_k(1/2), h_k(2)\right) = \begin{cases} 0.38629 & k=1 \\ 0.11111 & k=2 \end{cases}$$

*and since $H_n^{(k)}$ is the mean of $h_k(.)$ over the transformed data, the support of statistics $H_n^{(k)}$, $k = 1, 2$, can be obtained as:*

$$\text{supp}(H_n^{(1)}) = [0, 0.38629], \qquad \text{supp}(H_n^{(2)}) = [0, 0.11111].$$

**Proposition 2.2.** *Let $F_1$ be an arbitrary continuous cdf in $H_1$. Then under the assumption that the observed sample have cdf $F_1$, the test based on $H_n$ is consistent.*

*Proof.* Based on Glivenko-Cantelli theorem, for $n$ large enough, we have that $F_n(x) \approx F_1(x)$, for all $x \in R$. Therefore,

$$\begin{aligned} H_n &= \tfrac{1}{n}\sum_{i=1}^{n} h\left(\tfrac{1+F_0(X_{(i)};\hat{\mu},\hat{\lambda})}{1+F_n(X_{(i)})}\right) = \tfrac{1}{n}\sum_{i=1}^{n} h\left(\tfrac{1+F_0(X_i;\hat{\mu},\hat{\lambda})}{1+F_n(X_i)}\right) \\ &\approx \tfrac{1}{n}\sum_{i=1}^{n} h\left(\tfrac{1+F_0(X_i;\hat{\mu},\hat{\lambda})}{1+F_1(X_i)}\right) \approx \tfrac{1}{n}\sum_{i=1}^{n} h\left(\tfrac{1+F_0(X_i;\hat{\mu},\hat{\lambda})}{1+F_1(X_i)}\right) \\ &\to E_{F_1}\left[h\left(\tfrac{1+F_0(X_i;\hat{\mu},\hat{\lambda})}{1+F_1(X_i)}\right)\right] = D(F_0, F_1), \ \ as\ n \to \infty. \end{aligned}$$

Note that the convergence holds by the law of large numbers and $D(F_0, F_1)$ is a divergence between $F_0$ and $F_1$. So, the test based on $H_n$ is consistent. □

## 3. Simulation Study

3.1. **Critical values and the actual sizes.** At the significance level $\alpha$, we reject $H_0$ if the value of the test statistic is greater than $C(\alpha)$, where the critical value $C(\alpha)$ is obtained by the $(1-\alpha)-$quantile of the distribution of the test statistic under the null hypothesis $H_0$.

Distribution of the test statistics $W^2, D, V, U^2, A^2, Z_A, Z_C, Z_K$, $H_n^{(1)}$ and $H_n^{(2)}$ under the null hypothesis cannot be evaluated analytically. Therefore, the critical values of the test statistics are computed by the Monte Carlo method. For each test statistic $\{W^2, D, V, U^2, A^2, Z_A, Z_C, Z_K, H_n^{(1)}, H_n^{(2)}\}$, its sample values is calculated for 100,000 simulated random samples of size $n$ from the IG with parameters 1 and 1. Since $\alpha = 0.05 = 5000/100000$, the 5000-th order statistic is evaluated and the critical value $C(\alpha)$ is specified. The critical values obtained for the statistics $W^2$-$H_n^{(2)}$ and sample sizes $10 \le n \le 100$ are given in Table 1.

TABLE 1. Critical values of the statistics at level 0.05%

| $n$ | $W^2$ | $D$ | $V$ | $U^2$ | $A^2$ | $Z_A$ | $Z_C$ | $Z_k$ | $H_n^{(1)}$ | $H_n^{(2)}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 0.8015 | 0.2859 | 0.4420 | 0.1213 | 0.1455 | 3.4978 | 6.5460 | 1.2207 | 0.003722 | 0.001996 |
| 20 | 0.8376 | 0.2094 | 0.3236 | 0.1241 | 0.1515 | 3.4446 | 8.8848 | 1.5360 | 0.001862 | 0.000968 |
| 30 | 0.8522 | 0.1736 | 0.2678 | 0.1250 | 0.1541 | 3.4154 | 10.3074 | 1.7162 | 0.001235 | 0.000635 |
| 40 | 0.8482 | 0.1511 | 0.2332 | 0.1254 | 0.1541 | 3.3943 | 11.2712 | 1.8166 | 0.000922 | 0.000471 |
| 50 | 0.8523 | 0.1355 | 0.2097 | 0.1247 | 0.1549 | 3.3815 | 12.1254 | 1.9161 | 0.000736 | 0.000374 |
| 60 | 0.8576 | 0.1240 | 0.1919 | 0.1254 | 0.1550 | 3.3718 | 12.7786 | 1.9936 | 0.000610 | 0.000309 |
| 70 | 0.8606 | 0.1155 | 0.1785 | 0.1263 | 0.1559 | 3.3638 | 13.2940 | 2.0712 | 0.000525 | 0.000266 |
| 80 | 0.8587 | 0.1082 | 0.1670 | 0.1262 | 0.1555 | 3.3576 | 13.8307 | 2.1221 | 0.000458 | 0.000232 |
| 90 | 0.8603 | 0.1020 | 0.1579 | 0.1257 | 0.1559 | 3.3524 | 14.2463 | 2.1671 | 0.000410 | 0.000207 |
| 100 | 0.8660 | 0.0970 | 0.1503 | 0.1264 | 0.1569 | 3.3476 | 14.6298 | 2.2225 | 0.000369 | 0.000186 |

TABLE 2. Type I error control of the tests for the nominal significance level $\alpha = 0.05$.

