Journal of Mahani Mathematical Research
Shahid Bahonar University of Kerman
Print ISSN: 2251-7952 Online ISSN: 2645-4505

(INVERSE) NEUTROSOPHIC SPECIAL n-DOMINATION IN NEUTROSOPHIC GRAPHS WITH APPLICATION IN DECISION MAKING

S. Banitalebi and R.A. Borzooei ✉

Article type: Research Article
(Received: 17 September 2022, Received in revised form 09 March 2023)
(Accepted: 29 April 2023, Published Online: 30 April 2023)

Abstract. In this paper the meanings of neutrosophic special n-dominating set, neutrosophic special n-domination number, inverse neutrosophic special domination set (number) and inverse neutrosophic special n-domination number are introduced and some of related results are investigated. Finally, an application of inverse neutrosophic special dominating set in decision making under ashy clauses between certainty and uncertainty is provided. In fact, we present a decision-making problem in real-world applied example which discusses the factors influencing a companys efficiency. The presented model is, in fact, a factor-based model wherein the impact score of each factor is divided into two types of direct and indirect influences.

Keywords: Neutrosophic graph, Neutrosophic special n-dominating set, Neutrosophic special n-domination number, Inverse neutrosophic special domination set (number), Inverse neutrosophic special n-domination number.
2020 MSC: Primary xxYxx, xxYxx, xxYxx

1. Introduction

For the very first time, Smarandache [17] offered the idea of neutrosophic sets (NSs) as an extension of the fuzzy sets [23], intuitionistic fuzzy sets [4], interval-valued fuzzy sets [20] and interval-valued intuitionists fuzzy sets [5] theories. The (NS) is a highly applicable to solve hybrid issues in various fields with incomplete, indeterminate and inconsistent information in real world. Smarandache [18, 19] defined two principal classes of neutrosophic graphs. Further, Satham Hussain, Jahir Hussain and Smarandache [16] offered the notion of domination in neutrosophic soft graphs. Banitalebi and Borzooei [11] presented the meaning of neutrosophic special domination in neutrosophic graphs. In some scientific studies, fuzzy sets do not have the necessary yield to display and resolve mental obscurity and neutrosophic sets show more flexibility

✉ borzooei@sbu.ac.ir, ORCID: 0000-0001-7538-7885
DOI: 10.22103/jmmr.2023.20250.1339
Publisher: Shahid Bahonar University of Kerman
How to cite: S. Banitalebi, R.A. Borzooei, *(Inverse) Neutrosophic special n-domination in neutrosophic graphs with application in decision making*, J. Mahani Math. Res. 2024; 13(1): 127-142.

and capability in this field. In recent years, research on Single-Valued Neutrosophic and neutrosophic soft graphs has been done by Akram et. al. [1–3] and interesting results have been obtained. In recent studies, various models and methods in the field of decision making Presented by Zhan et. al. [13,14,21,22]. Characteristics of neutrosophic graphs urged us to examine different meanings regarding their domination sets and in the following, expand the factor-based decision modeling technique Banitalebi et. al. [6–11], this time in neutrosophic cognitive maps. In this modeling method, optimization and modeling solutions are proposed using the concept of governing sets, and by reducing the dominating set size, the effective weight of the graph of factors on the control goal increases. The principal aim of this article is to discuss the meanings of neutrosophic special $n-$dominating set, neutrosophic special $n-$domination number, inverse neutrosophic special $n-$dominating set and inverse neutrosophic special $n-$domination number in neutrosophic graphs and finally, a model for optimizing the neutrosophic special domination parameter will be offered, where in it will be feasible to optimize the neutrosophic special domination number parameter more precisely in a partial way.

2. Preliminaries

A *fuzzy graph* $\mathbb{G}_{\mathbb{F}} = (\kappa, \tau)$ on crisp graph $\mathbb{G}_{\mathbb{C}} = (\mathbb{A}_{\mathbb{V}}, \mathbb{B}_{\mathbb{E}})$ is a pair of functions $\kappa : \mathbb{A}_{\mathbb{V}} \to [0,1]$ and $\tau : \mathbb{B}_{\mathbb{E}} \to [0,1]$ wherever, for any $qp \in \mathbb{B}_{\mathbb{E}}$, $\tau(qp) \preceq \min\{\kappa(q), \kappa(p)\}$.

Definition 2.1. [17] If $\mathbb{A}_{\mathbb{V}}$ is a space of nodes with universal elements in $\mathbb{A}_{\mathbb{V}}$ marked by q, then the neutrosophic set $\mathbb{H}_{Ne}$ is an object having the form

$$\mathbb{H}_{Ne} = \{< q : \xi_{\mathbb{H}_{Ne}}(q), \varpi_{\mathbb{H}_{Ne}}(q), \varrho_{\mathbb{H}_{Ne}}(q) >, q \in \mathbb{A}_{\mathbb{V}}\},$$

wherever the functions $\xi, \varpi, \varrho : \mathbb{A}_{\mathbb{V}} \to]^{-}0, 1^{+}[$ describe respectively, the truth-membership function, the indeterminacy-membership function and the falsity-membership function of the element $q \in \mathbb{A}_{\mathbb{V}}$ to the set $\mathbb{H}_{Ne}$.

Definition 2.2. [12] A neutrosophic graph ($\mathbb{NG}$) on crisp graph $\mathbb{G}_{\mathbb{C}} = (\mathbb{A}_{\mathbb{V}}, \mathbb{B}_{\mathbb{E}})$ is marked by $\mathbb{G}_{Ne} = (\chi, \pi)$, wherever $\chi = (\xi_\chi, \varpi_\chi, \varrho_\chi)$ so that $\xi_\chi, \varpi_\chi, \varrho_\chi : \mathbb{A}_{\mathbb{V}} \to [0,1]$ with the clause

$$0 \leq \xi_\chi(q) + \varpi_\chi(q) + \varrho_\chi(q) \leq 3,$$

for all $q \in \mathbb{A}_{\mathbb{V}}$ and $\pi = (\xi_\pi, \varpi_\pi, \varrho_\pi)$ wherever $\xi_\pi, \varpi_\pi, \varrho_\pi : \mathbb{B}_{\mathbb{E}} \to [0,1]$. With clauses

$$\xi_\pi(qp) \leq \xi_\chi(q) \wedge \xi_\chi(p),$$

$$\varpi_\pi(qp) \geq \varpi_\chi(q) \vee \varpi_\chi(p),$$

$$\varrho_\pi(qp) \geq \varrho_\chi(q) \vee \varrho_\chi(p),$$

and $0 \leq \xi_\pi(qp) + \varpi_\pi(qp) + \varrho_\pi(qp) \leq 3$ for all $qp \in \mathbb{B}_{\mathbb{E}}$.

Definition 2.3. [15] Let $\mathbb{G}_{Ne} = (\chi, \pi)$ be a $\mathbb{NG}$ on crisp graph $\mathbb{G}_{\mathbb{C}} = (\mathbb{A}_{\mathbb{V}}, \mathbb{B}_{\mathbb{E}})$ and $i, j \in \mathbb{A}_{\mathbb{V}}$. Afterward,
(1) ξ-*strength* of connectedness between i and j is

$$\xi_\pi^\infty(ij) = \sup\{\xi_\pi^s(ij)\big|s = 1, 2, \ldots, r\},$$

and

$$\xi_\pi^s(ij) = \min\{\xi_\pi(it_1), \xi_\pi(t_1t_2), \ldots, \xi_\pi(t_{s-1}j)\big|i, t_1, \ldots, t_{s-1}, j \in \mathbb{A}_{\mathbb{V}}, s = 1, 2, \ldots, r\}.$$

(2) ϖ-*strength* of connectedness between i and j is

$$\varpi_\pi^\infty(ij) = \inf\{\varpi_\pi^s(ij)\big|s = 1, 2, \ldots, r\},$$

and

$$\varpi_\pi^k(ij) = \max\{\varpi_\pi(it_1), \varpi_\pi(t_1t_2), \ldots, \varpi_\pi(t_{s-1}j)\big|i, t_1, \ldots, t_{n-1}, j \in \mathbb{A}_{\mathbb{V}}, s = 1, 2, \ldots, r\}.$$

(3) ϱ-*strength* of connectedness between i and j is

$$\varrho_\pi^\infty(ij) = \inf\{\varrho_\pi^s(ij)\big|s = 1, 2, \ldots, r\},$$

and

$$\varrho_\pi^n(ij) = \max\{\varrho_\pi(it_1), \varrho_\pi(t_1t_2), \ldots, \varrho_\pi(t_{s-1}v)\big|i, t_1, \ldots, t_{n-1}, j \in \mathbb{A}_{\mathbb{V}}, s = 1, 2, \ldots, r\},$$

wherever $it_1t_2t_{s-1}j$ a path from i to j in $\mathbb{G}_{\mathbb{C}}$.

