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RICCI-BOURGUIGNON FLOW ON OPEN SURFACE

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Abstract. In this paper, we investigate the normalized Ricci-Bourguignon flow with incomplete initial metric on an open surface. We show that such a flow converges exponentially to a metric with constant Gaussian curvature if the initial metric is suitable. In particular, if the initial metric is complete, then the metrics converge to the standard hyperbolic metric.

Keywords: Ricci-Bourguignon flow, incomplete surface, uniformization theorem.
2020 MSC: Primary 53E20, 30F10, 53C40.

1. Introduction

Let $(M, g(t))$, $t \in [0, T)$ be an n-dimensional manifold evolving along the normalized Ricci-Bourguignon flow

$$\frac{\partial g}{\partial t} = -2Ric + 2\rho R g + (1 - n\rho)\frac{2r}{n} g, \qquad g(0) = g_0, \tag{1}$$

where Ric is the Ricci curvature tensor, R is the scalar curvature, $r = \frac{\int_M R d\mu}{\int_M d\mu}$ is the average of the scalar curvature R, and ρ is a real constant. The Ricci-Bourguignon flow introduced by J. P. Bourguignon in [2]. It should be noted that the Ricci-Bourguignon flow is a generalization of some other geometric flows. For instance, when $\rho = 0$ the tensor $Ric - \rho R g - (1 - n\rho)\frac{r}{n} g$ corresponds to the tensor $Ric - \frac{r}{n} g$ and the normalized Ricci-Bourguignon flow (1) becomes the normalized Ricci flow. When $\rho = \frac{1}{n}$, the tensor $Ric - \rho R g - (1 - n\rho)\frac{r}{n} g$ corresponds to the traceless Ricci tensor. The short time existence and uniqueness of the solution of the Ricci-Bourguignon flow (1) as a system of partial differential equations on $[0, T)$ have been established by Catino et al. [3] for the case $\rho < \frac{1}{2(n-1)}$. Other recent studies on the flow include [1, 4, 13].

A smooth surface with a complete metric of Gaussian curvature -1 is called a hyperbolic surface. Hamilton [9] and Chow [5] proved the normalized Ricci flow with any initial metric on a compact surface converges to a constant Gaussian curvature metric. Then Shi [15] showed that on complete manifolds with bounded curvature, a complete Ricci flow $g(t)$ exists for $t \in [0, T]$, for some

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$T > 0$. In 2009, Ji et al. [10] and Yin [17] studied normalized Ricci flow on hyperbolic cusp and non parabolic surfaces with additional conditions, respectively. In fact, Ji et al. [10] considered the normalized Ricci flow $\frac{\partial g}{\partial t} = (r - R)g$ with the complete metric g_0 on an open surface (that is, a non-compact Riemann surface) M, where M is conformal to a punctured compact Riemannian surface and g_0 has ends which are asymptotic to hyperbolic cusps. Then, they showed that if the Euler number of M, $\mathcal{X}(M)$, is less than zero and $r < 0$, the flow $g(t)$ converges exponentially to a unique complete metric of constant curvature $\frac{r}{2}$ in the conformal class. Yin [17], considered the normalized Ricci flow on a Riemann surface M, where M is obtained from a compact Riemann surface by removing finitely many disjoint point and/or closed disks. Also, if no disk is removed, then he further assumed that $\mathcal{X}(M) < 0$ and he proved that there exists on M, a complete hyperbolic metric compatible with the conformal structure. Note that a conformal structure on a manifold is the structure of a Riemannian metric modulo rescaling of the metric tensor by the some real valued function on the manifold. Giesen and Topping [8] considered the Ricci flow of negatively curved incomplete surfaces and showed such a flow exists for all time and the normalized Ricci flow converges. In 2013, X. Zhu [18], studied the normalized Ricci flow with incomplete initial metric on open surfaces. In 2019, Cortissoz and Murcia [6] investigated the Ricci flow on surfaces with boundary and in 2022, Dubedat and Shen [7] studied stochastic Ricci flow on compact surfaces. Also, Katsinis et al. [12] considered geometric flow on minimal surfaces.

