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ANALYTICAL INVESTIGATION OF FRACTIONAL DIFFERENTIAL INCLUSION WITH A NONLOCAL INFINITE–POINT OR RIEMANN–STIELTJES INTEGRAL BOUNDARY CONDITIONS

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Abstract. Here, we investigate the existence of solutions for the initial value problem of fractional-order differential inclusion containing a nonlocal infinite-point or Riemann–Stieltjes integral boundary conditions. A sufficient condition for the uniqueness of the solution is given. The continuous dependence of the solution on the set of selections and on some data is studied. At last, examples are designed to illustrate the applicability of the theoretical results.

Keywords: Functional integro-differential inclusion, Fixed point theorem, ϕ-Caputo fractional operator, Riemann–Stieltjes Integral boundary conditions, infinite-point boundary conditions.
2020 MSC: 26A33, 34K45, 47G10

1. Introduction

Differential and integral equation models have appeared in a variety of applications (see [3], [5], [6], [8], [9]- [12]). In physical sciences and applied mathematics, boundary value problems involving fractional differential equations occur. Subsidiary conditions are imposed locally in some of these issues. Nonlocal conditions are imposed in other cases. Nonlocal conditions are frequently preferable to local conditions because the measurements required by a nonlocal condition are sometimes more precise than the measurements provided by a local condition. As a result, a number of outstanding results on fractional boundary value problems (abbreviated BVPs) with resonant requirements have been obtained. Bai [4] investigated a class of fractional differential equations with $m-$point boundary conditions. Using the same technique Kosmatov [7] investigated the fractional order three points BVP with resonant case. Considering the fact that the study of fractional BVPs at resonance has yielded fruitful results, it should be highlighted that problems involving Riemann-Stieltjes

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integrals are very scarce. As a consequence, the study of fractional BVPs at resonance has yielded fruitful results, it should be mentioned that such problems with Riemann-Stieltjes integrals are very scarce, so it is worthy of additional study. Riemann-Stieltjes integral has been considered as both multipoint and integral in one frame, which is more common, see the relevant works due to Ahmad et. al. [1, 2].

Boundary value problems for nonlinear differential equations issues can occur in a variety of areas, including applied mathematics, physics, and variations problems of control theory, we refer the reader to the papers [10, 12]. In recent years, several scientists and academics have become interested in the study of boundary value problems of fractional order, and the topic has grown across several academic disciplines. The existence of continuous solutions to the nonlocal first-order boundary value problem (BVP) using the Liouville-Caputo fractional derivative was demonstrated in [15]

$$\frac{dx}{d\mathfrak{r}} = f(\mathfrak{r}, D^{\alpha}x(\mathfrak{r})),\ \mathfrak{r} \in (0,1),\ 0 < \alpha < 1,$$

together with either the infinite-point boundary conditions given by

$$\sum_{k=1}^{\infty} a_k\ \mu(\tau_k) = \mu_\circ,\ a_k\ > 0, \tau_k \in (0,1],$$

or the Riemann-Stieltjes functional integral boundary conditions

$$\int_0^T \mu(\varsigma)\ dh(\varsigma)\ =\ \mu_\circ.$$

Some authors have investigated boundary value issues with nonlocal, integral, and infinite points boundary conditions, we refer the reader to the monographs (see [6], [8], [13]- [16]).

Based on the above contributions, in this paper, we consider a modified version of the problem investigated in [15]. Precisely, we study the existence of solutions for a Caputo type fractional differential inclusion

$$(1) \qquad {}^{c}D^{\eta}\mu(\mathfrak{r}) \ \in\ \Phi_1(\mathfrak{r}, I^{\sigma}\phi_2(\mathfrak{r}, \mu(\varphi(\mathfrak{r})))),\ \eta,\ \sigma\ \in (0,1),\ \mathfrak{r} \in (0,T],$$

provided with Riemann–Stieltjes integro boundary conditions

$$(2) \qquad \mu(0)\ +\ \int_0^T \mu(\varsigma)\ dh(\varsigma)\ =\ \mu_\circ,$$

where $h : [0,T] \to \mathbb{R}$ is nondecreasing function, or provided with the infinite-point boundary conditions with the nonlocal condition

$$(3) \qquad \mu(0)\ +\ \sum_{k=1}^{\infty} a_k\ \mu(\tau_k) = \mu_\circ,\ a_k\ > 0, \tau_k \in (0,T],$$

where $\mathfrak{r} \in I = [0,T]$, ${}^{c}D^{\eta}$ is the Caputo fractional derivative of order η, I^{σ} is the Riemann-Liouville fractional integral operator of order σ, and $\Phi_1 : [0,T] \times \mathbb{R}^+ \to P(\mathbb{R})$ is a multivalued map, with $P(\mathbb{R})$ is the family of all nonempty

subsets of $\mathbb{R}$. Our investigation is built on the selections of the set-valued function Φ_1 by modifying the inclusion of the functional integral into a coupled system.
We first find the continuous solution of the problem (1) with the m-point BCs given by

$$\mu(0) \ + \ \sum_{k=1}^{m} a_k \ \mu(\tau_k) = \mu_\circ, \ a_k \ > 0, \tau_k \in [0, T] \tag{4}$$

and after that, by applying the characteristics of the Riemann sum for continuous functions, we study the solutions of the BVP with the Riemann-Stieltjes integral presented by (1) and (2) in addition the BVP with infinite points presented by (1) and (3). As a part of the process to achieve the main aim, the designed problem is transformed into an equivalent integral equation, and the existences result is proved by applying the Schauder fixed point theorem.

The remainder of the paper is organized as follows: Section 2 contains our principal result regarding the problems (1)–(4). In light of the developments conclusion, we investigate the BVP provided by (1)-(2) and by (1)-(3). We demonstrate sufficient conditions in each for the problem (1) under the Riemann-Stieltjes functional integral BC (2) and under infinite-point BC (3), while Section 3 covers the continuous dependence and the uniqueness of solutions. Example is provided in Section 4 to illustrate our results. Conclusion is mentioned in the last Section 5.

2. **Existence of solution**

Take into account the following assumptions:

(i) The Lipschitzian set-valued map $\Phi_1 : I \times \mathbb{R} \to 2^{\mathbb{R}}$ has a nonempty compact convex subset of $2^{\mathbb{R}}$, utilizing the Lipschitz constant $k > 0$

$$\|\Phi_1(\mathfrak{r}, \mu) - \Phi_1(\mathfrak{r}, \nu)\| \leq k \ |\mu - \nu|.$$

Note: The set of Lipschitz selections for Φ_1 is not empty and there exists $\phi_1 \in \Phi_1$ (see [3]), with

$$|\phi_1(\mathfrak{r}, \mu) - \phi_1(\mathfrak{r}, \nu)| \leq k \ |\mu - \nu|.$$

(ii) The function $\varphi : I \to I$ is continuous.

(iii) The Caratheodory requirement is satisfied for function $\phi_2 : I \times \mathbb{R} \to \mathbb{R}$, i.e., ϕ_2 is measurable in t for any $\mu \in \mathbb{R}$ and continuous in μ for almost all $t \in I$. There exists a function $a(t)$ that is measurable bounded and there is a positive constant $b > 0$, with

$$|\phi_2(\mathfrak{r}, \mu)| \leq \ a(\mathfrak{r}) + \ b|\mu|, \ \forall \ \mathfrak{r} \ \in I \ and \ x \in \mathbb{R}.$$

(iv) $[a \sum_{k=1}^{m} \ |a_k| \ + 1] \frac{k \ T^{\eta}}{\Gamma(\eta+1)} < 1, \ \frac{b \ T^{\sigma}}{\Gamma(\sigma+1)} < 1,$ and $I_c^{\gamma} a(.) \leq M \ \forall \ \gamma \ \leq \sigma, \ c \geq 0.$

Lemma 2.1. *For any $\mu \in C(I,\mathbb{R})$, the solution of the linear fractional boundary value problem*

$$D^{\eta}\mu(\mathfrak{r}) = \phi_1(\mathfrak{r}, I^{\sigma}\phi_2(\mathfrak{r},\mu(\varphi(\mathfrak{r})))),\ \sigma \in (0,1),\ \mathfrak{r} \in I, \tag{5}$$

supplemented with the non-local condition (4), is equivalence to the integral equation

$$\begin{aligned}\mu(\mathfrak{r}) = a\left(\mu_\circ - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\varsigma, I^{\sigma}\phi_2(\varsigma,\mu(\varphi(\varsigma))))d\varsigma\right) \\ + \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-s)^{\eta-1}}{\Gamma(\eta)}\phi_1(\varsigma, I^{\sigma}\phi_2(\varsigma,\mu(\varphi(\varsigma))))d\varsigma,\end{aligned} \tag{6}$$

where $a = (1+\sum_{k=1}^{m} a_k)^{-1}$.

Proof. We start by looking at problem (5) with $m-$point BCs in (4). Integrating both sides of (5), we obtain

$$\mu(\mathfrak{r}) = \mu(0) + I^{\eta}\phi_1(\mathfrak{r}, I^{\sigma}\phi_2(\mathfrak{r},\mu(\varphi(\mathfrak{r})))). \tag{7}$$

Use condition (4), we get

$$\mu(\mathfrak{r}) = \mu_\circ - \sum_{k=1}^{m} a_k\ \mu(\tau_k) + \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\varsigma, I^{\sigma}\phi_2(\varsigma,\mu(\varphi(\varsigma))))\ d\varsigma. \tag{8}$$

In fact, when we set $\mathfrak{r} = \tau_k \in [0,T]$ in Equation (8), we have

$$\mu(\tau_k) = \mu_\circ - \sum_{k=1}^{m} a_k\ \mu(\tau_k) + \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\varsigma, I^{\sigma}\phi_2(\varsigma,\mu(\varphi(\varsigma))))\ d\varsigma. \tag{9}$$

So, we have

$$\begin{aligned}\mu(\tau_k) = \mu(\mathfrak{r}) + \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\varsigma, I^{\sigma}\phi_2(\varsigma,\mu(\varphi(\varsigma))))d\varsigma \\ - \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\varsigma, I^{\sigma}\phi_2(\varsigma,\mu(\varphi(\varsigma))))d\varsigma.\end{aligned} \tag{10}$$

From (8) and (10), we have

$$\begin{aligned}\mu(\mathfrak{r}) = \mu_\circ - \sum_{k=1}^{m} a_k\ (\mu(\mathfrak{r}) &+ \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\varsigma, I^{\sigma}\phi_2(\varsigma,\mu(\varphi(\varsigma))))\ d\varsigma \\ &- \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\varsigma, I^{\sigma}\phi_2(\varsigma,\mu(\varphi(\varsigma))))\ d\varsigma) \\ &+ \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\varsigma, I^{\sigma}\phi_2(\varsigma,\mu(\varphi(\varsigma))))\ d\varsigma.\end{aligned}$$