Definition 2.4. [15] Let $\mathbb{G}_{Ne} = (\chi, \pi)$ be a $\mathbb{NG}$ on crisp graph $\mathbb{G}_{\mathbb{C}} = (\mathbb{A}_{\mathbb{V}}, \mathbb{B}_{\mathbb{E}})$. An arc $qp \in \mathbb{B}_{\mathbb{E}}$ is named a neutrosophic strong arc if

$$\xi_\pi(qp) \geq \xi_\pi^\infty(qp)\ ,\ \varpi_\pi(qp) \leq \varpi_\pi^\infty(qp)\ and\ \varrho_\pi(qp) \leq \varrho_\pi^\infty(qp).$$

Definition 2.5. [11] Let $\mathbb{G}_{Ne} = (\chi, \pi)$ be a $\mathbb{NG}$. Afterward,

(1) the neutrosophic order of $\mathbb{G}_{Ne}$ is as follows,

$$|\mathbb{A}_{\mathbb{V}}| = \sum_{q_i \in \mathbb{A}_{\mathbb{V}}} \left(\frac{3 + \xi_\chi(q_i) - (\varpi_\chi(q_i) + \varrho_\chi(q_i))}{2} \right),$$

(2) the neutrosophic size of $\mathbb{G}_{Ne}$ is as follows,

$$|\mathbb{B}_{\mathbb{E}}| = \sum_{q_iq_j \in \mathbb{E}} \left(\frac{3 + \xi_\pi(q_iq_j) - (\varpi_\pi(q_iq_j) + \varrho_\pi(q_iq_j))}{2} \right),$$

(3) the neutrosophic cardinality of $\mathbb{G}_{Ne}$ is as follows,

$$|\mathbb{G}| = |\mathbb{A}_{\mathbb{V}}| + |\mathbb{B}_{\mathbb{E}}|,$$

(4) for any $\mathbb{M} \subset \mathbb{A}_{\mathbb{V}}$, the neutrosophic node cardinality of $\mathbb{M}$ is marked by $\mathbb{O}(\mathbb{M})$ and is as follows,

$$\mathbb{O}(\mathbb{M}) = \sum_{q_i \in \mathbb{M}} \left(\frac{3 + \xi_\chi(q_i) - (\varpi_\chi(q_i) + \varrho_\chi(q_i))}{2} \right),$$

(5) for any $\mathfrak{P} \subset \mathbb{B}_{\mathbb{E}}$, the neutrosophic arc cardinality of $\mathfrak{P}$ is marked by $\mathbb{S}(\mathfrak{P})$ and is as follows,

$$\mathbb{S}(\mathfrak{P}) = \sum_{q_i q_j \in \mathfrak{P}} \left(\frac{3 + \xi_\pi(q_i q_j) - (\varpi_\pi(q_i q_j) + \varrho_\pi(q_i q_j))}{2} \right).$$

Definition 2.6. [11] Let $\mathbb{G}_{Ne} = (\chi, \pi)$ be a $\mathbb{NG}$. Afterward,
(1) an arc $a = qp$ in $\mathbb{G}_{Ne}$ is named a *neutrosophic highly strong arc ($\mathcal{HS}$)*, if

$$\xi_\pi(qp) > \xi_\pi^\infty(qp)\ ,\ \varpi_\pi(qp) < \varpi_\pi^\infty(qp)\ ,\ \varrho_\pi(qp) < \varrho_\pi^\infty(qp).$$

(2) The *neutrosophic highly strong neighborhood* of $q \in \mathbb{A}_{\mathbb{V}}$ is marked by $\mathbb{N}_{hs}(q)$ and is as follows,

$$\mathbb{N}_{hs}(q) = \{p \in \mathbb{A}_{\mathbb{V}} \mid qp \text{ is a highly strong arc in } \mathbb{G}_{Ne}\}.$$

(3) A node $q \in \mathbb{A}_{\mathbb{V}}$ of a $\mathbb{NG}$ $\mathbb{G}_{Ne}$ said to be a *neutrosophic slightly isolated node ($\check{\mathcal{I}}_n$)* if $\mathbb{N}_{hs}(q) = \emptyset$.

Definition 2.7. [11] Let $\mathbb{G}_{Ne}$ be a $\mathbb{NG}$ on crisp graph $\mathbb{G}_{\mathbb{C}} = (\mathbb{A}_{\mathbb{V}}, \mathbb{B}_{\mathbb{E}})$ and $q, p \in \mathbb{A}_{\mathbb{V}}$. Afterward:
(1) q *specially dominate* p in $\mathbb{G}_{Ne}$, if there exists a $\mathcal{HS}$ between q and p.
(2) $\mathbb{S} \subset \mathbb{A}_{\mathbb{V}}$ is named a *neutrosophic special dominating set ($\widetilde{\mathbf{SpD}}$)* in $\mathbb{G}_{Ne}$, if for any $p \in \mathbb{A}_{\mathbb{V}} \setminus \mathbb{S}$, there exists $q \in \mathbb{S}$ wherever q specially dominates w.
(3) A $\widetilde{\mathbf{SpD}}$ $\mathbf{M}$ in $\mathbb{G}_{Ne}$ is named a *minimal* $\widetilde{\mathbf{SpD}}$ if no proper subset of $\mathbf{M}$ is a $\widetilde{\mathbf{SpD}}$.
(4) Minimum neutrosophic node cardinality amidst all minimal $\widetilde{\mathbf{SpD}}$*s* of $\mathbb{G}_{Ne}$ is defined *lower neutrosophic special domination number* of $\mathbb{G}_{Ne}$ and is marked by $\check{\mathfrak{N}}_{\mathbb{A}_{\mathbb{V}}}(\mathbb{G}_{Ne})$.
(5) Maximum neutrosophic node cardinality amidst all minimal $\widetilde{\mathbf{SpD}}$*s* of $\mathbb{G}_{Ne}$ is defined *upper neutrosophic special domination number* of $\mathbb{G}_{Ne}$ and is marked by $\hat{\mathfrak{N}}_{\mathbb{A}_{\mathbb{V}}}(\mathbb{G}_{Ne})$.
(6) The *neutrosophic special domination number* of $\mathbb{G}_{Ne}$ is marked by $\bar{\mathfrak{N}}(\mathbb{G}_{Ne})$ and interpreted with

$$\bar{\mathfrak{N}}(\mathbb{G}_{Ne}) = \frac{\check{\mathfrak{N}}_{\mathbb{A}_{\mathbb{V}}}(\mathbb{G}_{Ne}) + \hat{\mathfrak{N}}_{\mathbb{A}_{\mathbb{V}}}(\mathbb{G}_{Ne})}{2}.$$

Definition 2.8. [11] Let $\mathbb{G}_{Ne}$ be a $\mathbb{NG}$. Afterward:
(1) Two nodes $q,\ p \in \mathbb{A}_{\mathbb{V}}$ are named *neutrosophic slightly independent* if there is not any $\mathcal{HS}$ between them.
(2) $\mathbb{S} \subset \mathbb{A}_{\mathbb{V}}$ is defined a *neutrosophic slightly independent set ($\check{\mathcal{I}}$S)* in $\mathbb{G}_{Ne}$ if for any$q, p \in \mathbb{S}$, $\xi_\pi(qp) \leq \xi_\pi^\infty(qp)$, $\varpi_\pi(qp) \geq \varpi_\pi^\infty(qp)$ and $\varrho_\pi(qp) \geq \varrho_\pi^\infty(qp)$.
(3) A $\check{\mathcal{I}}$S $\mathbf{M}$ in $\mathbb{G}_{Ne}$ said to be a *maximal* $\check{\mathcal{I}}$S if for any node $q \in \mathbb{A}_{\mathbb{V}} \setminus \mathbf{M}$, the set $\mathbf{M} \cup \{q\}$ is not $\check{\mathcal{I}}$S.

(4) Minimum neutrosophic node cardinality amidst all maximal $\check{\mathcal{I}}$Ss is defined *lower neutrosophic slightly independent number* of $\mathbb{G}_{Ne}$ and is marked by $\check{\mathrm{N}}(\check{\mathcal{I}})(\mathbb{G}_{Ne})$.
(5) Maximum neutrosophic node cardinality amidst all maximal $\check{\mathcal{I}}$Ss is defined *upper neutrosophic slightly independent number* of $\mathbb{G}_{Ne}$ and is marked by $\hat{\mathrm{N}}(\check{\mathcal{I}})(\mathbb{G}_{Ne})$.
(6) The *neutrosophic slightly independent number* of $\mathbb{G}_{Ne}$ is marked by $\bar{\mathrm{N}}(\check{\mathcal{I}})(\mathbb{G}_{Ne})$ and interpreted as follows,

$$\bar{\mathrm{N}}(\check{\mathcal{I}})(\mathbb{G}_{Ne}) = \frac{\check{\mathrm{N}}(\check{\mathcal{I}})(\mathbb{G}_{Ne}) + \hat{\mathrm{N}}(\check{\mathcal{I}})(\mathbb{G}_{Ne})}{2}.$$

From now on, in this article we suppose that $\mathbb{G}_{Ne} = (\chi, \pi)$ be a neutrosophic graph on crisp graph $\mathbb{G}_{\mathbb{C}} = (\mathbb{A}_{\mathbb{V}}, \mathbb{B}_{\mathbb{E}})$ and marked by $\mathbb{NG}$.