| $IG(\mu,\lambda)$ | $n$ | $W^2$ | $D$ | $V$ | $U^2$ | $A^2$ | $Z_A$ | $Z_C$ | $Z_K$ | $H_n^{(1)}$ | $H_n^{(2)}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $IG(0.5, 0.5)$ | 10 | 0.0506 | 0.0499 | 0.0506 | 0.0505 | 0.0509 | 0.0515 | 0.0514 | 0.0492 | 0.0516 | 0.0516 |
| | 20 | 0.0483 | 0.0498 | 0.0486 | 0.0491 | 0.0493 | 0.0497 | 0.0493 | 0.0495 | 0.0486 | 0.0487 |
| | 30 | 0.0497 | 0.0498 | 0.0510 | 0.0511 | 0.0489 | 0.0496 | 0.0497 | 0.0493 | 0.0498 | 0.0495 |
| | 50 | 0.0492 | 0.0503 | 0.0506 | 0.0499 | 0.0498 | 0.0492 | 0.0511 | 0.0501 | 0.0496 | 0.0497 |
| $IG(0.5, 1)$ | 10 | 0.0376 | 0.0380 | 0.0451 | 0.0440 | 0.0407 | 0.0495 | 0.0508 | 0.0371 | 0.0383 | 0.0386 |
| | 20 | 0.0369 | 0.0393 | 0.0466 | 0.0452 | 0.0406 | 0.0530 | 0.0537 | 0.0419 | 0.0367 | 0.0368 |
| | 30 | 0.0372 | 0.0399 | 0.0470 | 0.0458 | 0.0399 | 0.0525 | 0.0524 | 0.0427 | 0.0369 | 0.0365 |
| | 50 | 0.0359 | 0.0368 | 0.0455 | 0.0437 | 0.0378 | 0.0524 | 0.0531 | 0.0429 | 0.0362 | 0.0362 |
| $IG(0.5, 2)$ | 10 | 0.0301 | 0.0312 | 0.0433 | 0.0411 | 0.0352 | 0.0510 | 0.0532 | 0.0305 | 0.0322 | 0.0327 |
| | 20 | 0.0298 | 0.0302 | 0.0428 | 0.0405 | 0.0348 | 0.0546 | 0.0543 | 0.0342 | 0.0285 | 0.0290 |
| | 30 | 0.0283 | 0.0306 | 0.0444 | 0.0401 | 0.0334 | 0.0540 | 0.0554 | 0.0362 | 0.0300 | 0.0299 |
| | 50 | 0.0278 | 0.0304 | 0.0430 | 0.0396 | 0.0326 | 0.0545 | 0.0559 | 0.0420 | 0.0298 | 0.0296 |
| $IG(1, 0.5)$ | 10 | 0.0693 | 0.0666 | 0.0576 | 0.0591 | 0.0668 | 0.0530 | 0.0515 | 0.0670 | 0.0650 | 0.0643 |
| | 20 | 0.0688 | 0.0666 | 0.0577 | 0.0580 | 0.0665 | 0.0493 | 0.0483 | 0.0679 | 0.0648 | 0.0641 |
| | 30 | 0.0717 | 0.0683 | 0.0604 | 0.0606 | 0.0692 | 0.0484 | 0.0480 | 0.0665 | 0.0671 | 0.0664 |
| | 50 | 0.0672 | 0.0674 | 0.0563 | 0.0557 | 0.0648 | 0.0459 | 0.0455 | 0.0600 | 0.0689 | 0.0687 |
| $IG(1, 1)$ | 10 | 0.0498 | 0.0503 | 0.0504 | 0.0510 | 0.0501 | 0.0495 | 0.0496 | 0.0501 | 0.0504 | 0.0504 |
| | 20 | 0.0500 | 0.0505 | 0.0495 | 0.0514 | 0.0498 | 0.0499 | 0.0494 | 0.0508 | 0.0488 | 0.0491 |
| | 30 | 0.0494 | 0.0504 | 0.0506 | 0.0498 | 0.0494 | 0.0493 | 0.0493 | 0.0502 | 0.0497 | 0.0495 |
| | 50 | 0.0492 | 0.0498 | 0.0505 | 0.0497 | 0.0488 | 0.0493 | 0.0501 | 0.0500 | 0.0489 | 0.0494 |
| $IG(1, 2)$ | 10 | 0.0378 | 0.0393 | 0.0457 | 0.0451 | 0.0405 | 0.0507 | 0.0511 | 0.0378 | 0.0405 | 0.0407 |
| | 20 | 0.0372 | 0.0397 | 0.0461 | 0.0458 | 0.0405 | 0.0514 | 0.0518 | 0.0408 | 0.0370 | 0.0373 |
| | 30 | 0.0348 | 0.0381 | 0.0451 | 0.0434 | 0.0384 | 0.0511 | 0.0522 | 0.0401 | 0.0377 | 0.0373 |
| | 50 | 0.0352 | 0.0368 | 0.0444 | 0.0432 | 0.0378 | 0.0517 | 0.0522 | 0.0420 | 0.0361 | 0.0365 |
| $IG(2, 0.5)$ | 10 | 0.0890 | 0.0826 | 0.0663 | 0.0681 | 0.0866 | 0.0567 | 0.0536 | 0.0886 | 0.0795 | 0.0776 |
| | 20 | 0.0919 | 0.0860 | 0.0655 | 0.0687 | 0.0885 | 0.0529 | 0.0493 | 0.0897 | 0.0820 | 0.0816 |
| | 30 | 0.0969 | 0.0887 | 0.0688 | 0.0713 | 0.0928 | 0.0512 | 0.0493 | 0.0901 | 0.0870 | 0.0858 |
| | 50 | 0.0935 | 0.0844 | 0.0680 | 0.0682 | 0.0902 | 0.0472 | 0.0456 | 0.0814 | 0.0872 | 0.0862 |
| $IG(2, 1)$ | 10 | 0.0683 | 0.0651 | 0.0566 | 0.0591 | 0.0661 | 0.0516 | 0.0504 | 0.0668 | 0.0649 | 0.0640 |
| | 20 | 0.0687 | 0.0655 | 0.0557 | 0.0581 | 0.0667 | 0.0520 | 0.0503 | 0.0677 | 0.0659 | 0.0658 |
| | 30 | 0.0700 | 0.0678 | 0.0581 | 0.0595 | 0.0669 | 0.0493 | 0.0492 | 0.0661 | 0.0653 | 0.0646 |
| | 50 | 0.0705 | 0.0668 | 0.0606 | 0.0594 | 0.0674 | 0.0474 | 0.0466 | 0.0625 | 0.0670 | 0.0670 |
| $IG(2, 2)$ | 10 | 0.0503 | 0.0500 | 0.0518 | 0.0519 | 0.0511 | 0.0501 | 0.0504 | 0.0506 | 0.0499 | 0.0497 |
| | 20 | 0.0485 | 0.0500 | 0.0476 | 0.0491 | 0.0488 | 0.0492 | 0.0501 | 0.0501 | 0.0494 | 0.0495 |
| | 30 | 0.0496 | 0.0504 | 0.0503 | 0.0492 | 0.0495 | 0.0493 | 0.0507 | 0.0497 | 0.0515 | 0.0509 |
| | 50 | 0.0482 | 0.0486 | 0.0513 | 0.0486 | 0.0490 | 0.0490 | 0.0497 | 0.0492 | 0.0494 | 0.0490 |