As a result, we obtain

$$\begin{aligned}(1+\sum_{k=1}^{m} a_k)\mu(\mathfrak{r}) &= \mu_\circ - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\varsigma, I^\sigma\phi_2(\varsigma,\mu(\varphi(\varsigma))))d\varsigma \\ &+ (1+\sum_{k=1}^{m} a_k) \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\varsigma, I^\sigma\phi_2(\varsigma,\mu(\varphi(\varsigma))))\, d\varsigma.\end{aligned}$$

Letting $a=(1+\sum_{k=1}^{m} a_k)^{-1}$, then

$$\begin{aligned}\mu(\mathfrak{r}) = a&\left(\mu_\circ - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\varsigma, I^\sigma\phi_2(\varsigma,\mu(\varphi(\varsigma))))d\varsigma\right) \\ &+ \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\varsigma, I^\sigma\phi_2(\varsigma,\mu(\varphi(\varsigma))))d\varsigma.\end{aligned}$$

Finally, in order to complete the proof of the above Lemma, we show that Equation (6) satisfies problem (5) together with the m-point BCs in (4). In fact, upon differentiating (6) with respect to $\mathfrak{r}$, we obtain

$$D^\eta\mu(\mathfrak{r}) = \phi_1(\mathfrak{r}, I^\sigma\phi_2(\mathfrak{r},\mu(\varphi(\mathfrak{r})))),$$

and

$$\begin{aligned}\mu(\tau_k) = a&\left(\mu_\circ - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\varsigma, I^\sigma\phi_2(\varsigma,\mu(\varphi(\varsigma)))d\varsigma\right) \\ &+ \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\varsigma, I^\sigma\phi_2(\varsigma,\mu(\varphi(\varsigma))))d\varsigma,\end{aligned} \tag{11}$$

so,

$$\begin{aligned}(1+\sum_{k=1}^{m} a_k)\mu(\tau_k) &= \mu_\circ - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\varsigma, I^\sigma\phi_2(\varsigma,\mu(\varphi(\varsigma))))\, d\varsigma \\ &+ (1+\sum_{k=1}^{m} a_k) \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\varsigma, I^\sigma\phi_2(\varsigma,\mu(\varphi(\varsigma))))d\varsigma \\ \mu(\tau_k)+\sum_{k=1}^{m} a_k\mu(\tau_k) &= \mu_\circ + \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\varsigma, I^\sigma\phi_2(\varsigma,\mu(\varphi(\varsigma))))d\varsigma,\end{aligned}$$

then

$$\sum_{k=1}^{m} a_k\mu(\tau_k) = \mu_\circ - \mu(\tau_k) + \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\varsigma, I^\sigma\phi_2(\varsigma,\mu(\varphi(\varsigma))))d\varsigma. \tag{12}$$

From (6) we have

$$\mu(0) = a\left(\mu_\circ - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\varsigma, I^\sigma\phi_2(\varsigma,\mu(\varphi(\varsigma))))d\varsigma\right).$$

Then

$$\mu(\tau_k) = \mu(0) \ + \int_0^{\tau_k} \frac{(\tau_k - \varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\varsigma, I^\sigma \phi_2(\varsigma, \mu(\varphi(\varsigma))))d\varsigma,$$

and

$$\mu(0) = \mu(\tau_k) - \int_0^{\tau_k} \frac{(\tau_k - \varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\varsigma, I^\sigma \phi_2(\varsigma, \mu(\varphi(\varsigma))))d\varsigma. \tag{13}$$

Now we obtain $m-$point BC (4), by adding (12) and (13),

$$\mu(0) \ + \sum_{k=1}^{m} a_k \ \mu(\tau_k) = \mu_\circ.$$

□

Remark 2.2. It is obvious from assumption (i), the set of Lipschitz selection of F_1 is nonempty. Moreover, there exists $\phi_1 \in S_{\Phi_1}$, such that

$$|\phi_1(\mathfrak{r}, \mu) - \phi_1(\mathfrak{r}, \nu)| \leq k|\mu - \nu|.$$

Hence, clearly, we have

$$|\phi_1(\mathfrak{r}, \mu)| \leq k|\mu| + \phi_1^*, \text{ where } \phi_1^* = \sup_{\mathfrak{r}\in[0,T]} |\phi_1(\mathfrak{r}, 0)|.$$

Let's go on to the next step

$$\nu(\mathfrak{r}) = I^\sigma \phi_2(\mathfrak{r}, \mu(\varphi(\mathfrak{r}))) \ \mathfrak{r} \in I. \tag{14}$$

The nonlinear functional integral equation (6) can thus be expressed as

$$\mu(\mathfrak{r}) = a \left(\mu_\circ - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k - \varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\varsigma, \nu(\varsigma))d\varsigma \right) + \int_0^{\mathfrak{r}} \frac{(\mathfrak{r} - \varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\varsigma, \nu(\varsigma))d\varsigma. \tag{15}$$

As a result, the coupled system (14) and (15) and the functional integral equation (6) are equal.

Now, we investigate the existence of a continuous solution of the Equation (6), that is a solution of inclusion (1) with nonlocal condition (4), by obtaining the continuous solution of the coupled system (14) and (15). Now for the existence of at least one solution, $u = (\mu, \nu), \ \mu, \nu \in C(I)$ of the coupled system (14), (15) we have the following theorem.

Theorem 2.3. *Assume that assumptions $(i) - (iv)$ hold. Then problems (14), (15) have at least one continuous solution $u = (\mu, \nu), \ \mu, \nu \in C(I, \mathbb{R})$.*

Proof. Let Q_r be defined as

$$Q_r = \{u = (\mu, \nu) \in \mathbb{R}^2, \|u\| \leq r\},$$

where

$$r=r_1+r_2=\frac{a|\mu_\circ|+[a\sum_{k=1}^m\ |a_k|\ +1]\frac{\phi_1^*\ T^\eta}{\Gamma(\eta+1)}}{1-[a\sum_{k=1}^m\ |a_k|\ +1]\frac{k\ T^\eta}{\Gamma(\eta+1)}}+(1-\frac{b\ T^\sigma}{\Gamma(\sigma+1)})^{-1}\frac{MT^{\sigma-\gamma}}{\Gamma(\sigma-\gamma+1)}.$$

It is clear that the set Q_r is nonempty, bounded, closed and convex.

Afterwards, let indicate by A the operator defined on the space $C(I,\mathbb{R})$ by

$$Au(\mathfrak{r})=A(\mu,\nu)(\mathfrak{r})=(A_1\nu(\mathfrak{r}),A_2\mu(\mathfrak{r})),$$

$$\begin{aligned}&A_1\nu(\mathfrak{r})\\&=a(\mu_\circ-\sum_{k=1}^m a_k\int_0^{\tau_k}\frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\varsigma,\nu(\varsigma))d\varsigma)+\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\varsigma,\nu(\varsigma)))d\varsigma,\end{aligned}$$

and

$$A_2\mu(\mathfrak{r})\ =\ \int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\varsigma)^{\sigma-1}}{\Gamma(\sigma)}\phi_2(\varsigma,\mu(\varphi(\varsigma)))\ d\varsigma\ \mathfrak{r}\in I.$$

Hence, according to $u=(\mu,\nu)\in Q_r$,

$$\begin{aligned}&|A_1\nu(\mathfrak{r})|\\=\ &|a(\mu_\circ-\sum_{k=1}^m a_k\int_0^{\tau_k}\frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\varsigma,\nu(\varsigma))d\varsigma)+\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\varsigma,\nu(\varsigma)))d\varsigma|\\\leq\ &a|\mu_\circ|+a\sum_{k=1}^m a_k\int_0^{\tau_k}\frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}|\phi_1(\varsigma,\nu(\varsigma))|d\varsigma+\int_0^{t}\frac{(t-\varsigma)^{\eta-1}}{\Gamma(\eta)}|\phi_1(\varsigma,\nu(\varsigma)))|d\varsigma\\\leq\ &a|\mu_\circ|+[a\sum_{k=1}^m|a_k|+1]\frac{(k|\nu|+\phi_1^*)T^\eta}{\Gamma(\eta+1)},\end{aligned}$$

then

$$\begin{aligned}\|A_1\nu\|\ &\leq\ a\ |\mu_\circ|+[a\ \sum_{k=1}^m\ |a_k|\ +1]\frac{(k|\nu|+\phi_1^*)T^\eta}{\Gamma(\eta+1)}=r_1,\\r_1\ &=\ \frac{a|\mu_\circ|+[a\sum_{k=1}^m\ |a_k|\ +1]\frac{\phi_1^*\ T^\eta}{\Gamma(\eta+1)}}{1-[a\sum_{k=1}^m\ |a_k|\ +1]\frac{k\ T^\eta}{\Gamma(\eta+1)}}.\end{aligned}$$

Also

$$\begin{aligned}|A_2\mu(\mathfrak{r})|\ &=\ |\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\varsigma)^{\sigma-1}}{\Gamma(\sigma)}\phi_2(\varsigma,\mu(\varphi(\varsigma)))d\varsigma|\\&\leq\ \int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\varsigma)^{\sigma-1}}{\Gamma(\sigma)}|\phi_2(\varsigma,\mu(\varphi(\varsigma)))|\ d\varsigma\\&\leq\ \int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\varsigma)^{\sigma-1}}{\Gamma(\sigma)}[a(\varsigma)+b\ |\mu(\varphi(\varsigma))|]d\varsigma.\end{aligned}$$

Hence

$$\begin{aligned}
\|A_2\mu\| &\leq \int_0^{\mathfrak{r}} a(\varsigma)\frac{(\mathfrak{r}-\varsigma)^{\sigma-1}}{\Gamma(\sigma)}d\varsigma + \int_0^{\mathfrak{r}} b\ |\mu(\varphi(\varsigma))|\frac{(\mathfrak{r}-\varsigma)^{\sigma-1}}{\Gamma(\sigma)}d\varsigma\\
&\leq I^{\sigma}a(\mathfrak{r}) + br_2\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\varsigma)^{\sigma-1}}{\Gamma(\sigma)}d\varsigma\\
&\leq I^{\sigma-\gamma}I^{\gamma}a(\mathfrak{r}) + br_2 I^{\sigma}(\mathfrak{r})\\
&\leq M\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\varsigma)^{\sigma-\gamma-1}}{\Gamma(\sigma-\gamma)}d\varsigma + br_2\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\varsigma)^{\sigma-1}}{\Gamma(\sigma)}d\varsigma\\
&\leq \frac{M\ \mathfrak{r}^{\sigma-\gamma}}{\Gamma(\sigma-\gamma+1)} + br_2\ \frac{\mathfrak{r}^{\sigma}}{\Gamma(\sigma+1)}\\
&\leq \frac{M\ T^{\sigma-\gamma}}{\Gamma(\sigma-\gamma+1)} + \frac{br_2\ T^{\sigma}}{\Gamma(\sigma+1)} = r_2,
\end{aligned}$$

$$r_2 = (1-\frac{b\ T^{\sigma}}{\Gamma(\sigma+1)})^{-1}\frac{M\ T^{\sigma-\gamma}}{\Gamma(\sigma-\gamma+1)}.$$

Now

$$\begin{aligned}
\|Au\|_X &= \|A_1\nu\|_C + \|A_2\mu\|_C \leq\ r_1 + r_2\\
&\leq \frac{a|\mu_\circ| + [a\sum_{k=1}^m |a_k| + 1]\frac{\phi_1^*}{\Gamma(\beta+1)}}{1-[a\sum_{k=1}^m |a_k|+1]\frac{k}{\Gamma(\beta+1)}} + (1-\frac{bT^{\sigma}}{\Gamma(\sigma+1)})^{-1}\frac{MT^{\sigma-\gamma}}{\Gamma(\sigma-\gamma+1)} = r.
\end{aligned}$$

Hence the class $\{Au\}$, $u\in Q_r$ is uniformly bounded for $AQ_r\subset Q_r$.