3. Neutrosophic special $n-$dominating set and Inverse $\widetilde{\mathbf{Sp}n-\mathbf{D}}$ in $\mathbb{NG}$s

Definition 3.1. $\mathbb{D}_{NSp} \subset \mathbb{A}_{\mathbb{V}}$ is named a neutrosophic special $n-$dominating set ($\widetilde{\mathbf{Sp}n-\mathbf{D}}$) of $\mathbb{G}_{Ne}$ if for any node $q \in \mathbb{A}_{\mathbb{V}} \setminus \mathbb{D}_{NSp}$ we get q specially dominate by at least n nodes in $\mathbb{D}_{NSp}$.

Definition 3.2. (1) An $\widetilde{\mathbf{Sp}n-\mathbf{D}}$ $\mathbb{D}_{NSp}$ of $\mathbb{G}_{Ne}$ is named a minimal $\widetilde{\mathbf{Sp}n-\mathbf{D}}$ if no proper subset of $\mathbb{D}_{NSp}$ is a $\widetilde{\mathbf{Sp}n-\mathbf{D}}$ of $\mathbb{G}_{Ne}$.
(2) Minimum neutrosophic node cardinality amidst all minimal $\widetilde{\mathbf{Sp}n-\mathbf{D}}s$ of $\mathbb{G}_{Ne}$ is defined lower neutrosophic special $n-$domination number of $\mathbb{G}_{Ne}$ and marked by $\check{\mathfrak{N}}^n_{\mathbb{A}_{\mathbb{V}}}(\mathbb{G}_{Ne})$.
(3) Maximum neutrosophic node cardinality amidst all minimal $\widetilde{\mathbf{Sp}n-\mathbf{D}}s$ of $\mathbb{G}_{Ne}$ is defined upper neutrosophic special $n-$domination number of $\mathbb{G}_{Ne}$ and marked by $\hat{\mathfrak{N}}^n_{\mathbb{A}_{\mathbb{V}}}(\mathbb{G}_{Ne})$.
(4) The neutrosophic special $n-$domination number of $\mathbb{G}_{Ne}$ is marked by $(\bar{\mathfrak{N}})^n(\mathbb{G}_{Ne})$ and interpreted with:

$$(\bar{\mathfrak{N}})^n(\mathbb{G}_{Ne}) = \frac{\check{\mathfrak{N}}^n_{\mathbb{A}_{\mathbb{V}}}(\mathbb{G}_{Ne}) + \hat{\mathfrak{N}}^n_{\mathbb{A}_{\mathbb{V}}}(\mathbb{G}_{Ne})}{2}.$$

Example 3.3. *Let a $\mathbb{NG}$ $\mathbb{G}_{Ne}$ as Figure 1. Afterward, u_1u_3 and u_3u_4 are neutrosophic highly strong arcs and It is obvious that $\mathbb{D}_2 = \{u_1, u_2, u_4\}$ is a neutrosophic special 2-dominating set of $\mathbb{G}_{Ne}$. Accordingly, $\check{\mathfrak{N}}^2_{\mathbb{A}_{\mathbb{V}}}(\mathbb{G}_{Ne}) = \hat{\mathfrak{N}}^2_{\mathbb{A}_{\mathbb{V}}}(\mathbb{G}_{Ne}) = (\bar{\mathfrak{N}})^2(\mathbb{G}_{Ne}) = 4.55$.*

Theorem 3.4. *Let $\mathbb{G}_{Ne}$ be a $\mathbb{NG}$ and $\mathbb{D}_{NSp}$ be a neutrosophic slightly independent and $\widetilde{\mathbf{Sp}n-\mathbf{D}}$ of $\mathbb{G}_{Ne}$. Afterward $\mathbb{D}_{NSp}$ is a minimal $\widetilde{\mathbf{SpD}}$ of $\mathbb{G}_{Ne}$.*

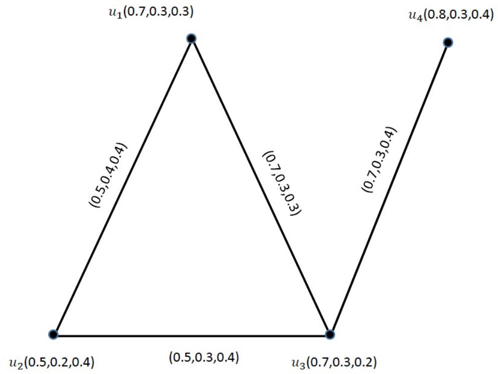

FIGURE 1. $\mathbb{NG}$ $\mathbb{G}_{Ne}$.

Proof. If $\mathbb{D}_{NSp}$ is a neutrosophic slightly independent and $\widetilde{\mathbf{Sp}n-\mathbf{D}}$ of neutrosophic graph $\mathbb{G}_{Ne}$, then $\mathbb{D}_{NSp}$ is a $\widetilde{\mathbf{SpD}}$ of $\mathbb{G}_{Ne}$ and $\mathbb{D}_{NSp}-\{d\}$ is not a $\widetilde{\mathbf{SpD}}$ for any $d \in \mathbb{D}_{NSp}$. Accordingly $\mathbb{D}_{NSp}$ is a minimal $\widetilde{\mathbf{SpD}}$ of $\mathbb{G}_{Ne}$. □

Theorem 3.5. *A $\widetilde{\mathbf{Sp}n-\mathbf{D}}$ $\mathbb{D}_{NSp}$ of a neutrosophic graph $\mathbb{G}_{Ne}$ is a minimal $\widetilde{\mathbf{Sp}n-\mathbf{D}}$ if and only if for each node $t \in \mathbb{D}_{NSp}$, one of the following qualifications holds.*
(1) $\left|\mathbb{N}_{hs}(t) \cap \mathbb{D}_{NSp}\right| \leq n-1$,
(2) there are nodes $s \in \mathbb{A}_{\mathbb{V}} \setminus \mathbb{D}_{NSp}$ and $u_1, u_2, u_3, \ldots, u_{n-1} \in \mathbb{D}_{NSp}$ so that $\mathbb{N}_{hs}(s) \cap (\mathbb{D}_{NSp}-\{t\}) = \{u_1, u_2, u_3, \ldots, u_{n-1}\}$.

Proof. Presume that $\mathbb{D}_{NSp}$ is a minimal $\widetilde{\mathbf{Sp}n-\mathbf{D}}$ of $\mathbb{G}_{Ne}$. Afterward, for any node $t \in \mathbb{D}_{NSp}$, $\mathbb{D}_{NSp}-\{t\}$ is not a $\widetilde{\mathbf{Sp}n-\mathbf{D}}$. Hence there is $s \in \mathbb{A}_{\mathbb{V}} \setminus (\mathbb{D}_{NSp}-\{t\})$ which is not specially dominated by n nodes in $\mathbb{D}_{NSp}-\{t\}$. If $s=t$, then $\left|\mathbb{N}_{hs}(t) \cap \mathbb{D}_{NSp}\right| \leq n-1$. If $s \neq t$, then s is not specially dominated by n nodes in $\mathbb{D}_{NSp}-\{t\}$, but is specially dominated by n nodes in $\mathbb{D}_{NSp}$. Accordingly there are nodes $u_1, u_2, u_3, \ldots, u_{n-1} \in \mathbb{D}_{NSp}$ so that $\mathbb{N}_{hs}(s) \cap (\mathbb{D}_{NSp}-\{t\}) = \{u_1, u_2, u_3, \ldots, u_{n-1}\}$.
Conversely, presume that $\mathbb{D}_{NSp}$ is not a minimal $\widetilde{\mathbf{Sp}n-\mathbf{D}}$. Afterward there is a node $t \in \mathbb{D}_{NSp}$ so that $\mathbb{D}_{NSp}-\{t\}$ is a $\widetilde{\mathbf{Sp}n-\mathbf{D}}$. Thus t is a neutrosophic highly strong neighbor of at least n nodes in $\mathbb{D}_{NSp}-\{t\}$, and so (1) does not hold. Also, every node in $\mathbb{A}_{\mathbb{V}} \setminus \mathbb{D}_{NSp}$ is a neutrosophic highly strong neighbor of at least n nodes in $\mathbb{D}_{NSp}-\{t\}$, and so (2) does not keep, which is a inconsistency. Hereupon at least one of the clauses must be kept. Accordingly, $\mathbb{D}_{NSp}$ is a minimal $\widetilde{\mathbf{Sp}n-\mathbf{D}}$. □

Proposition 3.6. (1) *If* $|\mathbb{N}_{hs}(v)| \leq n-1$, *for* $v \in \mathbb{A}_{\mathbb{V}}$, *then* v *in any* $\widetilde{\mathbf{Sp}n-\mathbf{D}}$ *of* $\mathbb{G}_{Ne}$.
(2) *Every* $\widetilde{\mathbf{Sp}n-\mathbf{D}}$ *of* $\mathbb{G}_{Ne}$ *is a* $\widetilde{\mathbf{SpD}}$.
(3) $\check{\mathfrak{N}}^n_{\mathbb{A}_{\mathbb{V}}}(\mathbb{G}_{Ne}) \geq \check{\mathfrak{N}}_{\mathbb{A}_{\mathbb{V}}}(\mathbb{G}_{Ne})$, $\hat{\mathfrak{N}}^n_{\mathbb{A}_{\mathbb{V}}}(\mathbb{G}_{Ne}) \geq \hat{\mathfrak{N}}_{\mathbb{A}_{\mathbb{V}}}(\mathbb{G}_{Ne})$ *and so*

$$(\bar{\mathfrak{N}})^n(\mathbb{G}_{Ne}) \geq \bar{\mathfrak{N}}(\mathbb{G}_{Ne}).$$

Proof. The proofs are obvious. □

Note. If $\mathbb{M}$ is a minimal $\widetilde{\mathbf{SpD}}$ of a neutrosophic graph $\mathbb{G}_{Ne} = (\chi, \pi)$ without $\check{\mathcal{I}}_n$, then $\mathbb{A}_{\mathbb{V}} \setminus \mathbb{M}$ is a $\widetilde{\mathbf{SpD}}$ of $\mathbb{G}_{Ne}$.