In Table 2 the estimated type I error control using the 0.05 percentiles of the tests are evaluated ($\alpha = 0.05$). The results are presented in Table 2. According to the results of Table 2, the value of the type I error increases with the increase of the value of $\mu/\lambda$, so that if $\mu/\lambda \approx 1$, then $\alpha$ is close to the nominal value. We see that the actual sizes of the tests based on $Z_A$ and $Z_C$ are acceptable and therefore we can use these tests in practice. The actual sizes of the other tests are more (or less) than the nominal size for different values of $\mu$ and $\lambda$. Moreover, it is evident that for all tests, when the parameters are equal the type I error are acceptable.

3.2. **Power comparison.** The power of each test is studied by means of Monte Carlo simulations. In power comparison, we considered the following alternatives.

- the exponential distribution $Exp(\theta)$ with density $\theta\exp(-\theta x)$;
- the Weibull distribution with density $\theta x^{\theta-1}\exp\left(-x^{\theta}\right)$, denoted by $W(\theta)$;
- the gamma distribution with density $\Gamma(\theta)^{-1}x^{\theta-1}\exp(-x)$, denoted by $\Gamma(\theta)$;
- the half-normal $HN$ distribution with density $\Gamma(2/\pi)^{1/2}\exp\left(-x^2/2\right)$;
- the lognormal distribution $LN(\theta)$ with density $(\theta x)^{-1}(2\pi)^{-1/2}\exp\left(-(\log x)^2\Big/(2\theta^2)\right)$;
- the Pareto distribution $Pa(\theta)$ with density $\theta\big/x^{\theta+1}$ ;
- the uniform distribution $U$ with density 1, $0 \le x \le 1$;
- the Beta distribution $Beta(\alpha,\beta)$ with density $x^{\alpha-1}(1-x)^{\beta-1}\Big/Beta(\alpha,\beta)$, $0 \le x \le 1$ ;
- the modified extreme value $EV(\theta)$, with distribution function $1-\exp\left(\theta^{-1}(1-e^{x})\right)$;
- the linear increasing failure rate law $LF(\theta)$ with density $(1+\theta x)\exp\left(-x-\theta x^2/2\right)$;
- Dhillon's (1981) distribution with distribution function $1-\exp\left(-(\log(x+1))^{\theta+1}\right)$;
- Chen's (2000) distribution $CH(\theta)$, with distribution function $1-\exp\left(2\left(1-e^{x^{\theta}}\right)\right)$.

The powers of the tests based on $W^2, D, V, U^2, A^2, Z_A, Z_C, Z_K,\ H_n^{(1)}$ and $H_n^{(2)}$ statistics are computed by Monte Carlo simulation. Under each alternative, 100,000 samples of size 10, 20, 30 and 50 are generated. Then, the power of the corresponding test was estimated by the frequency of the event "the test statistic is smaller than the critical point". The power estimates are presented in Tables 3–6.

For each alternative, the bold type in these tables indicates the test achieving the maximal power.