Currently, for $u=(\mu,\nu)\in Q_r$, for all $\epsilon>0$, $\delta>0$ and for each $\mathfrak{r}_1,\mathfrak{r}_2\in[0,T]$, $\mathfrak{r}_1<\mathfrak{r}_2$ such that $|\mathfrak{r}_2-\mathfrak{r}_1|<\delta$, we get

$$\begin{aligned}
&|A_1\nu(\mathfrak{r}_2) - A_1\nu(\mathfrak{r}_1)|\\
= &|a(\mu_\circ - \sum_{k=1}^m\ a_k\int_0^{\tau_k}\frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\varsigma,\nu(\varsigma))d\varsigma) + \int_0^{\mathfrak{r}_2}\frac{(\mathfrak{r}_2-\varsigma)^{\eta-1}}{\Gamma(\eta)}f_1(\varsigma,\nu(\varsigma))d\varsigma\\
- &a(\mu_\circ - \sum_{k=1}^m a_k\int_0^{\tau_k}\frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\varsigma,\nu(\varsigma))d\varsigma) + \int_0^{\mathfrak{r}_1}\frac{(\mathfrak{r}_1-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\varsigma,\nu(\varsigma))d\varsigma|\\
\leq &\int_{\mathfrak{r}_1}^{\mathfrak{r}_2}\frac{(\mathfrak{r}_2-\varsigma)^{\eta-1}}{\Gamma(\eta)}|\phi_1(\varsigma,\nu(\varsigma))|d\varsigma\\
+ &\int_0^{\mathfrak{r}_1}\left[\frac{(\mathfrak{r}_2-\varsigma)^{\eta-1}}{\Gamma(\eta)} - \frac{(\mathfrak{r}_1-\varsigma)^{\eta-1}}{\Gamma(\eta)}\right]|\phi_1(\varsigma,\nu(\varsigma))|d\varsigma\\
\leq &(k|\nu| + \phi_1^*)\frac{(\mathfrak{r}_2-\mathfrak{r}_1)^{\eta}}{\Gamma(\eta+1)} + (k|\nu|+\phi_1^*)\left(\frac{-(\mathfrak{r}_2-\mathfrak{r}_1)^{\eta}}{\Gamma(\eta+1)} + \frac{\mathfrak{r}_2^{\sigma}}{\Gamma(\eta+1)} - \frac{\mathfrak{r}_1^{\eta}}{\Gamma(\eta+1)}\right)\\
\leq &(k|\nu|+\phi_1^*)(\frac{\mathfrak{r}_2^{\eta}-\mathfrak{r}_1^{\eta}}{\Gamma(\eta+1)}),
\end{aligned}$$

and

$$
\begin{aligned}
& |A_2\mu(\mathfrak{r}_2) - A_2\mu(\mathfrak{r}_1)| \\
\leq\ & |\int_0^{\mathfrak{r}_2} \frac{(\mathfrak{r}_2-\varsigma)^{\sigma-1}}{\Gamma(\sigma)}\phi_2(\varsigma,\mu(\varphi(\varsigma)))d\varsigma - \int_0^{\mathfrak{r}_1} \frac{(\mathfrak{r}_1-\varsigma)^{\sigma-1}}{\Gamma(\sigma)}\phi_2(\varsigma,\mu(\varphi(\varsigma)))d\varsigma| \\
\leq\ & |\int_0^{\mathfrak{r}_2} \frac{(\mathfrak{r}_2-\varsigma)^{\sigma-1}}{\Gamma(\sigma)}\phi_2(\varsigma,\mu(\varphi(\varsigma)))d\varsigma - \int_0^{\mathfrak{r}_1} \frac{(\mathfrak{r}_2-\varsigma)^{\sigma-1}}{\Gamma(\sigma)}\phi_2(\varsigma,\mu(\varphi(\varsigma)))d\varsigma| \\
+\ & |\int_0^{\mathfrak{r}_1} \frac{(\mathfrak{r}_2-\varsigma)^{\sigma-1}}{\Gamma(\sigma)}\phi_2(\varsigma,\mu(\varphi(\varsigma)))d\varsigma - \int_0^{\mathfrak{r}_1} \frac{(\mathfrak{r}_1-\varsigma)^{\sigma-1}}{\Gamma(\sigma)}\phi_2(\varsigma,\mu(\varphi(\varsigma)))d\varsigma| \\
\leq\ & |\int_{\mathfrak{r}_1}^{\mathfrak{r}_2} \frac{(\mathfrak{r}_2-\varsigma)^{\sigma-1}}{\Gamma(\sigma)}\phi_2(\varsigma,\mu(\varphi(\varsigma)))d\varsigma| + |\int_0^{\mathfrak{r}_1} \frac{(\mathfrak{r}_2-\varsigma)^{\sigma-1}}{\Gamma(\sigma)}\phi_2(\varsigma,\mu(\varphi(\varsigma)))d\varsigma| \\
-\ & |\int_0^{\mathfrak{r}_1} \frac{(\mathfrak{r}_1-\varsigma)^{\sigma-1}}{\Gamma(\sigma)}\phi_2(\varsigma,\mu(\varphi(\varsigma)))d\varsigma| \\
\leq\ & \int_{\mathfrak{r}_1}^{\mathfrak{r}_2} \frac{(\mathfrak{r}_2-\varsigma)^{\sigma-1}}{\Gamma(\sigma)}|\phi_2(\varsigma,\mu(\varphi(\varsigma)))|d\varsigma \\
+\ & \int_0^{\mathfrak{r}_1} \frac{(\mathfrak{r}_2-\varsigma)^{\sigma-1}-(\mathfrak{r}_1-\varsigma)^{\sigma-1}}{\Gamma(\sigma)}|\phi_2(\varsigma,\mu(\varphi(\varsigma)))|d\varsigma \\
\leq\ & \int_{\mathfrak{r}_1}^{\mathfrak{r}_2} [a+b|\mu(\varphi(\varsigma))|]\frac{(\mathfrak{r}_2-\varsigma)^{\sigma-1}}{\Gamma(\sigma)}d\varsigma \\
+\ & \int_0^{\mathfrak{r}_1} [a+b|\mu(\varphi(s))|]\frac{(\mathfrak{r}_2-\varsigma)^{\sigma-1}-(\mathfrak{r}_1-\varsigma)^{\sigma-1}}{\Gamma(\sigma)}d\varsigma \\
\leq\ & (a+br_2)\int_{\mathfrak{r}_1}^{\mathfrak{r}_2} \frac{(\mathfrak{r}_2-\varsigma)^{\sigma-1}}{\Gamma(\sigma)}ds + (a+br_2)\int_0^{\mathfrak{r}_1} \frac{(\mathfrak{r}_2-\varsigma)^{\sigma-1}-(\mathfrak{r}_1-\varsigma)^{\sigma-1}}{\Gamma(\sigma)}d\varsigma \\
\leq\ & (a+br_2)\frac{(\mathfrak{r}_2-\mathfrak{r}_1)^{\sigma}}{\Gamma(\sigma+1)} + (a+br_2)\left(\frac{-(\mathfrak{r}_2-\mathfrak{r}_1)^{\sigma}}{\Gamma(\sigma+1)} + \frac{\mathfrak{r}_2^{\sigma}}{\Gamma(\sigma+1)} - \frac{\mathfrak{r}_1^{\sigma}}{\Gamma(\sigma+1)}\right) \\
\leq\ & (a+br_2)\frac{(\mathfrak{r}_2^{\sigma}-\mathfrak{r}_1^{\sigma})}{\Gamma(\sigma+1)}.
\end{aligned}
$$

For the operator A and $u \in Q_r$, we have

$$
\begin{aligned}
Au(\mathfrak{r}_2) - Au(\mathfrak{r}_1) &= A(\mu,\nu)(\mathfrak{r}_2) - A(\mu,\nu)(\mathfrak{r}_1) \\
&= (A_2\mu(\mathfrak{r}_2), A_1\nu(\mathfrak{r}_2)) - (A_2\mu(\mathfrak{r}_1), A_1\nu(\mathfrak{r}_1)) \\
&= (A_2\mu(\mathfrak{r}_2) - A_2\mu(\mathfrak{r}_1), A_1\nu(\mathfrak{r}_2) - A_1\nu(\mathfrak{r}_1)),
\end{aligned}
$$

then

$$
\begin{aligned}
|Au(\mathfrak{r}_2) - Au(\mathfrak{r}_1)|_X &= |A(\mu,y)(\mathfrak{r}_2) - A(\mu,y)(\mathfrak{r}_1)|_X, \\
&= |A_1\nu(\mathfrak{r}_2) - A_1\nu(\mathfrak{r}_2)|_C + |A_2\mu(\mathfrak{r}_2) - A_2\mu(\mathfrak{r}_1)|_C \\
&= (k|\nu| + \phi_1^*)\frac{(\mathfrak{r}_2^{\eta}-\mathfrak{r}_1^{\eta})}{\Gamma(\eta+1)} \ +\ (a+br_2)\ \frac{(\mathfrak{r}_2^{\sigma}-\mathfrak{r}_1^{\sigma})}{\Gamma(\sigma+1)}.
\end{aligned}
$$

