Journal of Mahani Mathematical Research
Shahid Bahonar University of Kerman
Print ISSN: 2251-7952 Online ISSN: 2645-4505

STABILITY OF DEEBA AND DRYGAS FUNCTIONAL EQUATIONS IN NON-ARCHIMEDEAN SPACES

D. Khatibi Aghda AND S.M.S. Modarres Mosaddegh ✉

Article type: Research Article
(Received: 26 October 2022, Received in revised form 20 May 2023)
(Accepted: 26 May 2023, Published Online: 26 May 2023)

Abstract. In this paper, we use new techniques to prove Hyers-Ulam and Hyers-Ulam-Rasiass stability of Deeba, Drygas and logarithmic functional equations in non-Archimedean normed spaces. We generalize some earlier results connected with the stability of these functional equations and inequalities. In addition, we provide some examples to clarify the definitions and theorems.

Keywords: Functional equations, Hyers-Ulam stability, Hyers-Ulam-Rasiass stability, non-Archimedean normed spaces.
2020 MSC: 39B82, 46S10.

1. Introduction

It was in the autumn of 1940, when S. M.Ulam gave a speech at a mathematics conference at the University of Wisconsin. He discussed several important problems which seemed to be unsolvable. Among these problems, there was the following problem about the stability of group homomorphisms [26]: Suppose a group $(G_1, \star)$ and a metric group $(G_2, \diamond, d)$ and also a positive ε are given. Does there exist a positive δ such that if a function $f : G_1 \to G_2$ satisfies the inequality $d(f(x \star y), f(x) \diamond f(y)) < \delta$ for every $x, y \in G_1$, then a homomorphism $T : G_1 \to G_2$, i.e., $T(x \star y) = T(x) \diamond T(y)$, exists with $d(f(x), T(x)) < \varepsilon$ for all $x \in G_1$?

And if so, it is said that the functional equation (by definition, an equation contains an unknown function) for homomorphisms from G_1 to G_2 is stable. After about a year, in 1941, Hyers solved the problem of approximately additive mappings [16], assuming G_1 and G_2 are Banach spaces. Some years later, in 1950, Aoki generalized Hyers theorem [3] for approximately additive mappings. 28 Years later, in 1978, Rassias generalized this theorem by taking unbounded Cauchy differences into account. Moreover, he introduced a stability phenomenon namely the Hyers-Ulam-Rassias stability [22]. Now based on Rassias theorem, assume E_1 is a normed vector space, E_2 is a Banach space, and suppose that a mapping $f : E_1 \to E_2$ satisfies the inequality

✉ smodarres@yazd.ac.ir, ORCID: 0000-0002-1371-3769
DOI: 10.22103/jmmr.2023.20433.1356

Publisher: Shahid Bahonar University of Kerman
How to cite: D. Khatibi Aghda, S.M.S. Modarres Mosaddegh, *Stability of Deeba and Drygas functional equations in non-Archimedean spaces*, J. Mahani Math. Res. 2024; 13(1): 167-180.

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p), \tag{1}$$

for all $x, y \in E_1$, where $p < 1$ and $\varepsilon > 0$ are given constants. Then a unique additive mapping $T : E_1 \to E_2$ exists such that

$$\|f(x) - T(x)\| \leq \frac{\varepsilon}{1 - 2^{p-1}} \|x\|^p, \tag{2}$$

for all $x \in E_1$. If $p < 1$ the inequality (1) holds for all $x, y \neq 0$, and (2) for all $x \neq 0$. Moreover, if for each fixed $x \in E_1$, the function $t \to f(tx)$ from $\mathbb{R} \to E_2$ is continuous, then the mapping T is $\mathbb{R}$–linear.

Taking this into account, it is said that the additive functional equation

$$f(x+y) = f(x) + f(y),$$

has the Hyers-Ulam-Rassias stability on (E_1, E_2). Several mathematicians have investigated a number of stability problems of functional equations during recent decades; see [5, 12–14, 20] and references therein for more details.

A mapping f is said to be bi-additive if f is additive in each variable, that is,

$$f(x+y, z) = f(x, z) + f(y, z),$$
$$f(x, y+z) = f(x, y) + f(x, z).$$

For example, $f(x, y) = xy$ is a bi-additive mapping [23].