TABLE 3. Monte Carlo power estimates of the tests for $n = 10$ and at the significance level $\alpha = 0.05$.

| *Alternative* | $W^2$ | $D$ | $V$ | $U^2$ | $A^2$ | $Z_A$ | $Z_C$ | $Z_K$ | $H_n^{(1)}$ | $H_n^{(2)}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $Exp(1)$ | 0.4182 | 0.3847 | 0.2977 | 0.3320 | **0.4223** | 0.3859 | 0.3646 | 0.3878 | 0.2822 | 0.2498 |
| $W(0.5)$ | 0.7950 | 0.7628 | 0.6818 | 0.6972 | **0.8079** | 0.7622 | 0.7255 | 0.7884 | 0.6762 | 0.6465 |
| $W(2)$ | 0.1811 | 0.1609 | 0.1391 | 0.1597 | 0.1916 | **0.2174** | 0.2144 | 0.1564 | 0.0938 | 0.0751 |
| $\Gamma(0.5)$ | 0.7527 | 0.7189 | 0.6341 | 0.6594 | **0.7606** | 0.7106 | 0.6822 | 0.7359 | 0.6301 | 0.5984 |
| $\Gamma(2)$ | 0.1731 | 0.1563 | 0.1220 | 0.1401 | 0.1783 | **0.1842** | 0.1772 | 0.1532 | 0.0924 | 0.0752 |
| $HN$ | 0.4271 | 0.3917 | 0.3146 | 0.3557 | **0.4330** | 0.4134 | 0.4002 | 0.3917 | 0.2867 | 0.2522 |
| $LN(0, 0.5)$ | 0.0380 | 0.0376 | 0.0468 | 0.0454 | 0.0430 | 0.0618 | **0.0624** | 0.0365 | 0.0322 | 0.0319 |
| $LN(0, 1)$ | **0.1045** | 0.0966 | 0.0711 | 0.0773 | 0.1041 | 0.0973 | 0.0884 | 0.0962 | 0.0672 | 0.0609 |
| $LN(0, 2)$ | 0.4827 | 0.4337 | 0.3221 | 0.3367 | **0.5037** | 0.4312 | 0.3696 | 0.4726 | 0.3220 | 0.2878 |
| $Pa(0.5)$ | 0.2353 | 0.1932 | 0.1865 | 0.1999 | 0.2596 | 0.2363 | 0.2405 | 0.2343 | 0.3423 | **0.3495** |
| $Pa(1)$ | 0.3581 | 0.2962 | 0.3025 | 0.3229 | 0.3607 | 0.3812 | 0.3908 | 0.2897 | 0.4913 | **0.5007** |
| $Pa(2)$ | 0.3338 | 0.2679 | 0.3251 | 0.3418 | 0.3673 | 0.4551 | **0.4584** | 0.2588 | 0.4476 | 0.4575 |
| $U$ | 0.5341 | 0.4795 | 0.4445 | 0.4849 | 0.5559 | **0.5633** | 0.5629 | 0.4804 | 0.3620 | 0.3189 |
| $Beta(2, 2)$ | 0.2535 | 0.2188 | 0.2098 | 0.2351 | 0.2739 | **0.3161** | 0.3164 | 0.2132 | 0.1292 | 0.1033 |
| $Beta(2, 0.5)$ | 0.6005 | 0.4831 | 0.6214 | 0.6238 | 0.6557 | 0.7539 | **0.7552** | 0.4920 | 0.3767 | 0.3239 |
| $Beta(0.5, 2)$ | 0.7666 | 0.7343 | 0.6496 | 0.6819 | **0.7755** | 0.7271 | 0.7079 | 0.7499 | 0.6430 | 0.6113 |
| $Beta(2, 5)$ | 0.1984 | 0.1764 | 0.1482 | 0.1713 | 0.2100 | **0.2300** | 0.2266 | 0.1722 | 0.1014 | 0.0815 |
| $CH(0.5)$ | 0.7736 | 0.7397 | 0.6559 | 0.6784 | **0.7833** | 0.7337 | 0.7053 | 0.7591 | 0.6529 | 0.6224 |
| $CH(1)$ | 0.4386 | 0.4029 | 0.3216 | 0.3631 | **0.4451** | 0.4191 | 0.4059 | 0.4038 | 0.2936 | 0.2589 |
| $CH(1.5)$ | 0.2908 | 0.2593 | 0.2208 | 0.2526 | 0.3025 | **0.3201** | 0.3160 | 0.2549 | 0.1670 | 0.1377 |
| $LF(2)$ | 0.4093 | 0.3758 | 0.3047 | 0.3443 | **0.4158** | 0.4020 | 0.3911 | 0.3750 | 0.2718 | 0.2390 |
| $LF(4)$ | 0.3877 | 0.3561 | 0.2927 | 0.3296 | **0.3958** | 0.3911 | 0.3821 | 0.3531 | 0.2588 | 0.2269 |
| $EV(0.5)$ | 0.4398 | 0.4024 | 0.3207 | 0.3625 | **0.4459** | 0.4202 | 0.4055 | 0.4040 | 0.2923 | 0.2580 |
| $EV(1.5)$ | 0.4444 | 0.4064 | 0.3421 | 0.3846 | **0.4539** | 0.4464 | 0.4390 | 0.4051 | 0.2975 | 0.2614 |
| $DL(1)$ | 0.1599 | 0.1460 | 0.1091 | 0.1250 | **0.1628** | 0.1627 | 0.1521 | 0.1442 | 0.0909 | 0.0756 |
| $DL(1.5)$ | 0.1203 | 0.1101 | 0.0900 | 0.1017 | 0.1264 | **0.1405** | 0.1352 | 0.1079 | 0.0628 | 0.0518 |

TABLE 4. Monte Carlo power estimates of the tests for $n = 20$ and at the significance level $\alpha = 0.05$.