As a result, the class of functions $\{Au\}$ is equi-continuous on Q_r. The operator A is compact as a result of the Arzela-Ascoli Theorem [14]. The continuity of $A : Q_r \to Q_r$ still needs to be proven. Let $u_n = (\mu_n, \nu_n)$ be a sequence in Qr with $\mu_n \to \mu$, and $y_n \to y$ and since $\phi_2(\mathfrak{r}, \mu(\mathfrak{r}))$ is continuous in $C(I, \mathbb{R})$, then $\phi_2(\mathfrak{r}, \mu_n(\mathfrak{r}))$ converges to $\phi_2(\mathfrak{r}, \mu(\mathfrak{r}))$, thus $\phi_2(\mathfrak{r}, \mu_n(\varphi(\mathfrak{r})))$ converges to $\phi_2(\mathfrak{r}, \mu(\varphi(\mathfrak{r})))$, using assumptions (iii)-(iv) and applying the Lebesgue Dominated Convergence Theorem, we get

$$\lim_{n\to\infty} \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\sigma-1}}{\Gamma(\sigma)}\ \phi_2(\varsigma, \mu_n(\varphi(\varsigma)))\ d\varsigma\ =\ \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\sigma-1}}{\Gamma(\sigma)}\ \phi_2(\varsigma, \mu(\varphi(\varsigma)))\ d\varsigma,$$

then

$$\begin{aligned}
\lim_{n\to\infty} A_2\mu_n(\mathfrak{r}) &= \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\sigma-1}}{\Gamma(\sigma)} \lim_{n\to\infty} \phi_2(\varsigma, \mu_n(\varphi(\varsigma)))d\varsigma \\
&= \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\sigma-1}}{\Gamma(\sigma)} \phi_2(\varsigma, \mu(\varphi(\varsigma)))d\varsigma = A_2\mu(\mathfrak{r}) \\
\lim_{n\to\infty} A_1\nu_n(\mathfrak{r}) &= a(\mu_\circ - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)} \lim_{n\to\infty} \phi_1(\varsigma, \nu_n(\varsigma))d\varsigma) \\
&+ \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)} \lim_{n\to\infty} \phi_1(\varsigma, \nu_n(\varsigma))d\varsigma \\
&= a(x_\circ - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\varsigma, \nu(\varsigma))\ d\varsigma) \\
&+ \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\varsigma, \nu(\varsigma))d\varsigma\ =\ A_1\nu(\mathfrak{r}).
\end{aligned}$$

So,

$$\begin{aligned}
\lim_{n\to\infty} Au_n(\mathfrak{r}) &= \lim_{n\to\infty} (A_1\nu_n(\mathfrak{r}), A_2\mu_n(\mathfrak{r})) \\
&= (\lim_{n\to\infty} A_1\nu_n(\mathfrak{r}), \lim_{n\to\infty} A_2\mu_n(\mathfrak{r})) = (A_1\nu(\mathfrak{r}), A_2\mu(\mathfrak{r}))\ =\ Au(\mathfrak{r}).
\end{aligned}$$

Then $Au_n \to Au$ as $n \to \infty$. The operator A is continuous as a result. While all criteria of the Schauder fixed-point theorem [6] have achieved, then A has a fixed point $u \in Q_r$, and then Problems (14)-(15) have at least one continuous solutions $u = (\mu, \nu) \in Q_r$, $\mu; \nu \in C(I, \mathbb{R})$.

Therefore, there is at least one solution $\mu \in C(I, \mathbb{R})$ to the functional integral Equation (1).

Conversely, by differentiating (6), we get

$$D^{\eta}\mu(\mathfrak{r}) = D^{\eta}\Bigg\{ a\left(\mu_{\circ} - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\varsigma,\nu(\varsigma))d\varsigma\right) + \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\varsigma,\nu(\varsigma))d\varsigma\Bigg\},$$

$$\nu(t) = I^{\sigma}\phi_2(\mathfrak{r},\mu(\varphi(\mathfrak{r}))).$$

Additionally, we derive from the integral equation (14)–(15)

$$\begin{aligned} \mu(\tau_k) =&\ a\left(\mu_{\circ} - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\varsigma,\nu(\varsigma))\ d\varsigma\right) \\ &+ \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\varsigma,\nu(\varsigma))\ d\varsigma, \\ \mu(0) =&\ a\left(\mu_{\circ} - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\varsigma,\nu(\varsigma))\ d\varsigma\right), \\ \nu(\mathfrak{r}) =&\ I^{\sigma}\phi_2(\mathfrak{r},\mu(\varphi(\mathfrak{r}))), \end{aligned} \tag{16}$$

and

$$\begin{aligned} \sum_{k=1}^{m} a_k\mu(\tau_k) =&\ a\sum_{k=1}^{m} a_k\left(\mu_{\circ} - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\beta)}\phi_1(\varsigma,\nu(\varsigma))d\varsigma\right) \\ &+ \sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\beta)}\phi_1(\varsigma,\nu(\varsigma))d\varsigma, \\ \nu(\mathfrak{r}) =&\ I^{\sigma}\phi_2(\mathfrak{r},\mu(\varphi(\mathfrak{r}))). \end{aligned} \tag{17}$$

From (16) and (17), we have

$$\mu(0) + \sum_{k=1}^{m} a_k\mu(\tau_k) = a(1+\sum_{k=1}^{m} a_k)\left(\mu_{\circ} - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\beta-1}}{\Gamma(\eta)}\phi_1(\varsigma,\nu(\varsigma))d\varsigma\right) + \sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\beta-1}}{\Gamma(\eta)}\phi_1(\varsigma,\nu(\varsigma))d\varsigma.$$

Then

$$\mu(0) + \sum_{k=1}^{m} a_k\ \mu(\tau_k) = \mu_{\circ}.$$

Consequently, the nonlocal problem of functional differential inclusions (1)-(4) have at least one solution $\mu \in C(I,\mathbb{R})$. □

2.1. **Riemann-Stieltjes integral BCs** (2)**.** Let $\mu \in C(I, \mathbb{R})$ represent the solution to the non localproblem of $(1)-(4)$. Let $a_k = h(\mathfrak{r}_k) - h(\mathfrak{r}_{k-1})$, the function h is nondecreasing, $\tau_k \in (\mathfrak{r}_{k-1}, \mathfrak{r}_k)$, $0 = \mathfrak{r}_0 < \mathfrak{r}_1 < \mathfrak{r}_2 \cdots < T$. The nonlocal condition (4) will then take the following form

$$\mu(0) + \sum_{k=1}^{m} \mu(\tau_k)\ (h(\mathfrak{r}_k) - h(\mathfrak{r}_{k-1}))\ = \mu_\circ.$$

We derive from [14] as $m \to \infty$ the continuation of the solution of the nonlocal problem $(1)-(4)$.

$$\lim_{m\to\infty} \sum_{k=1}^{m} \mu(\tau_k)\ (h(\mathfrak{r}_k) - h(\mathfrak{r}_{k-1}))\ = \int_0^T \mu(\varsigma)\ dh(\varsigma),$$

that is, the nonlocal conditions (4) is a modification to the Riemann-Steltjes integral condition as $m \to \infty$

$$\mu(0) + \lim_{m\to\infty} \sum_{k=1}^{m} \mu(\tau_k)\ (h(\mathfrak{r}_k) - h(\mathfrak{r}_{k-1}))\ = \mu(0) + \int_0^T \mu(\varsigma)\ dh(\varsigma) = \mu_\circ.$$

Theorem 2.4. *Assume that assumptions (i)–(iv) of Theorem 2.3 hold and $h : I \to I$ is an increasing function, then the Riemann-Stieltjes functional integral condition* (2) *and the nonlocal problems* (1) *have a solution $\mu \in C(I, \mathbb{R})$ that is represented by*

$$\begin{aligned}
(18)\qquad \mu(\mathfrak{r}) &= (1 + h(T) - h(0))^{-1}\mu_\circ - (1 + h(T) - h(0))^{-1} \\
(19)\qquad &\times \int_0^T \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\varsigma, I^\sigma \phi_2(\varsigma, \mu(\varphi(\varsigma))))d\varsigma\ dh(\varsigma) \\
&+ \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\varsigma, I^\sigma \phi_2(\varsigma, \mu(\varphi(\varsigma))))ds.
\end{aligned}$$

Proof. The nonlocal problem (1) – (4) will have the following solution as $m \to \infty$:

$$\begin{aligned}
&\mu(\mathfrak{r}) \\
= &\lim_{m\to\infty} \frac{1}{(1+\sum_{k=1}^{m} a_k)}\big(\mu_\circ - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\varsigma, I^\sigma\phi_2(\varsigma,\mu(\varphi(\varsigma))))d\varsigma\big) \\
+ &\int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\varsigma, I^\sigma\phi_2(\varsigma,\mu(\varphi(\varsigma))))d\varsigma \\
= &\frac{1}{(1+h(T)-h(0))}\big(\mu_\circ - \lim_{m\to\infty}\sum_{k=1}^{m}(h(\mathfrak{r}_k)-h(\mathfrak{r}_{k-1})) \\
\times &\int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\varsigma, I^\sigma\phi_2(\varsigma,\mu(\varphi(\varsigma))))d\varsigma\big) \\
+ &\int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\varsigma, I^\sigma\phi_2(\varsigma,\mu(\varphi(\varsigma))))\ d\varsigma \\
= &\frac{1}{(1+h(T)-h(0))}\big(\mu_\circ - \int_0^{T}\int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\varsigma, I^\sigma\phi_2(\varsigma,\mu(\varphi(\varsigma))))d\varsigma dh\big) \\
+ &\int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\varsigma, I^\sigma\phi_2(\varsigma,\mu(\varphi(\varsigma))))\ d\varsigma.
\end{aligned}$$

□

As a result, the solution $\mu \in C(I,\mathbb{R})$ of the first-order nonlinear differential Equation (1) with the Riemann-Stieltjes integral condition (2) is represented by (18).
Consequently, there exists at least one solution $\mu \in C(I,\mathbb{R})$ of the nonlocal problem of functional differential inclusion (1)-(2).

2.2. **Infinite-point boundary condition** (3)**.** Take into account that $\mu \in C(I,\mathbb{R})$ is the solution to the nonlocal problem presented by (1) and (3).

Theorem 2.5. *Assume assumptions (i)–(iv) of Theorem 2.3 hold and let $B_m^{-1} = 1+\sum_{k=1}^{m} a_k$ is convergent sequence. Then the nonlocal problem of (1)-(3) represented by the integral equation*

$$\begin{aligned}
\mu(\mathfrak{r}) = B_m\mu_\circ - B_m\sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\varsigma, I^\sigma\phi_2(\varsigma,\mu(\varphi(\varsigma))))d\varsigma \\
+ \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\varsigma, I^\sigma\phi_2(\varsigma,\mu(\varphi(\varsigma))))d\varsigma,
\end{aligned} \tag{20}$$

has at least one solution $\mu \in C(I,\mathbb{R})$.