Theorem 3.7. *Put* $\mathbb{D}_{NSp}$ $(n > 1)$ *be a minimal* $\widetilde{\mathbf{Sp}n-\mathbf{D}}$ *of* $\mathbb{G}_{Ne}$. *Afterward* $\mathbb{A}_{\mathbb{V}} \setminus \mathbb{D}_{NSp}$ *is not necessarily* $\widetilde{\mathbf{Sp}n-\mathbf{D}}$ *of* $\mathbb{G}_{Ne}$.

Proof. Suppose that $q \in \mathbb{A}_{\mathbb{V}}$ and $|\mathbb{N}_{hs}(q)| \leq n-1$. Afterward q in any $\widetilde{\mathbf{Sp}n-\mathbf{D}}$ of $\mathbb{G}_{Ne}$. Hereupon $q \in \mathbb{D}_{NSp}$ and so q does not belong to $\mathbb{A}_{\mathbb{V}} \setminus \mathbb{D}_{NSp}$. Accordingly, $\mathbb{A}_{\mathbb{V}} \setminus \mathbb{D}_{NSp}$ not included a $\widetilde{\mathbf{Sp}n-\mathbf{D}}$ of $\mathbb{G}_{Ne}$. □

Definition 3.8. Put $\mathbb{D} \subset \mathbb{A}_{\mathbb{V}}$ be a minimal $\widetilde{\mathbf{SpD}}$ of $\mathbb{G}_{Ne}$. Afterward:
(1) $\mathbb{D}^{-1} \subseteq \mathbb{A}_{\mathbb{V}} \setminus \mathbb{D}$ is defined an inverse $\widetilde{\mathbf{SpD}}$ of $\mathbb{G}_{Ne}$ regarding to $\mathbb{D}$ if $\mathbb{D}^{-1}$ is a $\widetilde{\mathbf{SpD}}$ of $\mathbb{G}_{Ne}$.
(2) An inverse $\widetilde{\mathbf{SpD}}$ $\mathbb{D}^{-1} \subseteq \mathbb{A}_{\mathbb{V}} \setminus \mathbb{D}$ said to be a minimal inverse $\widetilde{\mathbf{SpD}}$ of $\mathbb{G}_{Ne}$ regarding to $\mathbb{D}$ if no proper subset of $\mathbb{D}^{-1}$ is a $\widetilde{\mathbf{SpD}}$.
(3) The lower inverse neutrosophic special domination number of $\mathbb{G}_{Ne}$ is marked by $\check{\mathfrak{N}}^{-1}_{\mathbb{A}_{\mathbb{V}}}(\mathbb{G}_{Ne})$ and clarified as is explained:

$$\check{\mathfrak{N}}^{-1}_{\mathbb{A}_{\mathbb{V}}}(\mathbb{G}_{Ne}) = \min\left\{\mathbb{O}(\mathbb{D}^{-1}) \middle| \mathbb{D}^{-1} \text{is a minimal inverse } \widetilde{\mathbf{SpD}} \text{ of } \mathbb{G}_{Ne}\right\}.$$

(4) The upper inverse neutrosophic special domination number of $\mathbb{G}_{Ne}$ is marked by $\hat{\mathfrak{N}}^{-1}_{\mathbb{A}_{\mathbb{V}}}(\mathbb{G}_{Ne})$ and clarified as is explained:

$$\hat{\mathfrak{N}}^{-1}_{\mathbb{A}_{\mathbb{V}}}(\mathbb{G}_{Ne}) = \max\left\{\mathbb{O}(\mathbb{D}^{-1}) \middle| \mathbb{D}^{-1} \text{is a minimal inverse } \widetilde{\mathbf{SpD}} \text{ of } \mathbb{G}_{Ne}\right\}.$$

(5) The inverse neutrosophic special domination number of $\mathbb{G}_{Ne}$ is marked by $\bar{\mathfrak{N}}^{-1}(\mathbb{G}_{Ne})$ and clarified as is explained:

$$\bar{\mathfrak{N}}^{-1}(\mathbb{G}_{Ne}) = \frac{\check{\mathfrak{N}}^{-1}_{\mathbb{A}_{\mathbb{V}}}(\mathbb{G}_{Ne}) + \hat{\mathfrak{N}}^{-1}_{\mathbb{A}_{\mathbb{V}}}(\mathbb{G}_{Ne})}{2}.$$

Example 3.9. *Let be a* $\mathbb{NG}$ $\mathbb{G}_{Ne}$ *as in Figure 2. Clearly,* $\mathbb{D} = \{a, d\}$ *is a minimal neutrosophic special dominating set in* $\mathbb{G}_{Ne}$. *Also,* $\mathbb{A}_{\mathbb{V}} \setminus \mathbb{D} = \{b, c\}$ *is a neutrosophic slightly independent dominating set of* $\mathbb{G}_{Ne}$. *Accordingly,* $\mathbb{A}_{\mathbb{V}} \setminus \mathbb{D}$ *is a minimal neutrosophic special dominating set of* $\mathbb{G}_{Ne}$. *Therefore,* $\mathbb{A}_{\mathbb{V}} \setminus \mathbb{D}$ *is*

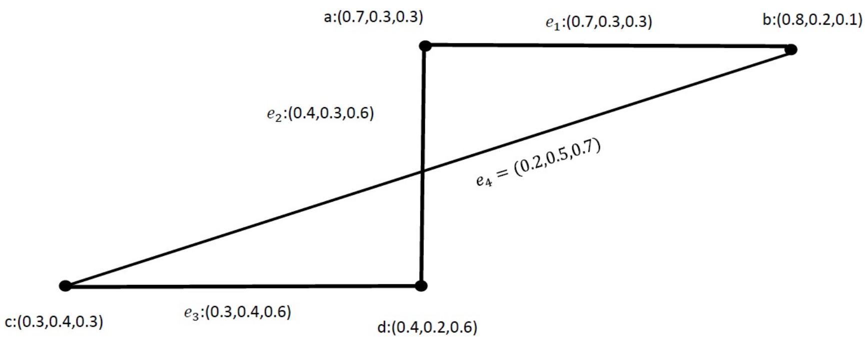

FIGURE 2. $\mathbb{NG}$ $\mathbb{G}_{Ne}$.

a minimal inverse neutrosophic special dominating set of $\mathbb{G}_{Ne}$ *regarding to* $\mathbb{D}$. *By simple computations,*

$$\check{\mathfrak{N}}^{-1}_{\mathbb{A}_{\mathbb{V}}}(\mathbb{G}_{Ne}) = \check{\mathfrak{N}}_{\mathbb{A}_{\mathbb{V}}}(\mathbb{G}_{Ne}) = 1.425,\ \hat{\mathfrak{N}}^{-1}_{\mathbb{A}_{\mathbb{V}}}(\mathbb{G}_{Ne}) = \hat{\mathfrak{N}}_{\mathbb{A}_{\mathbb{V}}}(\mathbb{G}_{Ne}) = 1.525,$$

and so

$$\bar{\mathfrak{N}}^{-1}(\mathbb{G}_{Ne}) = \bar{\mathfrak{N}}(\mathbb{G}_{Ne}) = 1.475.$$

Theorem 3.10. *A minimal* $\widetilde{\textbf{SpD}}$ $\mathbb{D}$ *of* $\mathbb{G}_{Ne}$ *is a minimal inverse* $\widetilde{\textbf{SpD}}$ *if and only if* $\mathbb{G}_{Ne}$ *has no* $\check{\mathcal{I}}_n$.