Motivated by the above results, we investigate the normalized Ricci-Bourguignon flow with incomplete initial metric on open surfaces using the same techniques as in [8, 18]. We prove that under certain condition on initial metric, the normalized Ricci-Bourguignon flow always converges exponentially to a metric of constant Gaussian curvature. In addition, if the initial metric is complete then the normalized Ricci-Bourguignon flow converges to the hyperbolic metric. We generalize the results of [18], because assuming $\rho = 0$, the results of [18] are obtained.
The main results of the paper are as follows.

Theorem 1.1. *Let M be a Riemann surface equipped with a smooth conformal metric g_0 of Gaussian curvature $K_0 \leq -1$. Then flow (2) converges exponentially to a conformal metric with Gaussian curvature -1 for $\rho < \frac{1}{2}$. Moreover, if g_0 is complete, then the solution of this flow converges to the standard hyperbolic metric for $\rho < \frac{1}{2}$.*

From [17] we have the following theorem.

Theorem 1.2 ([17])**.** *Suppose that M is a Riemann surface obtained from a compact Riemann surface by removing finitely many disjoint closed disks and/or points. If no disk is removed, we assume that the Euler number of M is less than zero. Then M is a hyperbolic surface.*

Using Theorems 1.1 and 1.2, we obtain the following corollary.

Corollary 1.3. *Let $N \subset M$ be a sub-domain (an open connected subset) of a punctured and bordered Riemann surface with $\mathcal{X}(M) < 0$. Then N is hyperbolic.*

A domain in a plane is called a planar domain. By Theorem 1.2, the surface $\mathbb{C} \setminus \{p, q\} = S^2 \setminus \{\infty, p, q\}$ is hyperbolic, so we get the following corollary.

Corollary 1.4. *Any hyperbolic planar domain N is covered by the unit disc $\mathbb{D}$. In particular, any simply connected planar hyperbolic domain is biholomorphic to the unit disc $\mathbb{D}$.*

2. Proof of the main results

On open surfaces the normalized Ricci-Bourguignon flow becomes

$$\frac{\partial g}{\partial t} = -(1-2\rho)(r-R)g, \qquad g(0) = g_0. \tag{2}$$

Without loss of generality we choose $r = -2$ through the whole paper. On open surfaces the normalized Ricci-Bourguignon flow preserves the conformal class of the initial metric. Then we can write $g(t) = e^{2u(t)}g_0$, $u(0) = 0$ for some function u and we have

$$\frac{\partial u(t)}{\partial t} = (1-2\rho)\left(e^{-2u(t)}\Delta_{g_0}u(t) - 1 - e^{-2u(t)}K\right), \; u(0) = 0.$$

If $g(t)$ is a solution to (2), then its Gaussian curvature $K = K_{g(t)}$ evolves by the equation

$$\frac{\partial u(t)}{\partial t} = (1-2\rho)\left(\Delta K + 2K^2 + 2K\right).$$

For proving our results we apply the following lemma from Schwarz-Yau [16]:

Lemma 2.1. *Let (X, h_1) and (Y, h_2) be two Riemann surfaces with two complete conformal metrics h_1 and h_2 and Gaussian curvature K_{h_1} and K_{h_2}, respectively, such that K_{h_1} has lower bound $-a_1 \leq 0$ and $K_{h_2} \leq -a_2 < 0$. Then for any conformal mapping $f : X \to Y$, we have $f^*h_2 \leq \frac{a_1}{a_2}h_1$.*

From [11] we have the following definition.

Definition 2.2. A bordered Riemann surface is a connected Hasudorff space X together with a covering with regions U and homeomorphisms f from U onto the unit circle $D = \{z \in \mathbb{C} | |z| < 1\}$ or onto the unit half-circle $\tilde{D} = \{z \in \mathbb{C} | |z| < 1, Im(z) \geq 0\}$ such that if U_1 and U_2 are any two overlapping regions of the covering and f_1 and f_2 are the corresponding homeomorphisms, then homeomorphism $f_2 \circ f_1^{-1} : f_1(U) \to f_2(U)$ is directly conformal.