2. Preliminaries

Definition 2.1. (see [4]) A valuation on a field K is a function $|\cdot|$ from K into $[0, \infty)$ such that 0 is the unique element that has the 0 valuation, $|rs| = |r||s|$ and the triangle inequality is satisfied, i.e.,

$$|r+s| \leq |r| + |s|$$

A field K is called a valued field if K carries a valuation.

The best-known examples of valuations are the usual absolute values of the fields of real and complex numbers [4, 13]. Now we consider a valuation that satisfies a condition that is stronger than the triangle inequality. If one replaces the **triangle inequality** by

$$|r+s| \leq \max\{|r|, |s|\},$$

for all $r, s \in K$, then the function $|\cdot|$ is called a non-Archimedean valuation on K, and the field K is called a non-Archimedean field [4].

Remark 2.2. Clearly according to the definition of a non-Archimedean field, $|1| = |-1| = 1$, $|n| \leq 1$ and $|r^n| = |r|^n$ for all $r, n \in \mathbb{N}$ [18].

Example 2.3. *Let K be a field. The function $|\cdot| : K \to \mathbb{R}_0^+ = [0, \infty)$ given by*

$$|x| = \begin{cases} 0 & x = 0 \\ 1 & x \neq 0 \end{cases}$$

is called trivial valuation [25].

Example 2.4. *If $x \in \mathbb{Q}^\star = \mathbb{Q} - \{0\}$ and p is a fixed prime number, then we can write $x = p^t x_0$, where $\gcd(p, x_0) = 1$ and $t \in \mathbb{Z}$. The function $|\cdot| : \mathbb{Q} \to \mathbb{R}_0^+$ given by*

$$|x| = \begin{cases} 0 & x = 0 \\ \mathrm{e}^{-t} & x \neq 0 \end{cases}$$

is a valuation [15].

Remark 2.5. The p-adic numbers have drawn the attention of many physicists due to their many applications in quantum. These numbers can be considered as one of the most widely used non-Archimedean fields [25, 27].

Definition 2.6. (see [18]) Consider a vector space X over a scalar field K with a non-Archimedean non-trivial valuation $\|\cdot\|$. A function $\|\cdot\| : X \to \mathbb{R}$ is a non-Archimedean norm (valuation) if it satisfies the three conditions:

(1) $\|x\| = 0$ if and only if $x = 0$;
(2) $\|rx\| = |r|\|x\| (r \in K,\, x \in X)$;
(3) The **strong triangle inequality**; namely,

$$\|x + y\| \leq \max\{\|x\|, \|y\|\},$$

for every $x, y \in X$.

$(X, \|\cdot\|)$ is called a non-Archimedean space if $\|\cdot\|$ is a non-Archimedean norm.

Remark 2.7. Due to the fact that if $n > m$ then

$$\|x_n - x_m\| \leq \max\{\|x_{j+1} - x_j\| \mid m \leq j \leq n-1\},$$

the sequence $\{x_n\}$ is Cauchy if and only if the sequence $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean space [17].

A complete non-Archimedean space is a space in which every Cauchy sequence is convergent. A Banach space is a complete normed vector space.

Example 2.8. (see [24]) *Let K be a non-Archimedean field with a non-Archimedean valuation $|\cdot|$.*

(1) *The vector space K^n with the norm $\|(a_1, a_2, \ldots, a_n)\| = \max_{1 \leq i \leq n} |a_i|$ is a complete non-Archimedean space.*
(2) *Let X be a non-empty set. The set*

$$L^\infty = \{f | f : X \to K, \quad f \text{ is bounded}\},$$

with scalar multiplication, pointwise addition, and the norm

$$\|f\| = \sup_{x \in X} |f(x)|,$$

is a non-Archimedean Banach space.

C. Perez-Garcia and W. H. Schikhof, presented an interesting example of non-Archimedean spaces in [21].

Definition 2.9. **Deeba functional equation** is in the form

$$f(x\cdot y\cdot z)+f(x)+f(y)+f(z)=f(x\cdot y)+f(x\cdot z)+f(y\cdot z),$$

and, **Drygas functional equation** is in the form

$$f(x\cdot y)+f(x\cdot y^{-1})=2f(x)+f(y)+f(y^{-1}).$$

Also, we define **logarithmic functional equation (L.F.E.)** as:

$$f(x.y)=f(x)+f(y),$$

when the unknown function f is a function with values in a non-Archimedean space[1].