Proof. Presume that $\mathbb{M}$ is a minimal inverse $\widetilde{\textbf{SpD}}$ and $\mathbb{G}_{Ne}$ has a $\check{\mathcal{I}}_n$ $u \in \mathbb{A}_{\mathbb{V}}$, by the contrary. Afterward u in any $\widetilde{\textbf{SpD}}$ of $\mathbb{G}_{Ne}$. Hereupon u does not belong to $\mathbb{A}_{\mathbb{V}} \backslash \mathbb{M}$. Accordingly, $\mathbb{A}_{\mathbb{V}} \backslash \mathbb{M}$ not included $\widetilde{\textbf{SpD}}$ of $\mathbb{G}_{Ne}$, which is a inconsistency. Accordingly, $\mathbb{G}_{Ne}$ has no $\check{\mathcal{I}}_n$.
Conversely, let $\mathbb{M}$ be a minimal $\widetilde{\textbf{SpD}}$ of $\mathbb{G}_{Ne}$ and $\mathbb{G}_{Ne}$ have no $\check{\mathcal{I}}_n$. Afterward any node in $\mathbb{M}$ is specially dominated by at least one node in $\mathbb{A}_{\mathbb{V}} \setminus \mathbb{M}$ and $\mathbb{A}_{\mathbb{V}} \setminus \mathbb{M}$ is a $\widetilde{\textbf{SpD}}$ and hereupon $\mathbb{A}_{\mathbb{V}} \setminus \mathbb{M}$ contains a minimal $\widetilde{\textbf{SpD}}$ $\mathbb{M}^{-1}$ of $\mathbb{G}_{Ne}$. Accordingly, $\mathbb{M}$ is a minimal inverse $\widetilde{\textbf{SpD}}$ of $\mathbb{G}_{Ne}$ regarding to $\mathbb{M}^{-1}$. □

Theorem 3.11. *If* $\mathbb{G}_{Ne}$ *has a* $\check{\mathcal{I}}_n$ $u \in \mathbb{A}_{\mathbb{V}}$, *then* $\bar{\mathfrak{N}}^{-1}(\mathbb{G}_{Ne}) = 0$.

Proof. Assume that $u \in \mathbb{A}_{\mathbb{V}}$, so that $\left|\mathbb{N}_{hs}(u)\right| = 0$. Afterward u in any $\widetilde{\textbf{SpD}}$ $\mathbb{D}$ of $\mathbb{G}_{Ne}$. Hereupon u does not belong to $\mathbb{A}_{\mathbb{V}} \setminus \mathbb{D}$. Hence, $\mathbb{A}_{\mathbb{V}} \setminus \mathbb{D}$ not included $\widetilde{\textbf{SpD}}$ of $\mathbb{G}_{Ne}$. Accordingly, $\bar{\mathfrak{N}}^{-1}(\mathbb{G}_{Ne}) = 0$. □

Note. For any $\mathbb{NG}$ $\mathbb{G}_{Ne}$ without $\check{\mathcal{I}}_n$, we have

$$\check{\mathfrak{N}}_{\mathbb{A}_{\mathbb{V}}}(\mathbb{G}_{Ne}) \leq \check{\mathfrak{N}}^{-1}_{\mathbb{A}_{\mathbb{V}}}(\mathbb{G}_{Ne}).$$

Theorem 3.12. *If $\mathbb{G}_{Ne}$ is an $\mathbb{NG}$ and $\mathbb{M}$ is a minimal $\widetilde{\mathbf{SpD}}$ of $\mathbb{G}_{Ne}$, afterward*

$$\mathbb{O}(\mathbb{M}) + \mathbb{O}(\mathbb{M}^{-1}) \leq |\mathbb{A}_{\mathbb{V}}|.$$

In addition, equality is established if $\mathbb{G}_{Ne}$ has no $\check{\mathcal{I}}_n$ and $\mathbb{A}_{\mathbb{V}} \setminus \mathbb{M}$ is a $\check{\mathcal{I}}$S.

Proof. Let $\mathbb{G}_{Ne}$ be an $\mathbb{NG}$, $\mathbb{M}$ be a minimal $\widetilde{\mathbf{SpD}}$ and $\mathbb{M}^{-1}$ be a minimal inverse $\widetilde{\mathbf{SpD}}$ regarding to $\mathbb{M}$ of $\mathbb{G}_{Ne}$. Afterward $\mathbb{M}^{-1} \subseteq \mathbb{A}_{\mathbb{V}} \setminus \mathbb{M}$. Thus $\mathbb{O}(\mathbb{M}^{-1}) \leq \mathbb{O}(\mathbb{A}_{\mathbb{V}} \setminus \mathbb{M})$, and so $\mathbb{O}(\mathbb{M}^{-1}) \leq |\mathbb{A}_{\mathbb{V}}| - \mathbb{O}(\mathbb{M})$. Accordingly,

$$\mathbb{O}(\mathbb{M}^{-1}) + \mathbb{O}(\mathbb{M}) \leq |\mathbb{A}_{\mathbb{V}}|.$$

If $\mathbb{G}_{Ne}$ has no $\check{\mathcal{I}}_n$ and $\mathbb{M}$ is a minimal $\widetilde{\mathbf{SpD}}$, then $\mathbb{A}_{\mathbb{V}} \setminus \mathbb{M}$ is a $\widetilde{\mathbf{SpD}}$ and so $\mathbb{A}_{\mathbb{V}} \setminus \mathbb{M}$ is an inverse $\widetilde{\mathbf{SpD}}$ regarding to $\mathbb{M}$. Since $\mathbb{A}_{\mathbb{V}} \setminus \mathbb{M}$ is a $\check{\mathcal{I}}$S of $\mathbb{G}_{Ne}$, we obtain $\mathbb{A}_{\mathbb{V}} \setminus \mathbb{M}$ is a minimal inverse $\widetilde{\mathbf{SpD}}$ of $\mathbb{G}_{Ne}$ regarding to $\mathbb{M}$. □

Corollary 3.13. *If $\mathbb{G}_{Ne}$ has an inverse $\widetilde{\mathbf{SpD}}$, afterward*

$$\check{\mathfrak{N}}_{\mathbb{A}_{\mathbb{V}}}(\mathbb{G}_{Ne}) + \check{\mathfrak{N}}^{-1}_{\mathbb{A}_{\mathbb{V}}}(\mathbb{G}_{Ne}) \leq |\mathbb{A}_{\mathbb{V}}|.$$

Note. If $\mathbb{M} \subset \mathbb{A}_{\mathbb{V}}$ is a minimal $\widetilde{\mathbf{SpD}}$ of $\mathbb{G}_{Ne}$, afterward $\widetilde{\mathbf{SpD}}$ $\mathbb{M}^{-1} \subseteq \mathbb{A}_{\mathbb{V}} \setminus \mathbb{M}$ is a minimal inverse $\widetilde{\mathbf{SpD}}$ of $\mathbb{G}_{Ne}$ regarding to $\mathbb{M}$ if and only if for any node $s \in \mathbb{M}^{-1}$, one of the following qualifications correct.
(1) $\mathbb{N}_{hs}(s) \subseteq \mathbb{A}_{\mathbb{V}} \setminus \mathbb{M}^{-1}$,
(2) there is a node $t \in \mathbb{A}_{\mathbb{V}} \setminus \mathbb{M}^{-1}$ where $\mathbb{N}_{hs}(t) \cap \mathbb{M}^{-1} = \{s\}$.

Theorem 3.14. *If $\check{\mathfrak{N}}_{\mathbb{A}_{\mathbb{V}}}(\mathbb{G}_{Ne}) \geq \dfrac{|\mathbb{A}_{\mathbb{V}}|}{2}$, then $\mathbb{G}_{Ne}$ does not have inverse $\widetilde{\mathbf{SpD}}$s.*

Proof. Assume that $\mathbb{M}$ is a minimal $\widetilde{\mathbf{SpD}}$ of $\mathbb{G}_{Ne}$ and $\mathbb{O}(\mathbb{M}) = d_{\mathbb{A}_{\mathbb{V}}}(\mathbb{G}) \geq \dfrac{|\mathbb{A}_{\mathbb{V}}|}{2}$. If $\mathbb{M}^{-1} \subseteq \mathbb{A}_{\mathbb{V}} \setminus \mathbb{M}$ is a minimal inverse $\widetilde{\mathbf{SpD}}$ regarding to $\mathbb{M}$ and $\check{\mathfrak{N}}^{-1}_{\mathbb{A}_{\mathbb{V}}}(\mathbb{G}_{Ne}) \leq \mathbb{O}(\mathbb{M}^{-1})$, by the contrary, afterward by Theorem 3.12,

$$\mathbb{O}(\mathbb{M}) + \mathbb{O}(\mathbb{M}^{-1}) \leq |\mathbb{A}_{\mathbb{V}}|,$$

and so,

$$\mathbb{O}(\mathbb{M}^{-1}) \leq |\mathbb{A}_{\mathbb{V}}| - \mathbb{O}(\mathbb{M}),$$

afterward,

$$\mathbb{O}(\mathbb{M}^{-1}) < \frac{|\mathbb{A}_{\mathbb{V}}|}{2},$$

hereupon,

$$\check{\mathfrak{N}}^{-1}_{\mathbb{A}_{\mathbb{V}}}(\mathbb{G}_{Ne}) < \check{\mathfrak{N}}_{\mathbb{A}_{\mathbb{V}}}(\mathbb{G}_{Ne}),$$

which is a a inconsistency. Accordingly, $\mathbb{G}_{Ne}$ has no inverse $\widetilde{\mathbf{SpD}}$. □