| *Alternative* | $W^2$ | $D$ | $V$ | $U^2$ | $A^2$ | $Z_A$ | $Z_C$ | $Z_K$ | $H_n^{(1)}$ | $H_n^{(2)}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $Exp(1)$ | 0.6787 | 0.6309 | 0.5197 | 0.5686 | **0.6804** | 0.6390 | 0.6314 | 0.6306 | 0.5826 | 0.5569 |
| $W(0.5)$ | 0.9637 | 0.9508 | 0.9106 | 0.9224 | **0.9659** | 0.9433 | 0.9355 | 0.9557 | 0.9389 | 0.9321 |
| $W(2)$ | 0.3415 | 0.2955 | 0.2573 | 0.2957 | 0.3613 | **0.4220** | 0.4194 | 0.3068 | 0.2412 | 0.2157 |
| $\Gamma(0.5)$ | 0.9521 | 0.9363 | 0.8887 | 0.9074 | **0.9535** | 0.9278 | 0.9228 | 0.9393 | 0.9174 | 0.9088 |
| $\Gamma(2)$ | 0.3034 | 0.2676 | 0.2067 | 0.2416 | 0.3141 | **0.3335** | **0.3335** | 0.2714 | 0.2181 | 0.1956 |
| $HN$ | 0.7046 | 0.6538 | 0.5689 | 0.6178 | **0.7112** | 0.6995 | 0.6954 | 0.6553 | 0.6048 | 0.5780 |
| $LN(0, 0.5)$ | 0.0362 | 0.0365 | 0.0465 | 0.0458 | 0.0417 | **0.0691** | **0.0691** | 0.0413 | 0.0326 | 0.0317 |
| $LN(0, 1)$ | **0.1423** | 0.1247 | 0.0867 | 0.0995 | 0.1418 | 0.1370 | 0.1306 | 0.1294 | 0.0964 | 0.0858 |
| $LN(0, 2)$ | 0.7217 | 0.6672 | 0.5238 | 0.5531 | **0.7307** | 0.6368 | 0.5987 | 0.6885 | 0.6126 | 0.5871 |
| $Pa(0.5)$ | 0.4541 | 0.3582 | 0.3473 | 0.3584 | 0.4808 | 0.4957 | 0.4716 | 0.3904 | 0.5873 | **0.5990** |
| $Pa(1)$ | 0.7189 | 0.6121 | 0.6635 | 0.6604 | 0.7428 | 0.8236 | 0.7994 | 0.6386 | 0.8190 | **0.8258** |
| $Pa(2)$ | 0.6754 | 0.5456 | 0.6733 | 0.6686 | 0.7338 | **0.8685** | 0.8454 | 0.6698 | 0.7806 | 0.7883 |
| $U$ | 0.8481 | 0.7826 | 0.7883 | 0.8033 | 0.8702 | **0.9081** | 0.8971 | 0.7964 | 0.7407 | 0.7142 |
| $Beta(2, 2)$ | 0.5009 | 0.4216 | 0.4257 | 0.4602 | 0.5415 | **0.6451** | 0.6298 | 0.4442 | 0.3602 | 0.3262 |
| $Beta(2, 0.5)$ | 0.9248 | 0.8252 | 0.9439 | 0.9304 | 0.9556 | **0.9905** | 0.9855 | 0.9588 | 0.8221 | 0.7974 |
| $Beta(0.5, 2)$ | 0.9606 | 0.9453 | 0.9066 | 0.9249 | **0.9633** | 0.9449 | 0.9425 | 0.9487 | 0.9286 | 0.9212 |
| $Beta(2, 5)$ | 0.3770 | 0.3252 | 0.2793 | 0.3182 | 0.3967 | **0.4483** | 0.4447 | 0.3334 | 0.2678 | 0.2410 |
| $CH(0.5)$ | 0.9606 | 0.9463 | 0.9030 | 0.9201 | **0.9621** | 0.9385 | 0.9336 | 0.9504 | 0.9292 | 0.9218 |
| $CH(1)$ | 0.7197 | 0.6679 | 0.5791 | 0.6276 | **0.7262** | 0.7099 | 0.7061 | 0.6684 | 0.6161 | 0.5892 |
| $CH(1.5)$ | 0.5413 | 0.4761 | 0.4278 | 0.4749 | 0.5619 | **0.6098** | 0.6029 | 0.4835 | 0.4165 | 0.3855 |
| $LF(2)$ | 0.6855 | 0.6355 | 0.5549 | 0.6034 | **0.6934** | 0.6888 | 0.6854 | 0.6361 | 0.5834 | 0.5563 |
| $LF(4)$ | 0.6630 | 0.6094 | 0.5367 | 0.5843 | 0.6727 | **0.6800** | 0.6754 | 0.6114 | 0.5621 | 0.5359 |
| $EV(0.5)$ | 0.7216 | 0.6685 | 0.5810 | 0.6297 | **0.7281** | 0.7103 | 0.7059 | 0.6702 | 0.6167 | 0.5901 |
| $EV(1.5)$ | 0.7407 | 0.6856 | 0.6242 | 0.6693 | 0.7522 | **0.7630** | 0.7568 | 0.6874 | 0.6377 | 0.6101 |
| $DL(1)$ | 0.2639 | 0.2328 | 0.1711 | 0.1994 | 0.2685 | **0.2757** | 0.2722 | 0.2372 | 0.1890 | 0.1701 |
| $DL(1.5)$ | 0.2050 | 0.1784 | 0.1417 | 0.1638 | 0.2150 | 0.2455 | **0.2456** | 0.1856 | 0.1388 | 0.1213 |

Table 5. Monte Carlo power estimates of the tests for $n = 30$ and at the significance level $\alpha = 0.05$.