Proof. Let assumptions of Theorem 2.3 be satisfied, and let $\sum_{k=1}^{m} a_k$ be convergent. Then

(21)

$$\mu_m(\mathfrak{t}) = \frac{1}{(1+\sum_{k=1}^{m} a_k)}\left(\mu_\circ - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\varsigma, I^\sigma\phi_2(\varsigma,\mu(\varphi(\varsigma))))d\varsigma\right) + \int_0^{\mathfrak{t}} \frac{(\mathfrak{t}-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\varsigma, I^\sigma\phi_2(\varsigma,\mu_m(\varphi(\varsigma))))d\varsigma.$$

Consider the limit to (21), as $m\to\infty$, we obtain

$$\begin{aligned} &\lim_{m\to\infty}\mu_m(\mathfrak{t}) \\ =\ & \lim_{m\to\infty}\Bigg[\frac{1}{(1+\sum_{k=1}^{m} a_k)}\Big(\mu_\circ - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\varsigma, I^\sigma\phi_2(\varsigma,\mu(\varphi(\varsigma))))d\varsigma\Big) \\ +\ & \int_0^{\mathfrak{t}} \frac{(\mathfrak{t}-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\varsigma, I^\sigma\phi_2(\varsigma,\mu_m(\varphi(\varsigma))))\ d\varsigma\Bigg] \\ =\ & \lim_{m\to\infty}\frac{1}{(1+\sum_{k=1}^{m} a_k)}\left[\mu_\circ - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\varsigma, I^\sigma\phi_2(\varsigma,\mu(\varphi(\varsigma))))d\varsigma\right] \\ +\ & \lim_{m\to\infty}\int_0^{\mathfrak{t}} \frac{(\mathfrak{t}-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\varsigma, I^\sigma\phi_2(\varsigma,\mu_m(\varphi(\varsigma))))\ d\varsigma. \end{aligned}$$

Now $|a_k\mu(\tau_k)| \le |a_k|\|\mu\|$, so using a comparison test $\sum_{k=1}^{m} a_k\mu(\tau_k)$ is convergent. Also

$$\begin{aligned} &\left|\int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\varsigma, I^\sigma\phi_2(\varsigma,\mu(\varphi(\varsigma))))\ d\varsigma\right| \\ \le\ & \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}(k\ |I^\sigma\phi_2(\varsigma,\mu(\varphi(\varsigma)))|\ + \phi_1^*)d\varsigma \\ \le\ & k\ \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}[I^\sigma a(\varsigma) + b\int_0^{\varsigma}\frac{(\varsigma-\theta)^{\sigma-1}}{\Gamma(\sigma)}|\mu(\varphi(\theta))|d\theta\ ]d\varsigma \\ +\ & \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1^* d\varsigma \\ \le\ & k\int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}[I^{\sigma-\gamma}I^\gamma a(\varsigma) + br_2\int_0^{\varsigma}\frac{(\varsigma-\theta)^{\sigma-1}}{\Gamma(\sigma)}d\theta]d\varsigma + \phi_1^*\frac{T^\eta}{\Gamma(\eta+1)} \\ \le\ & k\ M\int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\int_0^{\varsigma}\frac{(\varsigma-\theta)^{\sigma-\gamma-1}}{\Gamma(\sigma-\gamma)}d\theta\ d\varsigma \\ +\ & k\ br_2\int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\int_0^{\varsigma}\frac{(\varsigma-\theta)^{\sigma-1}}{\Gamma(\sigma)}d\theta\ d\varsigma + \frac{\phi_1^*\ T^\eta}{\Gamma(\eta+1)} \\ \le\ & \frac{T^\eta}{\Gamma(\eta+1)}\left[\frac{k\ M\ T^{\sigma-\gamma+1}}{\Gamma(\sigma-\gamma+1)} + \frac{k\ br_2\ T^{\sigma+1}}{\Gamma(\sigma+1)} +\ \phi_1^*\right] \le N, \end{aligned}$$

then

$$|a_k| \mid \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\varsigma, I^\sigma \phi_2(\varsigma, \mu(\varphi(\varsigma))))\ d\varsigma \mid \leq |a_k|\ N,$$

and by the comparison test $\sum_{k=1}^m a_k \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\varsigma, I^\sigma \phi_2(\varsigma, \mu(\varphi(\varsigma))))\ d\varsigma$ is convergent.

Using assumptions $(i)-(iii)$ and applying Lebesgue Dominated convergence Theorem [2], from (20) we obtain (22). Furthermore, from (20), we have

$$\begin{aligned}
&(1+\sum_{k=1}^m a_k)\mu(\tau_k) \\
= \ & B_m^{-1}B_m x_\circ - B_m^{-1}B_m \sum_{k=1}^m a_k \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\varsigma, I^\sigma \phi_2(\varsigma, \mu(\varphi(\varsigma))))d\varsigma \\
+ \ & (1+\sum_{k=1}^m a_k) \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\varsigma, I^\sigma \phi_2(\varsigma, \mu(\varphi(\varsigma))))d\varsigma \\
= \ & \mu_\circ - \sum_{k=1}^m a_k \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\varsigma, I^\sigma \phi_2(\varsigma, \mu(\varphi(\varsigma))))\ d\varsigma \\
+ \ & \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\varsigma, I^\sigma \phi_2(\varsigma, \mu(\varphi(\varsigma))))d\varsigma \\
+ \ & \sum_{k=1}^m a_k \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\varsigma, I^\sigma \phi_2(\varsigma, \mu(\varphi(\varsigma))))d\varsigma,
\end{aligned}$$

$$\sum_{k=1}^m a_k\ \mu(\tau_k) = \mu_\circ - \mu(\tau_k) + \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\varsigma, I^\sigma \phi_2(\varsigma, \mu(\varphi(\varsigma))))d\varsigma. \tag{22}$$

From (6), we obtain

$$\mu(0) = a\left(\mu_\circ\ -\ \sum_{k=1}^m a_k \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\varsigma, I^\sigma \phi_2(\varsigma, \mu(\varphi(\varsigma))))d\varsigma\right)$$

and

$$\begin{aligned}
\mu(\tau_k) = a\Big(\mu_\circ - \sum_{k=1}^m a_k \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\varsigma, I^\sigma \phi_2(\varsigma, \mu(\varphi(\varsigma))))d\varsigma\Big) \\
+ \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\varsigma, I^\sigma \phi_2(\varsigma, \mu(\varphi(\varsigma))))d\varsigma.
\end{aligned}$$

So

$$\mu(0) = \mu(\tau_k) - \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\varsigma, I^\sigma \phi_2(\varsigma, \mu(\varphi(\varsigma))))d\varsigma.$$

Going back to (22) we obtain infinite-point BC (3)

$$\mu(0) \ + \ \sum_{k=1}^{\infty} a_k \ \mu(\tau_k) = \mu_\circ.$$

Consequently, the nonlocal problem of functional differential inclusion (1)-(3) has at least one solution $x \in C(I, \mathbb{R})$. □

3. Existence of unique solutions

The necessary condition for the uniqueness result for nonlocal problems (1)-(4) is provided in this section. Assume the following assumption.

$(iii)^*$ Suppose that $\phi_2 : I \times \mathbb{R} \to \mathbb{R}$, is a continuous function that satisfies the Lipschitz condition, with $|\phi_2(\mathfrak{r}, \mu) - \phi_2(\mathfrak{r}, \nu)| \leq c \ |\mu - \nu|$.

Theorem 3.1. *Assume that assumptions of Theorem 2.3 hold with condition (iii) replaced by $(iii)^*$, if*

$$\frac{(\ a \ \sum_{k=1}^{m} \ a_k + 1) \ T^{\eta+1} \ T^{\sigma+1} \ k \ c}{\Gamma(\eta+1)\Gamma(\sigma+1)} < 1.$$

Then the nonlocal problem (1)-(4) has a unique solution $x \in C(I, \mathbb{R})$.

Proof. let $\mu_1(\mathfrak{r})$ and $\mu_2(\mathfrak{r})$ be two solutions of the functional integral equation (6). Then

$$\begin{aligned}
&\mu_1(\mathfrak{r}) - \mu_2(\mathfrak{r}) \\
= \ & a \left(\mu_\circ - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k - \varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\varsigma, I^\sigma \phi_2(\varsigma, \mu_1(\varphi(\varsigma))))d\varsigma \right) \\
+ \ & \int_0^{\mathfrak{r}} \frac{(\mathfrak{r} - \varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\varsigma, I^\sigma \phi_2(\varsigma, \mu_1(\varphi(\varsigma))))d\varsigma \\
- \ & a \left(\mu_\circ - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k - \varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\varsigma, I^\sigma \phi_2(\varsigma, \mu_2(\varphi(\varsigma))))d\varsigma \right) \\
- \ & \int_0^{\mathfrak{r}} \frac{(\mathfrak{r} - \varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\varsigma, I^\sigma \phi_2(\varsigma, \mu_2(\varphi(\varsigma))))d\varsigma \\
&|\mu_1(\mathfrak{r}) - x_2(\mathfrak{r})| \\
\leq \ & a \Big| \sum_{k=1}^{m} a_k \ (\int_0^{\tau_k} \frac{(\tau_k - \varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\varsigma, I^\sigma \phi_2(\varsigma, \mu_2(\varphi(\varsigma))))d\varsigma \\
- \ & \int_0^{\tau_k} \frac{(\tau_k - \varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\varsigma, I^\sigma \phi_2(\varsigma, \mu_1(\varphi(\varsigma)))) \ d\varsigma) \Big| \\
+ \ & | \int_0^{\mathfrak{r}} \frac{(\mathfrak{r} - \varsigma)^{\eta-1}}{\Gamma(\eta)} \big[\phi_1(\varsigma, I^\sigma \phi_2(\varsigma, \mu_1(\varphi(\varsigma)))) - \phi_1(\varsigma, I^\sigma \phi_2(\varsigma, \mu_2(\varphi(\varsigma)))) \big] d\varsigma |
\end{aligned}$$

$$
\begin{aligned}
\leq\ & a\sum_{k=1}^{m} a_k\Big(\int_0^{\tau_k}\frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}|\phi_1(\varsigma, I^\sigma\phi_2(\varsigma,\mu_2(\varphi(\varsigma))))\\
-\ & \phi_1(\varsigma, I^\sigma\phi_2(\varsigma,\mu_1(\varphi(\varsigma))))|d\varsigma\Big)\\
+\ & \int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)}|\phi_1(\varsigma, I^\sigma\phi_2(\varsigma,\mu_1(\varphi(\varsigma))))-\phi_1(\varsigma, I^\sigma\phi_2(\varsigma,\mu_2(\varphi(\varsigma))))|d\varsigma.
\end{aligned}
$$