Remark 2.10. It's obvious that for any L.F.E., we have

$$f(x^n)=\underbrace{f(x)+f(x)+\cdots+f(x)}_{n\ term}=nf(x). \tag{3}$$

Example 2.11. *Suppose a function $H_c:\mathbb{R}-\{0\}\to\mathbb{R}$ is given by*

$$H_c(x)=c\ln|x|,$$

where c is any real constant. The function H_c is an L.F.E. that satisfies the Deeba and Drygas equations.

Some algebraic definitions are collected in the following definition.

Definition 2.12. A semigroup can be defined as an algebraic structure consisting of an internal binary operation and a set in which the binary operation is associative. A group is a semigroup with identity and inverse elements. An abelian group is a group with a commutative binary operation.

In this paper, we show how the stability problem can be solved for the functional equations of Deeba, Drygas and logarithmic functional equations in non-Archimedean spaces.

3. Main results

In this section we investigate the stability of several functional equations.

Lemma 3.1. *Let G and X be a normed semigroup and a complete non-Archimedean space, respectively. Assume that $\phi: G\times G\to[0,\infty)$ satisfies*

$$\lim_{n\to\infty}\frac{\phi(x^{2^n},y^{2^n})}{|2|^n}=0, \tag{4}$$

[1] The reader can see more about functional equations in [23].

for all $x, y \in G$, and for each $x \in G$ the following limit exists and is denoted by $\Phi(x)$.

$$\lim_{n\to\infty} \max\left\{ \frac{\phi(x^{2^i},\, x^{2^i})}{|2|^i} \,\Big|\, 0 \le i < n \right\}. \tag{5}$$

If a mapping $f : G \to X$ satisfies the inequality

$$\|f(x\cdot y) - f(x) - f(y)\| \le \phi(x,\, y), \tag{6}$$

then there exists an L.F.E. $H : G \to X$ such that

$$\|f(x) - H(x)\| \le \frac{1}{|2|}\Phi(x), \tag{7}$$

for any $x \in G$. Also, with the assumption that

$$\lim_{k\to\infty}\lim_{n\to\infty} \max\left\{ \frac{\phi(x^{2^i}, x^{2^i})}{|2|^i} \,\Big|\, k \le i < n+k \right\} = 0, \tag{8}$$

H is a unique L.F.E. which satisfies (7).

Proof. Putting $y = x$ in the inequality (6) and dividing by $|2|$ we get

$$\left\|\frac{1}{2}f(x^2) - f(x)\right\| \le \frac{1}{|2|}\phi(x, x),$$

for all $x \in G$. Replacing x by $x^{2^{n-1}}$ in the above inequality and dividing by $|2|^{n-1}$ yields

$$\left\|\frac{1}{2^n}f(x^{2^n}) - \frac{1}{2^{n-1}}f(x^{2^{n-1}})\right\| \le \frac{\phi(x^{2^{n-1}}, x^{2^{n-1}})}{|2|^n},$$

for all $x \in G$. Using (4) for $n \to \infty$ and Remark 2.7, we can easily calculate that

$$\left\{\frac{f(x^{2^n})}{2^n}\right\}$$

is a Cauchy sequence. On the other hand, since X is a complete non-Archimedean space, it implies that the sequence $\left\{\frac{f(x^{2^n})}{2^n}\right\}$ is convergent for each $x \in X$. Let us put:

$$H(x) = \lim_{n\to\infty}\frac{f(x^{2^n})}{2^n}.$$

Applying an induction to n, we can prove that

$$\left\|\frac{1}{2^n}f(x^{2^n}) - f(x)\right\| \le \frac{1}{|2|}\max\left\{ \frac{\phi(x^{2^k},\, x^{2^k})}{|2|^k} \,\Big|\, 0 \le k < n \right\},$$

for all $x \in G$. Taking the limit in the above inequality as $n \to \infty$ and using equation (5) in non-Archimedean spaces, we obtain (7). By replacing x^{2^n} and