| *Alternative* | $W^2$ | $D$ | $V$ | $U^2$ | $A^2$ | $Z_A$ | $Z_C$ | $Z_K$ | $H_n^{(1)}$ | $H_n^{(2)}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $Exp(1)$ | 0.8294 | 0.7864 | 0.6874 | 0.7323 | **0.8316** | 0.7968 | 0.7951 | 0.7882 | 0.7673 | 0.7506 |
| $W(0.5)$ | 0.9945 | 0.9912 | 0.9784 | 0.9825 | **0.9948** | 0.9881 | 0.9864 | 0.9922 | 0.9892 | 0.9880 |
| $W(2)$ | 0.4821 | 0.4159 | 0.3698 | 0.4166 | 0.5093 | **0.5835** | 0.5781 | 0.4545 | 0.3894 | 0.3643 |
| $\Gamma(0.5)$ | 0.9902 | 0.9853 | 0.9684 | 0.9754 | **0.9904** | 0.9819 | 0.9810 | 0.9856 | 0.9832 | 0.9810 |
| $\Gamma(2)$ | 0.4259 | 0.3742 | 0.2948 | 0.3398 | 0.4393 | 0.4658 | **0.4679** | 0.3945 | 0.3386 | 0.3145 |
| $HN$ | 0.8588 | 0.8151 | 0.7481 | 0.7884 | 0.8651 | **0.8622** | 0.8594 | 0.8184 | 0.7961 | 0.7800 |
| $LN(0, 0.5)$ | 0.0360 | 0.0369 | 0.0470 | 0.0462 | 0.0426 | 0.0726 | **0.0748** | 0.0482 | 0.0338 | 0.0325 |
| $LN(0, 1)$ | 0.1721 | 0.1497 | 0.1052 | 0.1202 | **0.1740** | 0.1704 | 0.1693 | 0.1617 | 0.1274 | 0.1154 |
| $LN(0, 2)$ | 0.8503 | 0.8077 | 0.6785 | 0.7078 | **0.8553** | 0.7705 | 0.7480 | 0.8173 | 0.7816 | 0.7661 |
| $Pa(0.5)$ | 0.6399 | 0.5167 | 0.5173 | 0.5076 | 0.6780 | 0.7494 | 0.6964 | 0.5790 | 0.7623 | **0.7708** |
| $Pa(1)$ | 0.8970 | 0.8137 | 0.8710 | 0.8563 | 0.9193 | **0.9712** | 0.9566 | 0.9078 | 0.9455 | 0.9485 |
| $Pa(2)$ | 0.8669 | 0.7469 | 0.8743 | 0.8563 | 0.9136 | 0.9829 | **0.9712** | 0.9369 | 0.9291 | 0.9317 |
| $U$ | 0.9572 | 0.9201 | 0.9361 | 0.9347 | 0.9692 | **0.9887** | 0.9833 | 0.9525 | 0.9150 | 0.9037 |
| $Beta(2, 2)$ | 0.6929 | 0.5933 | 0.6137 | 0.6404 | 0.7402 | **0.8515** | 0.8276 | 0.6752 | 0.5722 | 0.5427 |
| $Beta(2, 0.5)$ | 0.9904 | 0.9623 | 0.9953 | 0.9910 | 0.9968 | **0.9999** | 0.9997 | 0.9995 | 0.9664 | 0.9601 |
| $Beta(0.5, 2)$ | 0.9937 | 0.9894 | 0.9782 | 0.9834 | **0.9943** | 0.9910 | 0.9906 | 0.9902 | 0.9878 | 0.9861 |
| $Beta(2, 5)$ | 0.5279 | 0.4575 | 0.4017 | 0.4498 | 0.5562 | **0.6255** | 0.6171 | 0.4869 | 0.4258 | 0.3978 |
| $CH(0.5)$ | 0.9929 | 0.9887 | 0.9741 | 0.9797 | **0.9933** | 0.9862 | 0.9854 | 0.9894 | 0.9872 | 0.9857 |
| $CH(1)$ | 0.8706 | 0.8265 | 0.7564 | 0.7966 | **0.8767** | 0.8693 | 0.8650 | 0.8292 | 0.8075 | 0.7910 |
| $CH(1.5)$ | 0.7183 | 0.6476 | 0.6028 | 0.6486 | 0.7422 | **0.7933** | 0.7843 | 0.6715 | 0.6194 | 0.5940 |
| $LF(2)$ | 0.8424 | 0.7979 | 0.7321 | 0.7726 | 0.8498 | **0.8531** | 0.8488 | 0.8027 | 0.7812 | 0.7636 |
| $LF(4)$ | 0.8236 | 0.7772 | 0.7159 | 0.7562 | 0.8320 | **0.8434** | 0.8398 | 0.7854 | 0.7604 | 0.7429 |
| $EV(0.5)$ | 0.8706 | 0.8270 | 0.7565 | 0.7959 | **0.8769** | 0.8694 | 0.8651 | 0.8287 | 0.8070 | 0.7910 |
| $EV(1.5)$ | 0.8891 | 0.8458 | 0.8045 | 0.8369 | 0.8987 | **0.9115** | 0.9056 | 0.8512 | 0.8301 | 0.8151 |
| $DL(1)$ | 0.3544 | 0.3118 | 0.2323 | 0.2705 | 0.3626 | 0.3701 | **0.3735** | 0.3302 | 0.2831 | 0.2629 |
| $DL(1.5)$ | 0.2844 | 0.2474 | 0.1945 | 0.2261 | 0.2999 | 0.3394 | **0.3439** | 0.2725 | 0.2139 | 0.1954 |

Table 6. Monte Carlo power estimates of the tests for $n = 50$ and at the significance level $\alpha = 0.05$.