Lipschitz condition for ϕ_1 allows us to obtain

$$
\begin{aligned}
& |\mu_1(\mathfrak{r})-\mu_2(\mathfrak{r})|\\
\leq\ & a\sum_{k=1}^{m} a_k\ k\int_0^{\tau_k}\frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}|I^\sigma\phi_2(\varsigma,\mu_1(\varphi(\varsigma)))-I^\sigma\phi_2(\varsigma,\mu_2(\varphi(\varsigma)))|\ d\varsigma\\
+\ & k\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)}|I^\sigma\phi_2(\varsigma,\mu_1(\varphi(\varsigma)))-I^\sigma\phi_2(\varsigma,\mu_2(\varphi(\varsigma)))|\ d\varsigma\\
\leq\ & a\sum_{k=1}^{m} a_k k\Big(\int_0^{\tau_k}\frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\int_0^{\varsigma}\frac{(\varsigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}|\phi_2(\tau,\mu_1(\varphi(\tau)))\\
-\ & \phi_2(\tau,\mu_2(\varphi(\tau)))|d\tau d\varsigma\\
+\ & k\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)}\int_0^{\varsigma}\frac{(\varsigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}|\phi_2(\tau,\mu_1(\varphi(\tau)))-\phi_2(\tau,\mu_2(\varphi(\tau)))|d\tau d\varsigma.
\end{aligned}
$$

Lipschitz condition for ϕ_2 allows us to obtain

$$
\begin{aligned}
& |\mu_1(\mathfrak{r})-\mu_2(\mathfrak{r})|\\
\leq\ & a\sum_{k=1}^{m} a_k k\ c\int_0^{\tau_k}\frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\int_0^{\varsigma}\frac{(\varsigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}|\mu_1(\varphi(\tau))-\mu_2(\varphi(\tau))|d\tau\ d\varsigma\\
+\ & k\ c\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)}\int_0^{\varsigma}\frac{(\varsigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}|\mu_1(\varphi(\tau))-\mu_2(\varphi(\tau))\ |d\tau\ d\varsigma\\
\leq\ & a\sum_{k=1}^{m} a_k kc\|\mu_1-\mu_2\|\int_0^{\tau_k}\frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\int_0^{\varsigma}\frac{(\varsigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}d\tau d\varsigma\\
+\ & k\ c\ \|\mu_1-\mu_2\|\int_0^{t}\frac{(t-\varsigma)^{\eta-1}}{\Gamma(\eta)}\int_0^{\varsigma}\frac{(\varsigma-\tau\ )^{\sigma-1}}{\Gamma(\sigma)}\ d\tau\ d\varsigma.
\end{aligned}
$$

Then

$$
\|\mu_1-\mu_2\| \quad\leq\quad \frac{(\ a\ \sum_{k=1}^{m}\ a_k+1)\ T^{\eta+1}\ T^{\sigma+1}\ k\ c}{\Gamma(\eta+1)\Gamma(\sigma+1)}\|\mu_1-\mu_2\|.
$$

Hence

$$
\left(1-\frac{(\ a\ \sum_{k=1}^{m}\ a_k+1)\ T^{\eta+1}\ T^{\sigma+1}\ k\ c}{\Gamma(\eta+1)\Gamma(\sigma+1)}\right)\|\mu_1-\mu_2\| \quad\leq\quad 0.
$$

Since $\frac{(\ a\ \sum_{k=1}^{m}\ a_k+1)\ T^{\eta+1}\ T^{\sigma+1}\ k\ c}{\Gamma(\eta+1)\Gamma(\sigma+1)} < 1$, we have $\mu_1(\mathfrak{r}) = \mu_2(\mathfrak{r})$, therefore the solution of the integral equation (6) is unique, and consequence the integral equation (6) has a unique solution, and as a result, this establishes the existence of unique solutions to the nonlocal problem (1)-(4). □

3.1. **Continuous dependence.**

Theorem 3.2. *Assume assumptions of Theorem 3.1 hold. Then the solution of the nonlocal problem* (1)-(4) *is continuously dependent on the* S_{Φ_1} *the set of all Lipschitzian selections of* Φ_1.

Proof. Let $\phi_1(\mathfrak{r}, \mu(\mathfrak{r}))$ and $\phi_1^*(\mathfrak{r}, \mu(\mathfrak{r}))$ be two separate Lipschitzian selections of $\Phi_1(\mathfrak{r}, \mu(\mathfrak{r}))$, so that

$$|\phi_1(\mathfrak{r}, \mu(\mathfrak{r})) - \phi_1^*(\mathfrak{r}, \mu(\mathfrak{r})| < \epsilon, \ \epsilon > 0, \ \mathfrak{r} \in I,$$

then we have the following for the two related solutions $\mu_{\phi_1}(\mathfrak{r})$ and $\mu_{\phi_1^*}(\mathfrak{r})$ of (6).

$$\begin{aligned}
& |\mu_{\phi_1}(\mathfrak{r}) - \mu^*_{\phi_1}(\mathfrak{r})| \\
= \ & |a\sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k - \varsigma)^{\eta-1}}{\Gamma(\eta)} \big[\phi_1(\varsigma, I^\sigma \phi_2(\varsigma, \mu(\varphi(\varsigma))) - \phi_1^*(\mathfrak{r}, I^\sigma \phi_2(\varsigma, \mu^*(\varphi(\varsigma)))\big] d\varsigma \\
+ \ & \int_0^{\mathfrak{r}} \frac{(\mathfrak{r} - \varsigma)^{\eta-1}}{\Gamma(\eta)} \big[\phi_1(\varsigma, I^\sigma \phi_2(\varsigma, \mu(\varphi(\varsigma))) - \phi_1^*(\mathfrak{r}, I^\sigma \phi_2(\varsigma, \mu^*(\varphi(\varsigma)))\big] \ d\varsigma \ | \\
\leq \ & a\sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k - \varsigma)^{\eta-1}}{\Gamma(\eta)} |\phi_1(\varsigma, I^\sigma \phi_2(\varsigma, \mu(\varphi(\varsigma))) - \phi_1^*(\mathfrak{r}, I^\sigma \phi_2(\varsigma, \mu^*(\varphi(\varsigma)))| d\varsigma \\
+ \ & \int_0^{\mathfrak{r}} \frac{(\mathfrak{r} - \varsigma)^{\eta-1}}{\Gamma(\eta)} |\phi_1(\varsigma, I^\sigma \phi_2(\varsigma, \mu^*(\varphi(\varsigma))) - \phi_1^*(\varsigma, I^\sigma \phi_2(\varsigma, \mu^*(\varphi(\varsigma)))| d\varsigma \\
\leq \ & a\sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k - \varsigma)^{\eta-1}}{\Gamma(\eta)} |\phi_1(\varsigma, I^\sigma \phi_2(\varsigma, \mu(\varphi(\varsigma))) - \phi_1(\varsigma, I^\sigma \phi_2(\varsigma, \mu^*(\varphi(\varsigma)))| \ d\varsigma \\
+ \ & a\sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k - \varsigma)^{\eta-1}}{\Gamma(\eta)} |\phi_1(\varsigma, I^\sigma \phi_1(\varsigma, \mu^*(\varphi(\varsigma))) - \phi_1^*(\varsigma, I^\sigma \phi_2(\varsigma, \mu^*(\varphi(\varsigma)))| d\varsigma \\
+ \ & \int_0^{\mathfrak{r}} \frac{(\mathfrak{r} - \varsigma)^{\eta-1}}{\Gamma(\eta)} |\phi_1(\varsigma, I^\sigma \phi_2(\varsigma, \mu(\varphi(\varsigma))) - \phi_1(\varsigma, I^\sigma \phi_2(\varsigma, \mu^*(\varphi(\varsigma)))| \ d\varsigma \\
+ \ & \int_0^{\mathfrak{r}} \frac{(\mathfrak{r} - \varsigma)^{\eta-1}}{\Gamma(\eta)} |\phi_1(\varsigma, I^\sigma \phi_1(\varsigma, \mu^*(\varphi(\varsigma))) - \phi_1^*(\varsigma, I^\sigma \phi_2(\varsigma, \mu^*(\varphi(\varsigma)))| \ d\varsigma \\
\leq \ & a\sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k - \varsigma)^{\eta-1}}{\Gamma(\eta)} (|\phi_1(\varsigma, I^\sigma \phi_2(\varsigma, \mu(\varphi(\varsigma))) - \phi_1(\varsigma, I^\sigma \phi_2(\varsigma, \mu^*(\varphi(\varsigma)))| + \delta) d\varsigma \\
+ \ & \int_0^{\mathfrak{r}} \frac{(\mathfrak{r} - \varsigma)^{\eta-1}}{\Gamma(\eta)} |\phi_1(\varsigma, I^\sigma \phi_2(\varsigma, \mu(\varphi(\varsigma))) - \phi_1(\varsigma, I^\sigma \phi_2(\varsigma, \mu^*(\varphi(\varsigma)))| \ d\varsigma \\
+ \ & \int_0^{\mathfrak{r}} \frac{(\mathfrak{r} - \varsigma)^{\eta-1}}{\Gamma(\eta)} \delta \ d\varsigma
\end{aligned}$$

$$
\begin{aligned}
\leq \quad & a \sum_{k=1}^{m} a_k \ k \ (\int_0^{\mathfrak{r}_k} \ |I^\sigma \phi_2(\mathfrak{r}, \mu(\varphi(\mathfrak{r}))) - I^\sigma \phi_2(\mathfrak{r}, \mu^*(\varphi(\mathfrak{r})))| \ d\varsigma + \delta \ \frac{T^\eta}{\Gamma(\eta+1)}) \\
+ \quad & k \ \int_0^{\mathfrak{r}} \ \frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)} |I^\sigma \phi_2(\mathfrak{r}, \mu(\varphi(\mathfrak{r}))) - I^\sigma \phi_2(\mathfrak{r}, \mu^*(\varphi(\mathfrak{r})))| \ d\varsigma + \ \frac{\delta \ T^\eta}{\Gamma(\eta+1)} \\
\leq \quad & a \sum_{k=1}^{m} a_k k (\int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)} \int_0^{\varsigma} \frac{(\varsigma-\tau)^{\sigma-1}}{\Gamma(\sigma)} |\phi_2(\tau, \mu(\varphi(\tau))) - \phi_2(\tau, \mu^*(\varphi(\tau)))| d\tau \ d\varsigma \\
+ \quad & \frac{\delta \ T^\eta}{\Gamma(\eta+1)}) \\
+ \quad & k \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)} \int_0^{\varsigma} \ \frac{(\varsigma-\tau)^{\sigma-1}}{\Gamma(\sigma)} |\phi_2(\tau, \mu(\varphi(\tau))) - \phi_2(\tau, \mu^*(\varphi(\tau)))| \ d\tau \ d\varsigma \\
+ \quad & \frac{\delta \ T^\eta}{\Gamma(\eta+1)} \\
\leq \quad & a \sum_{k=1}^{m} a_k k \ c(\int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)} \int_0^{\varsigma} \frac{(\varsigma-\tau)^{\sigma-1}}{\Gamma(\sigma)} |\mu(\varphi(\varsigma)) - \mu^*(\varphi(\varsigma))| d\tau d\varsigma \\
+ \quad & \frac{\delta T^\eta}{\Gamma(\eta+1)}) \\
+ \quad & kc \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)} \int_0^{\varsigma} \ \frac{(\varsigma-\tau)^{\sigma-1}}{\Gamma(\sigma)} |\mu(\varphi(\varsigma)) - \mu^*(\varphi(\varsigma))| d\tau d\varsigma + \frac{\delta \ T^\eta}{\Gamma(\eta+1)} \\
\leq \quad & \|\mu - \mu^*\| (a \sum_{k=1}^{m} a_k kc \int_0^{\tau_k} \ \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)} \int_0^{\varsigma} \frac{(\varsigma-\tau)^{\sigma-1}}{\Gamma(\sigma)} d\tau d\varsigma \\
+ \quad & kc \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)} \int_0^{\varsigma} \ \frac{(s-\tau)^{\sigma-1}}{\Gamma(\sigma)} d\tau d\varsigma) + (a \sum_{k=1}^{m} a_k kc + 1) \frac{\delta \ T^\eta}{\Gamma(\eta+1)}.
\end{aligned}
$$