Corollary 3.15. *If $\mathbb{G}_{Ne}$ has an inverse $\widetilde{\mathbf{SpD}}$, then $\check{\mathfrak{N}}_{\mathbb{A}_{\mathbb{V}}}(\mathbb{G}_{Ne}) \leq \dfrac{|\mathbb{A}_{\mathbb{V}}|}{2}$.*

If in a graph $\mathbb{G}_{\mathbb{C}} = (\mathbb{A}_{\mathbb{V}}, \mathbb{B}_{\mathbb{E}})$ we add an arc e to $\mathbb{B}_{\mathbb{E}}$, afterward we denote it by $\mathbb{B}_{\mathbb{E}e} = \mathbb{B}_{\mathbb{E}} \cup \{e\}$ and $\mathbb{G}_{\mathbb{C}e} = (\mathbb{A}_{\mathbb{V}}, \mathbb{B}_{\mathbb{E}e})$. Moreover, if a $\mathbb{NG}$ $\mathbb{G}_{Ne} = (\chi, \pi)$ on $\mathbb{G}_{\mathbb{C}}$ extends on $\mathbb{G}_{\mathbb{C}e}$, then we characterize it by $\mathbb{G}^{e}_{Ne} = (\chi_e, \pi_e)$. If arc e in $\mathbb{NG}$ $\mathbb{G}^{e}_{Ne}$ is a $\mathcal{HS}$, afterward we characterize $\mathbb{G}^{hs}_{Ne} = (\chi^{hs}_e, \pi^{hs}_e)$ instead of $\mathbb{G}^{e}_{Ne} = (\chi_e, \pi_e)$.

Theorem 3.16. *Let $\mathbb{G}_{Ne}$ have no $\check{\mathcal{I}}_n$ and $a = uv$ be an additional $\mathcal{HS}$ in $\mathbb{G}_{\mathbb{C}a}$. Afterward,*

$$(\bar{\mathfrak{N}})^{-1}(\mathbb{G}^{a}_{Ne}) \leq \bar{\mathfrak{N}}^{-1}(\mathbb{G}_{Ne}).$$

Proof. If $\mathbb{M}$ is a minimal inverse $\widetilde{\mathbf{SpD}}$ of $\mathbb{G}_{Ne}$ and $u, v \in \mathbb{M}$ so that $\mathbb{N}_{hs}(u) \subseteq \mathbb{A}_{\mathbb{V}} \setminus \mathbb{M}$, afterward by adding $a = uv$, $\mathbb{M} \setminus \{u\}$ is a minimal inverse $\widetilde{\mathbf{SpD}}$ in $\mathbb{G}^{a}_{Ne}$. Otherwise, $\mathbb{M}$ is a minimal inverse $\widetilde{\mathbf{SpD}}$ in $\mathbb{G}^{a}_{Ne}$. Accordingly,

$$(\bar{\mathfrak{N}})^{-1}(\mathbb{G}^{a}_{Ne}) \leq (\bar{\mathfrak{N}})^{-1}(\mathbb{G}_{Ne}).$$

□

Definition 3.17. Let $\mathbb{D}_{NSp} \subset \mathbb{A}_{\mathbb{V}}$ be a minimal $\widetilde{\mathbf{Sp}n - \mathbf{D}}$ of $\mathbb{G}_{Ne}$. Afterward:
(1) $\mathbb{D}^{-1}_{NSp} \subseteq \mathbb{A}_{\mathbb{V}} \setminus \mathbb{D}_{NSp}$ is named an inverse $\widetilde{\mathbf{Sp}n - \mathbf{D}}$ of $\mathbb{G}_{Ne}$ regarding to $\mathbb{D}_{NSp}$ if $\mathbb{D}^{-1}_{NSp}$ is a $\widetilde{\mathbf{Sp}n - \mathbf{D}}$ of $\mathbb{G}_{Ne}$.
(2) An inverse $\widetilde{\mathbf{Sp}n - \mathbf{D}}$ $\mathbb{D}^{-1}_{NSp} \subseteq \mathbb{A}_{\mathbb{V}} \setminus \mathbb{D}_{NSp}$ said to be a minimal inverse $\widetilde{\mathbf{Sp}n - \mathbf{D}}$ of $\mathbb{G}_{Ne}$ regarding to $\mathbb{D}_{NSp}$ if no proper subset of $\mathbb{D}^{-1}_{NSp}$ is a $\widetilde{\mathbf{Sp}n - \mathbf{D}}$.
(3) The lower inverse neutrosophic special $n-$domination number of $\mathbb{G}_{Ne}$ is marked by $(\check{\mathfrak{N}}^{n}_{\mathbb{A}_{\mathbb{V}}})^{-1}(\mathbb{G}_{Ne})$ and clarified as is explained:

$$(\check{\mathfrak{N}}^{n}_{\mathbb{A}_{\mathbb{V}}})^{-1}(\mathbb{G}_{Ne}) = \min \left\{ \mathbb{O}(\mathbb{D}^{-1}_{NSp}) \middle| \mathbb{D}^{-1}_{NSp} \text{is a minimal inverse } \widetilde{\mathbf{Sp}n - \mathbf{D}} \text{ of } \mathbb{G}_{Ne} \right\}.$$

(4) The upper inverse neutrosophic special domination number of $\mathbb{G}_{Ne}$ is marked by $(\hat{\mathfrak{N}}^{n}_{\mathbb{A}_{\mathbb{V}}})^{-1}(\mathbb{G}_{Ne})$ and clarified as is explained:

$$(\hat{\mathfrak{N}}^{n}_{\mathbb{A}_{\mathbb{V}}})^{-1}(\mathbb{G}_{Ne}) = \max \left\{ \mathbb{O}(\mathbb{D}^{-1}_{NSp}) \middle| \mathbb{D}^{-1}_{NSp} \text{is a minimal inverse } \widetilde{\mathbf{Sp}n - \mathbf{D}} \text{ of } \mathbb{G}_{Ne} \right\}.$$

(5) The inverse neutrosophic special $n-$domination number of $\mathbb{G}_{Ne}$ is marked by $(\bar{\mathfrak{N}})^{n})^{-1}(\mathbb{G}_{Ne})$ and clarified as is explained:

$$((\bar{\mathfrak{N}})^{n})^{-1}(\mathbb{G}_{Ne}) = \frac{(\check{\mathfrak{N}}^{n}_{\mathbb{A}_{\mathbb{V}}})^{-1}(\mathbb{G}_{Ne}) + (\hat{\mathfrak{N}}^{n}_{\mathbb{A}_{\mathbb{V}}})^{-1}(\mathbb{G}_{Ne})}{2}.$$

Theorem 3.18. *A minimal $\widetilde{\mathbf{Sp}n - \mathbf{D}}$ $\mathbb{D}_{NSp}$ of $\mathbb{G}_{Ne}$ is a minimal inverse $\widetilde{\mathbf{Sp}n - \mathbf{D}}$ if $\mathbb{G}_{Ne}$ has no node $q \in \mathbb{A}_{\mathbb{V}}$ wherever $\left|\mathbb{N}_{hs}(q)\right| \leq n - 1$.*

Proof. Suppose that $\mathbb{D}_{NSp}$ is a minimal inverse $\widetilde{\mathbf{Sp}n - \mathbf{D}}$ and $\mathbb{G}_{Ne}$ has a node $q \in \mathbb{A}_{\mathbb{V}}$ so that $\left|\mathbb{N}_{hs}(q)\right| \leq n - 1$, by the contrary. Afterward q in any $\widetilde{\mathbf{Sp}n - \mathbf{D}}$

of $\mathbb{G}_{Ne}$. Hereupon $q \in \mathbb{D}_{NSp}$ and so q does not belong to $\mathbb{A}_{\mathbb{V}} \setminus \mathbb{D}_{NSp}$. Accordingly, $\mathbb{A}_{\mathbb{V}} \setminus \mathbb{D}_{NSp}$ not included $\widetilde{\mathbf{Sp}n - \mathbf{D}}$ of $\mathbb{G}_{Ne}$, which is a inconsistency. Accordingly, $\mathbb{G}_{Ne}$ has no node $q \in \mathbb{A}_{\mathbb{V}}$ so that $\left|\mathbb{N}_{hs}(q)\right| \leq n-1$. □

Theorem 3.19. *If $\mathbb{G}_{Ne}$ has a node $q \in \mathbb{A}_{\mathbb{V}}$ wherever $\left|\mathbb{N}_{hs}(q)\right| \leq n-1$, then* $((\bar{\mathfrak{N}})^n)^{-1}(\mathbb{G}_{Ne}) = 0$.