| *Alternative* | $W^2$ | $D$ | $V$ | $U^2$ | $A^2$ | $Z_A$ | $Z_C$ | $Z_K$ | $H_n^{(1)}$ | $H_n^{(2)}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $Exp(1)$ | 0.9554 | 0.9324 | 0.8777 | 0.9052 | **0.9558** | 0.9412 | 0.9405 | 0.9317 | 0.9343 | 0.9288 |
| $W(0.5)$ | 0.9998 | 0.9997 | 0.9988 | 0.9992 | **0.9998** | 0.9995 | 0.9994 | 0.9997 | 0.9998 | 0.9998 |
| $W(2)$ | 0.7032 | 0.6209 | 0.5682 | 0.6205 | 0.7306 | **0.8014** | 0.7915 | 0.6837 | 0.6260 | 0.6062 |
| $\Gamma(0.5)$ | 0.9997 | 0.9993 | 0.9977 | 0.9985 | **0.9997** | 0.9992 | 0.9992 | 0.9992 | 0.9994 | 0.9993 |
| $\Gamma(2)$ | 0.6186 | 0.5477 | 0.4489 | 0.5063 | 0.6319 | 0.6626 | **0.6643** | 0.5812 | 0.5491 | 0.5291 |
| $HN$ | 0.9717 | 0.9523 | 0.9237 | 0.9416 | 0.9740 | **0.9756** | 0.9732 | 0.9546 | 0.9551 | 0.9506 |
| $LN(0.0.5)$ | 0.0379 | 0.0379 | 0.0483 | 0.0475 | 0.0446 | 0.0800 | **0.0847** | 0.0580 | 0.0354 | 0.0343 |
| $LN(0.1)$ | 0.2295 | 0.1937 | 0.1360 | 0.1584 | 0.2333 | 0.2272 | **0.2366** | 0.2179 | 0.1841 | 0.1720 |
| $LN(0.2)$ | 0.9572 | 0.9358 | 0.8620 | 0.8815 | **0.9580** | 0.9115 | 0.9022 | 0.9356 | 0.9357 | 0.9309 |
| $Pa(0.5)$ | 0.8684 | 0.7541 | 0.7878 | 0.7384 | 0.9024 | **0.9703** | 0.9388 | 0.9123 | 0.9331 | 0.9368 |
| $Pa(1)$ | 0.9906 | 0.9655 | 0.9874 | 0.9804 | 0.9950 | **0.9998** | 0.9992 | 0.9989 | 0.9971 | 0.9973 |
| $Pa(2)$ | 0.9857 | 0.9422 | 0.9884 | 0.9806 | 0.9946 | **0.9999** | 0.9997 | 0.9995 | 0.9952 | 0.9954 |
| $U$ | 0.9980 | 0.9918 | 0.9964 | 0.9950 | 0.9990 | **0.9999** | 0.9998 | 0.9995 | 0.9933 | 0.9923 |
| $Beta(2, 2)$ | 0.9039 | 0.8222 | 0.8527 | 0.8627 | 0.9331 | **0.9842** | 0.9747 | 0.9372 | 0.8370 | 0.8231 |
| $Beta(2, 0.5)$ | 0.9999 | 0.9994 | **1.0000** | 0.9999 | **1.0000** | **1.0000** | **1.0000** | **1.0000** | 0.9994 | 0.9993 |
| $Beta(0.5, 2)$ | 0.9999 | 0.9997 | 0.9993 | 0.9995 | **0.9999** | **0.9999** | **0.9999** | 0.9997 | 0.9997 | 0.9997 |
| $Beta(2, 5)$ | 0.7565 | 0.6718 | 0.6159 | 0.6655 | 0.7820 | **0.8521** | 0.8364 | 0.7267 | 0.6742 | 0.6560 |
| $CH(0.5)$ | 0.9998 | 0.9997 | 0.9988 | 0.9993 | **0.9998** | 0.9995 | 0.9995 | 0.9997 | **0.9998** | 0.9997 |
| $CH(1)$ | 0.9750 | 0.9561 | 0.9274 | 0.9448 | 0.9774 | **0.9772** | 0.9748 | 0.9580 | 0.9594 | 0.9558 |
| $CH(1.5)$ | 0.9091 | 0.8536 | 0.8265 | 0.8572 | 0.9229 | **0.9530** | 0.9448 | 0.8874 | 0.8605 | 0.8494 |
| $LF(2)$ | 0.9653 | 0.9439 | 0.9130 | 0.9338 | 0.9683 | **0.9707** | 0.9683 | 0.9469 | 0.9464 | 0.9413 |
| $LF(4)$ | 0.9574 | 0.9321 | 0.9013 | 0.9238 | 0.9614 | **0.9674** | 0.9650 | 0.9388 | 0.9357 | 0.9303 |
| $EV(0.5)$ | 0.9754 | 0.9564 | 0.9271 | 0.9448 | 0.9771 | **0.9775** | 0.9754 | 0.9576 | 0.9604 | 0.9567 |
| $EV(1.5)$ | 0.9834 | 0.9673 | 0.9544 | 0.9653 | 0.9861 | **0.9912** | 0.9890 | 0.9739 | 0.9696 | 0.9664 |
| $DL(1)$ | 0.5117 | 0.4537 | 0.3520 | 0.4016 | 0.5198 | 0.5283 | **0.5372** | 0.4874 | 0.4515 | 0.4331 |
| $DL(1.5)$ | 0.4260 | 0.3664 | 0.2901 | 0.3374 | 0.4426 | 0.4908 | **0.4985** | 0.4187 | 0.3623 | 0.3434 |