For $\mathfrak{r} \in I$, we have

$$
\begin{aligned}
& \|\mu_{\phi_1} - \mu_{\phi_1^*}\| \\
\leq \quad & \frac{(a \sum_{k=1}^{m} \ a_k + 1) kc T^{\sigma+\eta}}{\Gamma(\eta+1)\Gamma(\sigma+1)} \|\mu_{\phi_1} - \mu_{\phi_1^*}\| + (a \sum_{k=1}^{m} \ a_k kc + 1) \frac{\delta \ T^\eta}{\Gamma(\eta+1)}.
\end{aligned}
$$

Then

$$
\begin{aligned}
& \|\mu_{\phi_1} - \mu_{\phi_1^*}\| \\
\leq \quad & \left(1 - \frac{(\ a \ \sum_{k=1}^{m} a_k + 1) kc T^{\sigma+\eta}}{\Gamma(\eta+1)\Gamma(\sigma+1)}\right)^{-1} (a \sum_{k=1}^{m} a_k \ kc + 1) \ \frac{\delta \ T^\eta}{\Gamma(\eta+1)} = \epsilon.
\end{aligned}
$$

Hence,

$$
\|\mu_{\phi_1} - \mu_{\phi_1^*}\| \leq \epsilon.
$$

It demonstrates the solution on the set S_{Φ_1} of all Lipschitzian selection of Φ_1 is continuous dependence. □

Theorem 3.3. *Assume assumptions of Theorem 3.1 hold. Then the solution of the nonlocal problem (1)-(4) depends continuously on the Lipschitz function* ϕ_2.

Proof. Let $\phi_2(\mathfrak{r},\mu(\mathfrak{r}))$ and $\phi_2^*(\mathfrak{r},\mu(\mathfrak{r}))$ be two different Lipschitz functions such that

$$|\phi_2(\mathfrak{r},\mu(\mathfrak{r}))-\phi_2^*(\mathfrak{r},\mu(\mathfrak{r})|<\delta,\ \delta>0,\ \mathfrak{r}\in I,$$

then for the two corresponding solutions μ and μ^* of (6), we have

$$\begin{aligned}
& |\mu(\mathfrak{r})-\mu^*(\mathfrak{r})| \\
\le\ & a\sum_{k=1}^{m} a_k\int_0^{\tau_k}\frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}|\phi_1(\varsigma,I^\sigma\phi_2(\varsigma,\mu_2(\varphi(\varsigma)))) \\
-\ & \phi_1(\varsigma,I^\sigma\phi_2^*(\varsigma,\mu^*(\varphi(\varsigma))))|d\varsigma \\
+\ & \int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)}\ |\phi_1(\varsigma,I^\sigma\phi_2(\varsigma,\mu_1(\varphi(\varsigma))))\ -\phi_1(\varsigma,I^\sigma\phi_2^*(\varsigma,\mu^*(\varphi(\varsigma))))|\ d\varsigma \\
\le\ & a\sum_{k=1}^{m} a_k\ k\int_0^{\mathfrak{r}_k}\frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}|I^\sigma\phi_2(\varsigma,\mu(\varphi(\varsigma)))-I^\sigma\phi_2^*(\varsigma,\mu^*(\varphi(\varsigma)))|\ d\varsigma \\
+\ & k\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)}|I^\sigma\phi_2(\varsigma,\mu^*(\varphi(\varsigma)))\ -I^\sigma\phi_2^*(\varsigma,\mu^*(\varphi(\varsigma)))|\ d\varsigma \\
\le\ & a\sum_{k=1}^{m} a_k\ k\int_0^{\tau_k}\frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}|I^\sigma\phi_2(\varsigma,\mu(\varphi(\varsigma)))-I^\sigma\phi_2(\varsigma,\mu^*(\varphi(\varsigma)))|\ d\varsigma \\
+\ & a\sum_{k=1}^{m} a_k\ k\int_0^{\tau_k}\frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}|I^\sigma\phi_2(\varsigma,\mu^*(\varphi(\varsigma)))-I^\sigma\phi_2^*(\varsigma,\mu^*(\varphi(\varsigma)))|\ d\varsigma \\
+\ & k\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)}|I^\sigma\phi_2(\varsigma,\mu(\varphi(\varsigma)))-I^\sigma\phi_2(\varsigma,\mu^*(\varphi(\varsigma)))|\ d\varsigma \\
+\ & k\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)}|I^\sigma\phi_2(\varsigma,\mu^*(\varphi(\varsigma)))-I^\sigma\phi_2^*(\varsigma,\mu^*(\varphi(\varsigma)))|\ d\varsigma \\
\le\ & a\sum_{k=1}^{m} a_k\Big[k\int_0^{\tau_k}\frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\int_0^{\varsigma}\frac{(\varsigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}|\phi_2(\tau,\mu(\varphi(\tau))) \\
-\ & \phi_2(\varsigma,\mu^*(\varphi(\varsigma)))|d\tau d\varsigma \\
+\ & \int_0^{\tau_k}\frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\int_0^{\varsigma}\frac{(\varsigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}|\phi_2(\tau,\mu^*(\varphi(\tau)))-\phi_2^*(\tau,\mu^*(\varphi(\tau)))|\ d\tau\ d\varsigma\Big] \\
+\ & k\Big[\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)}\int_0^{\varsigma}\frac{(\varsigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}|\phi_2(\tau,\mu(\varphi(\tau)))-\phi_2(\varsigma,\mu^*(\varphi(\varsigma)))|\ d\tau\ d\varsigma \\
+\ & \int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)}\int_0^{\varsigma}\frac{(\varsigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}|\phi_2(\tau,\mu^*(\varphi(\tau)))-\phi_2^*(\tau,\mu^*(\varphi(\tau)))|\ d\tau\ d\varsigma\Big]
\end{aligned}$$

$$
\begin{aligned}
&\leq a\sum_{k=1}^{m} a_k k c\int_0^{\tau_k}\frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\int_0^{\varsigma}\frac{(\varsigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}\big[|\mu(\varphi(\tau))-\mu^*(\varphi(\tau))|+k\delta\big]d\tau d\varsigma\\
&+ k\ c\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)}\int_0^{\varsigma}\frac{(\varsigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}\big[|\mu(\varphi(\tau))-\mu^*(\varphi(\tau))|+k\ \delta\big]\ d\tau\ d\varsigma\\
&\leq a\sum_{k=1}^{m} a_k[k\ c\ \|\mu-\mu^*\|+k\ \delta]\int_0^{\tau_k}\frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\int_0^{\varsigma}\frac{(\varsigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}\ d\tau\ d\varsigma\\
&+ [k\ c\ \|\mu-\mu^*\|+k\ \delta]\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)}\int_0^{\varsigma}\frac{(\varsigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}\ d\tau\ d\varsigma\\
&\leq a\sum_{k=1}^{m} a_k\ k\ [\ c\ \|\mu-\mu^*\|+\ \delta]\frac{T^{\sigma+\eta}}{\Gamma(\sigma+1)\Gamma(\eta+1)}\\
&+ k\ [\ c\ \|\mu-\mu^*\|+\ \delta]\frac{T^{\sigma+\eta}}{\Gamma(\sigma+1)\Gamma(\eta+1)}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\|\mu-\mu^*\| &\leq \frac{(a\sum_{k=1}^{m} a_k+1)k\ cT^{\sigma+\eta}}{\Gamma(\sigma+1)\Gamma(\eta+1)}\|\mu-\mu^*\|+\frac{(a\sum_{k=1}^{m}\ a_k+1)k\ cT^{\sigma+\eta}}{\Gamma(\sigma+1)\Gamma(\beta+1)}\\
\|\mu-\mu^*\| &\leq (1-\frac{(a\sum_{k=1}^{m} a_k+1)\ k\ c\ T^{\sigma+\eta}}{\Gamma(\sigma+1)\Gamma(\eta+1)})^{-1}\frac{(a\sum_{k=1}^{m} a_k+1)k\ cT^{\sigma+\eta}}{\Gamma(\sigma+1)\Gamma(\eta+1)}=\epsilon.
\end{aligned}
$$

Then

$$\|\mu-\mu^*\|\leq\epsilon,$$

which proves the continuous dependence of the solution on the Lipschitz function ϕ_2. □

Theorem 3.4. *Assume assumptions of Theorem 3.1 hold. Then the solution of the nonlocal problem (1)-(4) depends continuously on initial data $\mu_\circ$.*