Proof of Theorem 1.1. Let M be a non-compact Riemann surface. From [14, Page 199], there exists a sequence $M_1, M_2, \cdots$ of compact bordered surfaces contained in M such that $M_k \subset M_{k+1}$ and $M = \cup_{k=1}^{\infty} M_k$. Thus each M_k

has a complete conformal metric g_k of Gaussian curvature $-k^2$. Assume that $u_k \in C^\infty(M_k)$ is a unique function such that $g_k = e^{2u_k} g_0$. Setting $X = M_k$, $Y = M_{k+1}$ in Lemma 2.1 and suppose that $f : X \to Y$ is the inclusion map, then we conclude $(k+1)^2 g_{k+1} \le k^2 g_k$ or equivalently $u_k - u_{k+1} \ge \ln \frac{k+1}{k} > 0$. Hence, the sequence u_k is pointwisiy decreasing when $k \to \infty$. In particular, this yields for any fixed k and $x \in M_k$, $u_{k+l}(x) \le u_k - \ln((k+1)(k+l))$ and $\lim\limits_{l\to\infty} u_{k+l}(x) = -\infty$.

For $\epsilon > 0$, we consider a smooth function $\phi_\epsilon : \mathbb{R} \to \mathbb{R}$ such that $\phi_\epsilon(s) = 0$ for $s \le -\epsilon$, $\phi_\epsilon(s) = s$ for $s \ge \epsilon$, and $\phi''_\epsilon(s) \ge 0$ for all $s \in \mathbb{R}$. Therefore $0 \le \phi'_\epsilon(s) \le 1$ and $\phi(s) \ge s$ for all $s \in \mathbb{R}$. Now, we define new metrics on M_k as $\tilde{g}_k = e^{2\phi_\epsilon(u_k)} g_0$. Since $\phi(s) \ge s$ for all $s \in \mathbb{R}$ we conclude that $\phi_k(u_k) \ge u_k$. This results $\tilde{g}_k \ge g_k$. Also, since $\lim\limits_{l\to\infty} u_{k+l}(x) = -\infty$ for $x \in M_k$, we get $\lim\limits_{k\to\infty} \tilde{g}_k = g_0$. On the other hand, $\lim\limits_{x\to\partial M_k} u_k(x) = +\infty$, thus $\tilde{g}_k$ is a complete metric on M_k. Now, by definition of Gaussian curvature we have

$$\begin{aligned} K_{\tilde{g}} &= -e^{-2\phi_\epsilon(u_k)-2u_0} \Delta\{\phi_\epsilon(u_k) + u_0\} \\ &= -e^{-2\phi_\epsilon(u_k)-2u_0} \{\phi''_\epsilon(u_k)|\partial_z u_k|^2 + \phi'_\epsilon(u_k)\Delta u_k + \Delta u_0\}. \end{aligned} \tag{3}$$

If $x \in M_k$ such that $u_k(x) \ge \epsilon$, then $\phi_\epsilon(u_k(x)) = u_k(x)$, $\phi'_\epsilon(u_k(x)) = 1$, and $\phi''_\epsilon(u_k(x)) = 0$. Hence, $K_{\tilde{g}_k} = K_{g_k} = -k^2$. When $x \in M_k$ such that $u_k(x) \le \epsilon$, because of the compactness reason we have uniform lower bounds for $K_{\tilde{g}_k}$. Therefore, in any cases, $K_{\tilde{g}_k}$ has a k-dependent lower bound $-C(k)$ on M_k for suitable $C(k) > 0$. Since $\phi''_\epsilon > 0$ and $0 \le \phi'_\epsilon \le 1$, from (3) we obtain

$$\begin{aligned} K_{\tilde{g}_k} &\le -e^{-2\phi_\epsilon(u_k)-2u_0} \left\{\phi'_\epsilon(u_k)\Delta(u_k + u_0) + (1 - \phi'_\epsilon(u_k))\Delta u_0\right\} \\ &= e^{-2\phi_\epsilon(u_k)-2u_0} \left\{\phi'_\epsilon(u_k) e^{2u_k+2u_0} K_{g_k} \Delta(u_k + u_0) \right. \\ &\quad \left. + (1 - \phi'_\epsilon(u_k)) e^{2u_0} K_{g_0} \right\} \\ &= e^{-2(\phi_\epsilon(u_k)-u_k)} \phi'_\epsilon(u_k) K_{g_k} + (1 - \phi'_\epsilon(u_k)) e^{-2\phi_\epsilon(u_k)} K_{g_0}. \end{aligned} \tag{4}$$