From Tables 3-6, we observe that no single test can be said to perform the best against all alternatives. However, for almost alternatives the tests based on $A^2$, $Z_A$, and $Z_C$ statistics have the most power. Differences of power between these tests with each other are small. So, generally, three tests $A^2$, $Z_A$, $Z_C$ have a good performance. Also, we can see that for small sample sizes and against alternatives $Pa(0.5)$ and $Pa(1)$ the test $H_n^{(2)}$ has the most power.
We can see that the power values of the tests increase when the sample sizes increase. Also, the test $Z_A$ has a good performance when $n$ increases.
Power study reveals the tests $A^2$, $Z_A$, and $Z_C$ have a high power and generally they outperform the other tests under the different alternatives. The power differences between these tests and the other tests are substantial.
In other hand, from Table 2, we found that the actual sizes of the tests based on $Z_A$ and $Z_C$ were acceptable. Consequently, the tests based on $Z_A$ and $Z_C$ statistics should be recommended in practice. Since for small sample sizes, the test based on $Z_A$ has a better performance than test based on $Z_C$, we can generally conclude that the test $Z_A$ has a good performance against almost alternatives and this test can be confidently recommended in practice.

## 4. An Illustrative Example

In this section, we illustrate how the tests can be applied to test the goodness-of-fit for the IG distribution when the observations are available.

**Example 4.1.** *Folks and Chhikara [9] considered the following dataset, consisting of 19 fracture toughness of MIG (metal inert gas) welds.*
*54.4, 62.6, 63.2, 67.0, 70.2, 70.5, 70.6, 71.4, 71.8, 74.1, 74.1, 74.3, 78.8, 81.8, 83.0, 84.4, 85.3, 86.9, 87.3.*
*They concluded by using the KS statistic that the IG distribution is a reasonable fit. The empirical distribution function of the considered data set is presented in Figure 1.*

*Here, we apply the tests to this data set. First, the ML estimates of and are computed as:*

$$\hat{\mu} = 74.3 \quad and \quad \hat{\lambda} = 4923.952\,.$$

*Then, the value of each test statistic is computed and also the critical value of each test at the significance level 0.05 is obtained by Monte Carlo simulation. Results are summarized in Table 7.*

*Because the value of each test statistic is smaller than the corresponding critical value, the IG hypothesis is not rejected for these data at the significance level of 0.05. Therefore, we can conclude that the underlying distribution of these data is an IG distribution.*

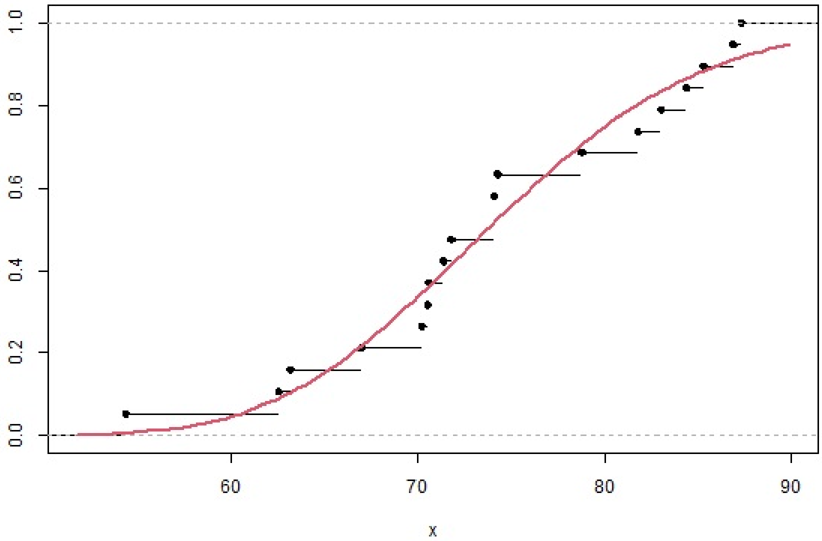


FIGURE 1. The empirical distribution function of data and a fitted IG distribution function.

TABLE 7. The value of the test statistics and critical values at 5% level.

| *Test* | *Value of the test statistic* | *Critical value* | *Decision* |
|---|---|---|---|
| $W^2$ | 0.05379 | 0.83473 | Not reject $H_0$ |
| $D$ | 0.13339 | 0.21478 | Not reject $H_0$ |
| $V$ | 0.24056 | 0.33114 | Not reject $H_0$ |
| $U^2$ | 0.05030 | 0.12381 | Not reject $H_0$ |
| $A^2$ | 0.37997 | 0.83192 | Not reject $H_0$ |
| $Z_A$ | 3.3847 | 3.4489 | Not reject $H_0$ |
| $Z_C$ | 5.6173 | 8.7055 | Not reject $H_0$ |
| $Z_K$ | 0.5517 | 1.52196 | Not reject $H_0$ |
| $H_n^{(1)}$ | 0.000576 | 0.00195 | Not reject $H_0$ |
| $H_n^{(2)}$ | 0.000292 | 0.00101 | Not reject $H_0$ |

## 5. **Conclusion**

In this paper, we have evaluated the empirical distribution function-based goodness-of-fit tests for the IG distribution, and have shown that the considered tests have a good performance. Critical points of the test statistics have computed and then the actual sizes of the considered tests have obtained. Through Monte Carlo simulations, we have carried out an extensive power study on the considered tests. It is shown that some of the tests outperform in most cases all other tests. Finally, we have used a real data set and have illustrated how the considered tests can be applied to test the goodness-of-fit for the IG distribution when a random sample is available.

## 6. Aknowledgement

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## 7. Conflict of interest

The authors declare no conflict of interest.

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