Proof. Let $\mu(\mathfrak{r})$ and $\mu^*(\mathfrak{r})$ be two solutions of the integral equation (6). Then we have

$$\begin{aligned}
& |\mu(\mathfrak{r}) - \mu^*(\mathfrak{r})| \\
= \ & |a\left(\mu_\circ - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)} |\phi_1(\varsigma, I^\sigma \phi_2(\varsigma, \mu_2(\varphi(\varsigma)))d\varsigma\right) \\
+ \ & \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\beta)} \phi_1(\varsigma, I^\sigma \phi_2(\varsigma, \mu(\varphi(\varsigma))))d\varsigma \\
- \ & a\left(\mu^*_\circ - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\beta)} \phi_1(\varsigma, I^\sigma \phi_2(\varsigma, \mu^*(\varphi(\varsigma)))d\varsigma\right) \\
- \ & \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\varsigma, I^\sigma \phi_2(\varsigma, \mu^*(\varphi(\varsigma))))d\varsigma| \\
\leq \ & a|\mu_\circ - \mu^*_\circ| \\
+ \ & a\sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)} |\phi_1(\varsigma, I^\sigma \phi_2(\varsigma, \mu(\varphi(\varsigma))) \\
- \ & \phi_1(\varsigma, I^\sigma \phi_2(\varsigma, \mu^*(\varphi(\varsigma)))|d\varsigma \\
+ \ & \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)} |\phi_1(\varsigma, I^\sigma \phi_2(\varsigma, \mu(\varphi(\varsigma))) - \phi_1(\varsigma, I^\sigma \phi_2(\varsigma, \mu^*(\varphi(\varsigma)))|d\varsigma \\
\leq \ & a\ \delta + a\sum_{k=1}^{m} a_k k \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)} |I^\sigma \phi_2(\varsigma, \mu(\varphi(\varsigma))) - I^\sigma \phi_2(\varsigma, \mu^*(\varphi(\varsigma)))|d\varsigma \\
+ \ & k \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\beta-1}}{\Gamma(\eta)} |I^\sigma \phi_2(\varsigma, \mu(\varphi(\varsigma))) - I^\sigma \phi_2(\varsigma, \mu^*(\varphi(\varsigma)))|\ d\varsigma \\
\leq \ & a\delta + a\sum_{k=1}^{m} a_k k \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)} \int_0^{\varsigma} \frac{(\varsigma-\tau)^{\sigma-1}}{\Gamma(\sigma)} |\phi_2(\tau, \mu(\varphi(\tau))) \\
- \ & \phi_2(\tau, \mu^*(\varphi(\tau)))|d\tau d\varsigma \\
+ \ & k\int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)} \int_0^{\varsigma} \frac{(\varsigma-\tau)^{\sigma-1}}{\Gamma(\sigma)} |\phi_2(\tau, \mu(\varphi(\tau))) - \phi_2(\tau, \mu^*(\varphi(\tau)))|\ d\tau d\varsigma \\
\leq \ & a\delta + a\sum_{k=1}^{m} a_k k c \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)} \int_0^{\varsigma} \frac{(\varsigma-\tau)^{\sigma-1}}{\Gamma(\sigma)} |\mu(\varphi(\tau)) - \mu^*(\varphi(\tau))|d\tau d\varsigma \\
+ \ & k\ c \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)} \int_0^{\varsigma} \frac{(\varsigma-\tau)^{\sigma-1}}{\Gamma(\sigma)} |\mu(\varphi(\tau)) - \mu^*(\varphi(\tau))|\ d\tau\ d\varsigma \\
\leq \ & a\ \delta\ +\ a\ \sum_{k=1}^{m} a_k\ k\ c\ \|\mu - \mu^*\| \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)} \int_0^{\varsigma} \frac{(\varsigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}\ d\tau\ d\varsigma \\
+ \ & k\ c\ \|\mu - \mu^*\| \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\beta-1}}{\Gamma(\eta)} \int_0^{\varsigma} \frac{(\varsigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}\ d\tau\ d\varsigma.
\end{aligned}$$

Then

$$\|\mu-\mu^*\| \leq a\ \delta\ +\ \frac{[a\ \sum_{k=1}^m\ a_k+1]\ k\ c\ T^{\sigma+\eta}}{\Gamma(\sigma+1)\Gamma(\eta+1)}\|\mu-\mu^*\|$$

$$\|\mu-\mu^*\| \leq (1-\frac{[a\ \sum_{k=1}^m\ a_k+1]\ k\ c\ T^{\sigma+\eta}}{\Gamma(\sigma+1)\Gamma(\eta+1)})^{-1}a\delta=\epsilon.$$

Hence

$$\|\mu-\mu^*\|\leq\epsilon.$$

Thus, the integral equation (6) has a continuous dependence on $\mu_\circ$. So the the nonlocal problem (1)-(4) its solution depends continuously on initial data $\mu_\circ$. □

4. **Example**

Consider the following Caputo fractional differential inclusion:

$$(23)\qquad {}^cD^\eta\mu(\mathfrak{r})\in\Phi_1(\mathfrak{r},I^\sigma\phi_2(\mathfrak{r},\mu(\varphi(\mathfrak{r})))),\ \mathfrak{r}\in[0,1],\ \sigma\in(0,1)$$

with infinite point boundary condition

$$(24)\qquad \mu(0)+\sum_{k=1}^{\infty}\frac{1}{k^2}\ \mu(\frac{k-1}{k})=\mu_\circ.$$

We choose $\Phi_1:[0,1]\times\mathbb{R}\to 2^{\mathbb{R}^+}$ in (23) as

$$\Phi_1(\mathfrak{r},I^{\frac14}\phi_2(\mathfrak{r},\mu(\mathfrak{r}))=\Big[0,\mathfrak{r}^3+\mathfrak{r}+1+\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\varsigma)^{-\frac34}}{2\ \Gamma(\frac34)}\big(\cos(\mu(\varsigma)+1)+\frac{\mu(\varsigma)}{e^\varsigma}\big)\ d\varsigma\Big],$$

and set

$$\phi_2(\mathfrak{r},\mu(\mathfrak{r}))=\frac12\big(\cos(\mu(\varsigma)+1)+\frac{\mu(\varsigma)}{e^\varsigma}\big).$$

Notice that for $\phi_1\in S_{\Phi_1}$, then we have

$$|\phi_1(\mathfrak{r},I^{\frac14}\phi_2(\mathfrak{r},\mu(\varphi(\mathfrak{r}))))-\phi_1(\mathfrak{r},I^{\frac14}\phi_2(\mathfrak{r},\nu(\varphi(\mathfrak{r}))))|\leq\frac{1+e}{2\ e\ \Gamma(\frac14)}|\mu-\nu|,$$

and

$$\big|\phi_2(\mathfrak{r},\mu(\mathfrak{r}))\big|\leq\frac12|\cos(\mu(\mathfrak{r})+1)|+\frac{|\mu(\mathfrak{r})|}{2e}.$$

As a result, conditions (i) and (iii) are held with $k=\frac{1+e}{2\ e\ \Gamma(\frac14)}\approx 0.1889<1$, $a(\mathfrak{r})=\frac12\ \cos(\mu(t)+1)\ \in L^1[0,1]$, $b=\frac{1}{2e}$ and the series $\sum_{k=1}^{\infty}\ \frac{1}{k^4}$ is convergent. Also, $[a\sum_{k=1}^m\ |a_k|\ +1]\frac{k\ T^\eta}{\Gamma(\eta+1)}\approx 0.6136<1$ and $\frac{b\ T^\sigma}{\Gamma(\sigma+1)}\approx 0.2029<1$. we deduce from Theorem 2.3 that the nonlocal problem (23)-(24) has at least one continuous solution.

5. Conclusion

We presented the existence criteria for solutions to Caputo fractional differential equations and inclusions of order in (0,1) complemented with nonlocal infinite-point and Riemann–Stieltjes integral boundary conditions (BCs). We first transformed the nonlinear Caputo type fractional boundary value problem into a fixed point problem. We have demonstrated that, if we can get the continuous solutions to boundary value problem with m-point BCs, we can easily get the solutions to these problems with integral BCs or with infinite-point BCs. For the single-valued case, we established the existence of a continuous solution using Schauder's fixed point theorem, the uniqueness solution, and the continuous dependence of the functional differential inclusion on the set of selections and some data were studied. To ensure the validity of all the obtained theoretical results, suitable examples were provided to support their validity. This study would make a significant contribution to the literature on qualitative theory. Which may include the expansion of the concept introduced in this study area and the likelihood of other generalizations in a wide range of exclusive outputs for applications and theories. Here, one suggestion is that it Future works may be expanded to discuss the existence and uniqueness of solutions for the other types of nonlinear Hadamard-type fractional differential inclusion with infinite-point boundary conditions or integral boundary conditions.

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References

[1] B. Ahmad, S. K. Ntouyas, Nonlocal fractional boundary value problems with slit-strips boundary conditions, Fract. Calc. Appl. Anal., 18 (2015), 261–280.

[2] B. Ahmad, A. Alsaedi, and Y. Alruwaily, *On Riemann-Stieltjes integral boundary value problems of Caputo-Riemann-Liouville type fractional integro-differential equations.* Filomat, **34**(8) (2020), 2723-2738.

[3] J. P. Aubin, A. Cellina, *Differential inclusions: set-valued maps and viability theory*, Springer Science & Business Media, **264** (2012),

[4] Z. Bai, T. Qiu, *Existence of positive solution for singular fractional differential equation.* Appl. Math. Comput. **215** (2009), 2761–2767.

[5] A. Cellina, S. Solimini, *Continuous extension of selection*, Bull. Polish Acad. Sci. Math., **35** (9) (1978).

[6] K. Deimling, *Nonlinear Functional Analysis*, Courier Corporation, 2010.

[7] N. Kosmatov, *A singular boundary value problem for nonlinear differential equations of fractional order.* J. Appl. Math. Comput. 29(1), 125–135 (2009).

[8] A.M.A. El-Sayed , W.G. El-Sayed , S.S. Amrajaa, *On a Boundary Value Problem of Hybrid Functional Differential Inclusion with Nonlocal Integral Condition.* Mathematics, **9**(21(2021) :2667. https://doi.org/10.3390/math9212667

[9] A.M.A. El-sayed, A.G. Ibrahim, *Set-valued integral equations of fractional-orders*, Appl. Math. Comput., **118** (2001), 113-121.

[10] A.M.A. El-Sayed, H.M.H. Hashem, Sh.M Al-Issa, *Existence of solutions for an ordinary secondorder hybrid functional differential equation*, Ad. Difference Equ. **2021** (2021), Paper No. 29.

[11] A.M.A. El-Sayed, H.H.G. Hashem, Sh.M. Al-Issa, *Analytical study of some self-referred or state dependent functional equations.* Methods Funct. Anal. Topol. **28**(1) (2022), 50-57.

[12] A.M.A. El-Sayed, Sh.M Al-Issa, *Monotonic integrable solution for a mixed type integral and differential inclusion of fractional orders*, Int. J. Differ. Equ. Appl., **18**(1) (2019), 1–9.

[13] V. Lakshmikantham, S. Leela, *Differential and Integral Inequalities*, Academic press, New York-London, **1** (1969).

[14] C. Nuchpong, S.K. Ntouyas, A. Samadi, *Boundary value problems for Hilfer type sequential fractional differential equations and inclusions involving Riemann–Stieltjes integral multi-strip boundary conditions.* Adv Differ Equ **2021**, 268 (2021).

[15] H.M. Srivastava , A.M.A El-Sayed, FM. A Gaafar , *Class of Nonlinear Boundary Value Problems for an Arbitrary Fractional-Order Differential Equation with the Riemann-Stieltjes Functional Integral and Infinite-Point Boundary Conditions.* Symmetry.**2018**; 10(10):508.

[16] H.M. Srivastava, A.M.A. El-Sayed, H.H.G. Hashem, Sh.M. Al-Issa, *Analytical investigation of nonlinear hybrid implicit functional differential inclusions of arbitrary fractional orders*, Rev. Real Acad. Cienc. Exactas Fis. Nat. -A: Mat. **2022** (2022) 116.

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