Proof. Suppose that $q \in \mathbb{A}_{\mathbb{V}}$ wherever $\left|\mathbb{N}_{hs}(q)\right| = n-1$. Afterward q in any $\widetilde{\mathbf{Sp}n - \mathbf{D}}$ of $\mathbb{G}_{Ne}$. Hereupon $q \in \mathbb{D}_{NSp}$ and so q does not belong to $\mathbb{A}_{\mathbb{V}} \setminus \mathbb{D}_{NSp}$. Accordingly, $\mathbb{A}_{\mathbb{V}} \setminus \mathbb{D}_{NSp}$ not included $\widetilde{\mathbf{Sp}n - \mathbf{D}}$ of $\mathbb{G}_{Ne}$. Accordingly, $((\bar{\mathfrak{N}})^n)^{-1}(\mathbb{G}_{Ne}) = 0$. □

Note. (1) If $\mathbb{G}_{Ne}$ has no node $q \in \mathbb{A}_{\mathbb{V}}$ wherever $\left|\mathbb{N}_{hs}(q)\right| \leq n-1$, we have

$$\check{\mathfrak{N}}^n_{\mathbb{A}_{\mathbb{V}}}(\mathbb{G}_{Ne}) \leq (\check{\mathfrak{N}}^n_{\mathbb{A}_{\mathbb{V}}})^{-1}(\mathbb{G}_{Ne}).$$

(2) If $\mathbb{G}_{Ne}$ is a neutrosophic graph and $\mathbb{D}_{NSp}$ is a minimal $\widetilde{\mathbf{Sp}n - \mathbf{D}}$ of $\mathbb{G}_{Ne}$, afterward

$$\mathbb{O}(\mathbb{D}_{NSp}) + \mathbb{O}(\mathbb{D}^{-1}_{NSp}) \leq |\mathbb{A}_{\mathbb{V}}|.$$

(3) If $\mathbb{G}_{Ne}$ has an inverse $\widetilde{\mathbf{Sp}n - \mathbf{D}}$, afterward

$$\check{\mathfrak{N}}^n_{\mathbb{A}_{\mathbb{V}}}(\mathbb{G}_{Ne}) + (\check{\mathfrak{N}}^n_{\mathbb{A}_{\mathbb{V}}})^{-1}(\mathbb{G}) \leq |\mathbb{A}_{\mathbb{V}}|.$$

(4) If $\check{\mathfrak{N}}^n_{\mathbb{A}_{\mathbb{V}}}(\mathbb{G}_{Ne}) \geq \dfrac{|\mathbb{A}_{\mathbb{V}}|}{2}$, afterward $\mathbb{G}_{Ne}$ has no inverse $\widetilde{\mathbf{Sp}n - \mathbf{D}}$.

Theorem 3.20. *$\mathbb{G}_{Ne}$ has at least two distinct $\widetilde{\mathbf{Sp}n - \mathbf{D}}s$ if and only if* $(\bar{\mathfrak{N}})^n)^{-1}(\mathbb{G}_{Ne}) > 0$.

$\widetilde{\mathbf{Sp}n - \mathbf{D}}s$

Proof. Assume that $\mathbb{M}_1$ and $\mathbb{M}_2$ are two distinct $\widetilde{\mathbf{Sp}n - \mathbf{D}}s$ of $\mathbb{G}_{Ne}$. Afterward $\mathbb{M}_1$ and $\mathbb{M}_2$ contain minimal $\widetilde{\mathbf{Sp}n - \mathbf{D}}s$ $\mathbb{M}^{'}_1$ and $\mathbb{M}^{'}_2$, respectively. Accordingly, $\mathbb{M}^{'}_1 \subseteq \mathbb{A}_{\mathbb{V}} \setminus \mathbb{M}^{'}_2$ is a minimal inverse $\widetilde{\mathbf{Sp}n - \mathbf{D}}$ of $\mathbb{G}_{Ne}$ regarding to $\mathbb{M}^{'}_2$ and $\mathbb{M}^{'}_2 \subseteq \mathbb{A}_{\mathbb{V}} \setminus \mathbb{M}^{'}_1$ is a minimal inverse $\widetilde{\mathbf{Sp}n - \mathbf{D}}$ of $\mathbb{G}_{Ne}$ regarding to $\mathbb{M}^{'}_1$. Accordingly, $(\bar{\mathfrak{N}})^n)^{-1}(\mathbb{G}_{Ne}) > 0$.
Conversely, suppose that $(\bar{\mathfrak{N}})^n)^{-1}(\mathbb{G}_{Ne}) > 0$, $\mathbb{M}_{NSp}$ is a minimal $\widetilde{\mathbf{Sp}n - \mathbf{D}}$ and $\mathbb{M}^{-1}_{NSp}$ is a minimal inverse $\widetilde{\mathbf{Sp}n - \mathbf{D}}$ regarding to $\mathbb{M}_{NSp}$ of $\mathbb{G}_{Ne}$. Afterward, it is obvious that $\mathbb{M}_{NSp}$ and $\mathbb{M}^{-1}_{NSp}$ distinct. □

4. Application of the inverse $\widetilde{\mathbf{SpD}}$ in decision making under ashy clauses between certainty and uncertainty

$\mathbb{NG}$ models are more flexible than other uncertainty models in dealing with human-collected data. In this study, we proffered the concept of a inverse $\widetilde{\mathbf{Sp}n - \mathbf{D}}$ in $\mathbb{NG}$ theory. The inverse $\widetilde{\mathbf{Sp}n - \mathbf{D}}$ in the neutrosophic network can be useful to solve many real-life problems. By understanding the concept of inverse neutrosophic special domination, it can be concluded that inverse $\widetilde{\mathbf{SpD}}s$ play a valuable role in helping to control and manage neutrosophic graphs.

Monitoring and confirming the decisions in various dimensions of a corporation with favorable and prearranged yield ideals is one of the principal missions of a corporation. It performs a substantial role in elevating the yield level as well as the influence rate of the corporation. Hence, designing, structuring and optimizing the map assist to advance the goals of a corporation. With regards to the affecting factors, the yield of a corporation is one of the substantial issues investigated by the chiefs of a corporation. The set of affecting agents and the yield of a corporation can be presumed as a $\mathbb{NG}$. We characterize the ξ-strength, ϖ-strength and ϱ-strength values in any node and arc (path) as is explained. For any $q, p \in \mathbb{A}_{\mathbb{V}}$ and $qp \in \mathbb{B}_{\mathbb{E}}$, we obtain:
$\xi_\chi(q)$: The heaviness of the direct influence of factor q on the corporation's yield in ashy clauses.
$\varpi_\chi(q)$: The heaviness of the inefficient of factor q on the corporation's yield in ashy clauses.
$\varrho_\chi(q)$: The heaviness of the indirect efficacy of factor q on the corporation's yield in ashy clauses.
$\xi_\pi(qp)$: The heaviness of direct capability qp on the corporation's yield in ashy clauses.
$\varpi_\pi(qp)$: The heaviness of the inefficient qp on the corporation's yield in ashy clauses.
$\varrho_\pi(qp)$: The heaviness of indirect capability qp on the corporation's yield in ashy clauses.
Hereon, the following relationships befit logically:

$$\xi_\pi(qp) \leq \xi_\chi(q) \wedge \xi_\chi(p), \;\; \varpi_\pi(qp) \geq \varpi_\chi(q) \vee \varpi_\chi(p), \;\; \varrho_\pi(qp) \geq \varrho_\chi(q) \vee \varrho_\chi(p).$$

The relevance between q and p is effectual when the qp is a $\mathcal{HS}$. Hence, the $\widetilde{\mathbf{SpD}}$ of this graph comprise factors that other factors are speciality dominated by at least one of the factors of this set. The $\widetilde{\mathbf{SpD}}$ creates a chance for officiares and commanders of the corporation to concentrate on the factors of the $\widetilde{\mathbf{SpD}}$ instead of paying attention and monitoring other decision agents. This helps the corporate commanders and officiares make the best decisions in a short time interval. For example, Figure 3, shows the graph of efficacious agents on the yield of a corporation, in which the set of $\{u_1, u_7\}$ is a minimal $\widetilde{\mathbf{SpD}}$ (with minimum neutrosophic node cardinality 2.3) and the set of $\{u_2, u_3, u_4, u_5, u_6, u_8\}$ is an inverse $\widetilde{\mathbf{SpD}}$ of $\mathbb{G}_{Ne}$ regarding to $\mathbb{D}$ (with neutrosophic node cardinality 7.05). In other words, instead of controlling the 7 agents, only agents u_2, u_4, u_7 can be controlled and observed and ensure desirable performance in the decision-making process. It is notable that some agents, as standard computational indices between two agents, dependent calculation formula, and relevance between the variables calculating the indices of the agents, play a substantial role in establishing effective relevance between them. For instance, in Figure 3, indicates that the most favorable effective weight of the agent graph($\mathbb{S}(\mathfrak{P})$ wherever $\mathfrak{P}$ is the set of all highly strong arcs of $\mathbb{G}_{Ne}$) is 5.5.