y^{2^n} with x and y in (6), respectively and dividing by $|2|^n$ we get

$$\Big\|\frac{f(x^{2^n}\cdot y^{2^n})}{2^n}-\frac{f(x^{2^n})}{2^n}-\frac{f(y^{2^n})}{2^n}\Big\|\leq\frac{\phi(x^{2^n},\,y^{2^n})}{|2|^n},$$

where x, y are in G.
Letting $n\to\infty$ and using equation (4), we conclude

$$H(x\cdot y)=H(x)+H(y).$$

Therefore, H is an L.F.E.. It remains to show that H is uniquely defined. Let $H^{'}$ be another L.F.E. satisfying (7). Then, we get

$$\begin{aligned}\Big\|H(x)-H^{'}(x)\Big\| &\overset{(3)}{=}\lim_{k\to\infty}\frac{\Big\|H(x^{2^k})-H'(x^{2^k})\Big\|}{|2|^k}\\ &\leq\lim_{k\to\infty}\frac{\max\Big\{\Big\|H(x^{2^k})-f(x^{2^k})\Big\|,\Big\|f(x^{2^k})-H^{'}(x^{2^k})\Big\|\Big\}}{|2|^k}\\ &\leq\frac{1}{|2|}\lim_{k\to\infty}\lim_{n\to\infty}\max\left\{\frac{\phi(x^{2^i},x^{2^i})}{|2|^i}\,\Big|\,k\leq i<n+k\right\}\\ &=0.\end{aligned}$$

Therefore $H=H^{'}$, and the proof is complete. □

According to Lemma 3.1, we can prove the Hyers-Ulam-Rassias stability and Hyers-Ulam stability of logarithmic functional equations (L.F.E.) as follows:

Corollary 3.2. *Suppose G is a normed group, X is a complete non-Archimedean space and $f:G\to X$ is a mapping satisfying the functional equation*

$$\|f(x\cdot y)-f(x)-f(y)\|\leq\frac{\varepsilon}{2}\Big(\mu(\|x\|)+\mu(\|y\|)\Big),$$

for some $\varepsilon>0$ and for any $x,y\in G$. If $\mu:\mathbb{R}_0^+\to\mathbb{R}_0^+$ is a function satisfying

$$\mu(\|x^2\|)\leq\mu(|2|)\,\mu(\|x\|),\quad \mu(|2|)<|2|,$$

then there exists a unique L.F.E. $H:G\to X$ such that

$$\|f(x)-H(x)\|\leq\varepsilon\mu(\|x\|).$$

Proof. Defining $\phi:G\times G\to\mathbb{R}_0^+$ by

$$\phi(x,y)=\frac{\varepsilon}{2}\big(\mu(\|x\|)+\mu(\|y\|)\big),$$

we get

$$\phi(x^{2^n},y^{2^n})=\frac{\varepsilon}{2}\Big(\mu(|2|)\Big)^n\Big(\|x\|+\|y\|\Big).$$

Therefore,

$$\lim_{n\to\infty} \frac{\phi(x^{2^n}, y^{2^n})}{|2|^n} \le \frac{\varepsilon}{2}\Big(\mu(\|x\|) + \mu(\|y\|)\Big) \lim_{n\to\infty}\left(\frac{\mu(|2|)}{|2|}\right)^n$$
$$= 0.$$

Also, we have

$$\begin{aligned}
&\max\left\{\frac{\phi(x^{2^i},\, x^{2^i})}{|2|^i} \,\Big|\, 0 \le i < n\right\} \\
&= \frac{\varepsilon}{2}\left(\max\left\{\frac{\mu(\|x^{2^i}\|) + \mu(\|x^{2^i}\|)}{|2|^i} \,\Big|\, 0 \le i < n\right\}\right) \\
&= \frac{\varepsilon}{2}\left(\max\left\{\frac{(\mu(|2|))^i\,(2\mu(\|x\|))}{|2|^i} \,\Big|\, 0 \le i < n\right\}\right) \\
&= \varepsilon\left(\max\left\{\left(\frac{\mu(|2|)}{|2|}\right)^i (\mu(\|x\|)) \,\Big|\, 0 \le i < n\right\}\right) \\
&\le \varepsilon\ \mu(\|x\|).
\end{aligned}$$