Since $K_{g_k} = -k^2 \le -1$ and $K_{g_0} \le -1$, we conclude

$$K_{\tilde{g}_k} \le -e^{-2(\phi_\epsilon(u_k)-u_k)} - (1 - \phi'_\epsilon(u_k)) e^{-2\phi_\epsilon(u_k)}.$$

Since $\phi_\epsilon(s) \le \epsilon + s$ when $\phi'_\epsilon(s) \ne 0$ and $\phi_\epsilon(s) \le \epsilon$ when $\phi'_\epsilon(s) \ne 1$, we infer $K_{\tilde{g}_k} \le -e^{-2\epsilon}$. Therefore, we show that $-C(k) \le K_{\tilde{g}_k} \le -e^{-2\epsilon}$ and obtain a sequence of complete conformal metric $\tilde{g}_k$ with bounded Gaussian curvature on M_k. Let $g_k(t)$ be the solution to the Ricci-Bourguignon flow with initial metric $g_k(0) = \tilde{g}_k$. From [3,15] there is a maximal existence interval $[0, T_k)$ for each flow, where $T_k > 0$ depends only on k and ϵ. Moreover, $g_k(t)$ are complete with bounded curvatures.
The corresponding ODE for

$$\partial_t K = (1 - 2\rho)(\Delta_t K + 2K^2 + 2K), \tag{5}$$

is $\frac{dy}{dt} = 2(1-2\rho)(y^2+y)$. This equation has solution as

$$y(t) = \frac{1}{-1+ce^{-2(1-2\rho)t}}.$$

Applying the maximum principle to the equation (5) we arrive at

$$\frac{1}{-1+e^{-2(1-2\rho)t}} \leq K_{g(t)} \leq \frac{1}{-1+(e^{2\epsilon}-1)e^{-2(1-2\rho)t}}, \tag{6}$$

for $0 < t < T_k$. This inequality shows that the maximal existence interval is $(0,\infty)$. Also, $\lim_{t\to\infty} K_{g_k(t)} = -1$ for $\rho \leq \frac{1}{2}$. Hence the metric $g_k(t)$ converges to a metric $g_k(\infty)$ of Gaussian curvature -1.
Set $u_k(x,t) \in C^\infty(M_k \times \mathbb{R}_+)$ such that $g_k(x,t) = e^{2u_k(x,t)}g_0(x)$. Curvature equation implies that

$$\frac{\partial u_k(t)}{\partial t} = (1-2\rho)(K_{g_k(t)}+1) \geq 0, \tag{7}$$

for $\rho \leq \frac{1}{2}$. This implies that $u_k(x,t) \geq u_k(x,0) = \phi_\epsilon(u_k) \geq 0$. Hence, taking the limit $k \to \infty$, $g_k(\infty) \geq g_0$. Next, we show that $g_k(\infty)$ is decreasing in k. For this purpose, we prove $u_{k+1}(t) \leq u_k(t)$ for all t. Setting

$$u_{k,\epsilon}(x,t) = u_k(x, \frac{1}{\epsilon}\ln(\epsilon t+1)) + \frac{1}{2}\ln(\epsilon t+1).$$

We have $u_{k,\epsilon}(x,0) = u_k(x)$ and

$$\begin{aligned}
&\left\{\partial_t u_{k,\epsilon} - (1-2\rho)\left(e^{-2u_{k,\epsilon}}\Delta u_{k,\epsilon} - 1 - e^{-2u_{k,\epsilon}}K_0\right)\right\}(x,t)\\
&= \frac{1}{\epsilon t+1}\left\{\partial_t u_k - (1-2\rho)\left(e^{-2u_k}\Delta u_k - 1 - e^{-2u_k}K_0\right)\right\}(x, \frac{1}{\epsilon}\ln(\epsilon t+1))\\
&+\frac{\epsilon}{2(\epsilon t+1)} + (1-2\rho)(1-\frac{1}{\epsilon t+1})\\
&\geq \frac{\epsilon}{2(\epsilon t+1)}\\
&> 0,
\end{aligned} \tag{8}$$