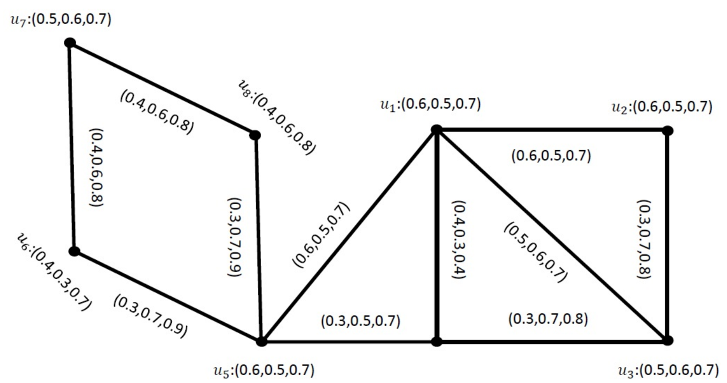

FIGURE 3. Neutrosophic graph $\mathbb{G}_{Ne}$.

As we checked in [11], it is feasible to increase the most favorable effective weight of the agent graph on favorable yield achievement by amplifying the relevance between the agents of the neutrosophic special dominating set. which leads to raise exactitude and certitude in the decision-making process and reducing the neutrosophic node cardinality of the $\widetilde{\mathbf{SpD}}$. Here, if feasible, to amplify the relevance between the agents of the $\widetilde{\mathbf{SpD}}$, the inverse $\widetilde{\mathbf{SpD}}$ can be used and with reinforcing the relevance between the factors of the inverse $\widetilde{\mathbf{SpD}}$, the most favorable effective weight of the agent graph can be increased. While the neutrosophic node cardinality of minimal $\widetilde{\mathbf{SpD}}$ is also fixed, in some cases it also reduces the neutrosophic special domination number of the agent graph on favorable yield achievement by substituting the minimal $\widetilde{\mathbf{SpD}}$. For instance, in Figure 4, by making an effective relevance with the coordinates $(0.4, 0.6, 0.8)$ between the agents u_2 and u_8 of $\mathbb{D}^{-1}$, the effective weight of the graph enhance to 6.5 while the minimal neutrosophic special dominating set is still constant and also neutrosophic special domination number of the agent graph reduced.

$\mathbb{D}$	$\mathbb{D}^{-1}$	$O(\mathbb{D})$	$\mathbb{O}(\mathbb{D}^{-1})$	$\mathfrak{N}(\mathbb{G}_{Ne})$	$\mathbb{S}(\mathfrak{P})$
$\mathbb{D} = \{u_1, u_7\}$	$\mathbb{D}^{-1} = \{u_2, u_3, u_4, u_5, u_6, u_8\}$	2.3	7.05	4.675	5.5
$\mathbb{D} = \{u_1, u_7\}$	$(\mathbb{D}^{-1})' = \{u_3, u_4, u_5, u_6, u_8\}$	2.3	5.85	4.175	6.5
$\mathbb{D} = \{u_1, u_7\}$	$(\mathbb{D}^{-1})'' = \{u_2, u_3, u_4, u_5, u_6\}$	2.3	6.05	4.175	6.5

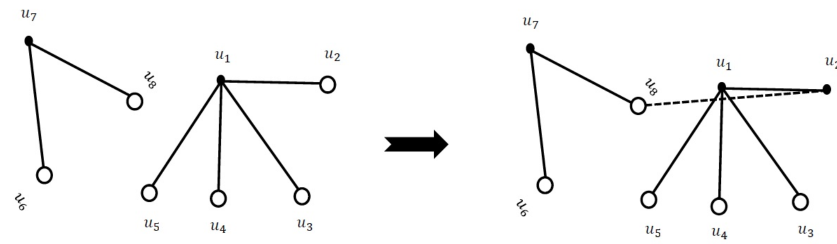

FIGURE 4. $\mathbb{D}^{-1}$ and $(\mathbb{D}^{-1})'$.

The advantage of using the inverse neutrosophic special n-dominating set technique in the presented method, compared to other methods, suggests the optimization, promotion and development solutions while the minimal neutrosophic special dominating set is still constant and neutrosophic special domination number decreased

5. Conclusion

Graphs are very suitable tools for displaying many issues in various fields. The concept of domination in graph is also very useful in theoretical and developmental researches. In this paper, we introduced for the first time the meanings of $\widetilde{\mathbf{Sp}n - \mathbf{D}}$ and inverse $\widetilde{\mathbf{Sp}n - \mathbf{D}}$ in a $\mathbb{NG}$. Finally, by applying the meaning of inverse $\widetilde{\mathbf{SpD}}$ and the reduction effect of an additional $\mathcal{HS}$ on the neutrosophic special domination number parameter, a model for optimizing the neutrosophic special domination parameter was presented. The advantage of this model over the model introduced in [11] is to optimize the neutrosophic special domination number parameter in specific and targeted sections more exactly while ensuring the stability or reduction of the neutrosophic special domination number parameter in neutrosophic network. In future works, we plan to study the notions of special regular and irregular NGs.

References

[1] Akram, M. Single-Valued Neutrosophic Graphs. Infosys Science Foundation Series in Mathematical Sciences, Springer, 2018, 213-237.

[2] Akram, M.; Shahzadi, S. Neutrosophic soft graphs with application. Journal of Intelligent & Fuzzy Systems. 2017, 32(1), 841-858, 2017.

[3] Akram, M.; Shahzadi, G. Operations on single-valued neutrosophic graphs, Journal of Uncertain Systems. 2017, 11(1), 1-26

[4] Atanassov, K. Intuitionistic fuzzy sets. Fuzzy Sets Syst. 1986, 20, 87-96.

[5] Atanassov, K.; Gargov, G. Interval valued intuitionistic fuzzy sets. Fuzzy Sets Syst. 1989, 31, 343-349.

[6] Banitalebi, S. Irregular vague graphs. J. algebr. hyperstrucres log. algebr. 2021, 2, 73-90.

[7] Banitalebi, S.; Ahn, S. S.; Jun, Y. B.; Borzooei, R. A. Normal m-domination and inverse m-domination in Pythagorean fuzzy graphs with application in decision making. J. Intell. Fuzzy Syst. (2022), 1-10.

[8] Banitalebi, S.; Borzooei, R. A. Domination in Pythagorean fuzzy graphs. Granul. Comput. 2023, 1-8.

[9] Banitalebi, S.; Borzooei, R. A. Domination of vague graphs by using of strong arcs. J. Math. Ext. 2022, 16(3), 1-22.

[10] Banitalebi, S.; Borzooei, R. A. 2-Domination in vague graphs. J. Algebr. Struct. their Appl. 2021, 8, 203-222.

[11] Banitalebi, S.; Borzooei, R. A. Neutrosophic special dominating set in neutrosophic graphs, NSS. 2021, 45, 26-39.

[12] Broumi, S.; Talea, M.; Bakali, A.; Smarandache, F. Single valued neutrosophic graphs. J. New Theory. 2016, 10, 861-101.

[13] Deng, J.; Zhan, J.; Herrera-Viedma, E.; Herrera, F. Regret theory-based three-way decision method on incomplete multi-scale decision information systems with interval fuzzy numbers. IEEE Transactions on Fuzzy Systems, 2022, doi: 10.1109/TFUZZ.2022.3193453.

[14] Deng, J.; Zhan, J.; Xu, Z.; Herrera-Viedma, E. Regret-Theoretic Multiattribute Decision-Making Model Using Three-Way Framework in Multiscale Information Systems. IEEE Transactions on Cybernetics, 2022, doi: 10.1109/TCYB.2022.3173374

[15] Fei, Y. Study on neutrosophic graph with application in wireless network. CAAI Trans. Intell. Technol. 2020, 5, 301-307.

[16] Hussain, S.; Satham, R.; Hussain, Florentin Smarandache. Domination number in neutrosophic soft graphs. NSS. 2019, 28, 228-244.

[17] Smarandache, F. Neutrosophic set - A generalization of the intuitionistic fuzzy set. Int J Pure Appl Math. 2005, 24(3), 287-297.

[18] Smarandache, F. Refined Literal Indeterminacy and the Multiplication Law of Sub-Indeterminacies. NSS. 2015, 9, 58-63.

[19] Smarandache, F. Symbolic Neutrosophic theory. Brussels, Europanova, 2015, 103-120.

[20] Turksen, I. Interval valued fuzzy sets based on normal forms. Fuzzy Sets Syst. 1986, 20, 191-210.

[21] Wang, W.; Zhan, J.; Herrera-Viedma, E. A three-way decision approach with a probability dominance relation based on prospect theory for incomplete information systems. Information Sciences. 2021, 611, 199-224.

[22] Wang, W.; Zhan, J.; Zhang, C.; Herrera-Viedma, E.; Kou, G. A regret-theory-based three-way decision method with a priori probability tolerance dominance relation in fuzzy incomplete information systems. Information Fusion. 2023, 89, 382-396.

[23] Zadeh, L. A. Fuzzy sets. Inf. Control. 1965, 8, 338-353.

Sadegh Banitalebi
Orcid number: 0009-0004-8803-7015
Researcher, Department of Cognitive Modeling and Simulation
Faculty of Artificial Intelligence and Cognitive Sciences
Imam Hossein University
Tehran, Iran
Email address: krbanitalebi@ihu.ac.ir

Rajab Ali Borzooei
Orcid number: 0000-0001-7538-7885
Department of Mathematics
Shahid Beheshti University
Tehran, Iran
Email address: borzooei@sbu.ac.ir