for $\rho \leq \frac{1}{2}$. Let $u_{k,\epsilon}(x,t) < u_{k+1}(x,t)$ at somewhere in $M_k \times [0,\infty)$. The completeness of the metric $g_k(t)$ on M_k, for every time $t \in [0,\infty)$ implies that $u_{k,\epsilon}(x,t) - u_{k+1}(x,t) \to +\infty$ as $x \to \partial M_k$. Thus $u_{k,\epsilon}(.,t) - u_{k+1}(.,t)$ achieves its infimum in M_k. We assume $(x_0,t_0) \in M_k \times [0,\infty)$ is one of the points at which $u_{k,\epsilon} - u_{k+1}$ first become negative. At the point (x_0,t_0), maximal principal yields

$$\begin{aligned}
&u_{k,\epsilon}(x_0,t_0) = u_{k+1}(x_0,t_0), \quad \Delta(u_{k,\epsilon} - u_{k+1})(x_0,t_0) \geq 0,\\
&\partial_t(u_{k,\epsilon} - u_{k+1}) \leq 0.
\end{aligned}$$

At this point, subtracting the normalized Ricci-Bourguignon flow equation (2) from (8), we obtain

$$\begin{aligned} 0 &< \partial_t u_{k,\epsilon} - (1-2\rho)\left(e^{-2u_{k,\epsilon}}\Delta u_{k,\epsilon} - 1 - e^{-2u_{k,\epsilon}}K_0\right) \\ &\quad - \left\{\partial_t u_{k+1} - (1-2\rho)\left(e^{-2u_{k+1}}\Delta u_{k+1} - 1 - e^{-2u_{k+1}}K_0\right)\right\} \\ &= \partial_t(u_{k,\epsilon} - u_{k+1}) - (1-2\rho)\left(e^{-2u_{k+1}}\Delta(u_{k,\epsilon} - u_{k+1})\right) \\ &\leq 0, \end{aligned}$$

which is a contradiction, then $u_{k+1}(t) \leq u_k(t)$ for all t and $g_k(\infty)$ is decreasing in k. Therefore, $g_\infty(t) = \lim_{k\to\infty} \tilde{g}_k(t)$ exists and satisfies the normalized Ricci-Bourguignon flow equation by regularity theory. Also, $g_\infty(\infty)$ is defined on the whole Riemann surface M, larger than g_0, and has constant Gaussian curvature -1. This completes the proof of theorem. □

Proof of Corollary 1.3. We can construct a sequence of bordered Riemann surfaces N_k as above. Then, by Theorem 1.2, we can find a hyperbolic metric g_k on each N_k. If g_0 is the restricted hyperbolic metric on M, then we can write $g_k = e^{2u_k} g_0$. Using Schwarz-Yau lemma we conclude that $u_k \geq u_{k+1}$ on N_k and $u_k \geq 0$. Let

$$u(x) = \mathit{lim}_{k\to\infty} u_k(x).$$

The function u is a well-defined function on N. Since

$$\frac{\partial u_k(t)}{\partial t} = -(1-2\rho)(K_{g_k(t)} + 1),$$

a standard regularity for elliptic operator shows that u satisfies the equation

$$\frac{\partial u(t)}{\partial t} = -(1-2\rho)(K_{g(t)} + 1)$$

and consequently $g(x) = e^{2u} g_0$ has Gaussian curvature -1 on N. Next, we prove that the metric g is complete. For this purpose, we assume that $\gamma : [0,T) \to N$ be a maximal geodesic with unit tangent vector g and $T < \infty$. the length of γ is finite for g_0 when $g \geq g_0$. Therefore, we can find a point $p \in \partial N$ such that

$$\mathit{lim}_{t\to T} \gamma(t) = p.$$

Since $\mathcal{X}(M) < 0$, by the Theorem 1.2 the open surface $M \setminus \{p\}$ has a hyperbolic metric h. We consider the inclusion map

$$i_k : (N_k, g_k) \to (M \setminus \{p\}, h).$$

It is easy to see that $g_k \geq h$ on N_k for all k and hence $g \geq h$. This yields γ has infinite length for g. □

Proof of Corollary 1.4. Theorem 1.2 yields $\mathbb{C} \setminus \{p,q\} = S^2 \setminus \{\infty, p, q\}$ is hyperbolic. The remainder of proof is exactly the same as above. □

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