Now, applying Lemma 3.1 we get the result. □

Lemma 3.3. *Let G and X be a normed group and a complete non-Archimedean space, respectively. Assume that $\phi : G \times G \to [0, \infty)$ satisfies*

$$\lim_{n\to\infty} \frac{\phi(x^{2^n},\, y^{-2^n})}{|2|^n} = 0.$$

If a mapping $f : G \to X$ satisfies the inequality

$$\|f(x \cdot y^{-1}) - f(x) - f(y^{-1})\| \le \phi(x,\, y^{-1}),$$

then there exists an L.F.E. $H : G \to X$ such that

$$\|f(x) - H(x)\| \le \frac{1}{|2|}\phi(x,\, x).$$

Proof. Using Lemma 3.1 and substituting y by y^{-1} we get the result. □

Theorem 3.4. *Let G and X be a normed group and a complete non-Archimedean spaces, respectively. Assume that $\phi : G \times G \to \mathbb{R}_0^+$ satisfies*

$$\lim_{n\to\infty} \frac{\phi(x^{2^n}, y^{2^n})}{|2|^n} = 0,$$
$$\lim_{n\to\infty} \frac{\phi(x^{2^n}, y^{-2^n})}{|2|^n} = 0, \tag{9}$$

for all $x, y \in G$, and let for each $x \in G$ the limit

$$\lim_{n\to\infty} \max\left\{ \max\left\{\frac{\phi(x^{2^i},\, x^{2^i})}{|2|^i}\right\}, \max\left\{\frac{\phi(x^{2^i},\, x^{-2^i})}{|2|^i}\right\} \,\middle|\, 0 \le i < n\right\}, \tag{10}$$

denoted by $\Phi(x)$, exist. If a mapping $f : G \to X$ satisfies the inequality

$$\begin{aligned} &f(x\cdot y)+f(x\cdot y^{-1}) - 2f(x) - f(y) - f(y^{-1}) \\ &\le \max\left\{\phi(x, y),\, \phi(x, y^{-1})\right\}. \end{aligned} \tag{11}$$

Then there exists a Drygas function $H : G \to X$ such that

$$\|f(x) - H(x)\| \le \frac{1}{|2|}\Phi(x).$$

Proof. In view of inequality (11) and using the third property of Definition 2.6, we obtain

$$\begin{aligned} &\|f(x\cdot y) + f(x\cdot y^{-1}) - 2f(x) - f(y) - f(y^{-1})\| \\ =&\|f(x\cdot y) - f(x) - f(y) + f(x\cdot y^{-1}) - f(x) - f(y^{-1})\| \\ \le&\max\left\{\|f(x\cdot y) - f(x) - f(y)\|,\right. \\ &\qquad \left.\| f(x\cdot y^{-1}) - f(x) - f(y^{-1})\|\right\}. \end{aligned}$$

By substituting x^{2^n} and y^{2^n} for x and y in above equation, respectively and dividing by $|2|^n$ we get

$$\begin{aligned} &\left\|\frac{f(x^{2^n}\cdot y^{2^n})}{2^n} + \frac{f(x^{2^n}\cdot y^{-2^n})}{2^n} - 2\left(\frac{f(x^{2^n})}{2^n}\right) - \frac{f(y^{2^n})}{2^n} - \frac{f(y^{-2^n})}{2^n}\right\| \\ &\le \max\left\{ \left\|\frac{f(x^{2^n}\cdot y^{2^n})}{2^n} - \frac{f(x^{2^n})}{2^n} - \frac{f(y^{2^n})}{2^n}\right\|,\right. \\ &\qquad \left.\left\|\frac{f(x^{2^n}\cdot y^{-2^n})}{2^n} - \frac{f(x^{2^n})}{2^n} - \frac{f(y^{-2^n})}{2^n}\right\| \right\} \\ &\le \max\left\{\frac{\phi(x^{2^n},\, y^{2^n})}{|2|^n},\, \frac{\phi(x^{2^n},\, y^{-2^n})}{|2|^n}\right\}. \end{aligned}$$

Let $n \to +\infty$ and

$$H(x) = \lim_{n\to\infty}\left\{\frac{f(x^{2^n})}{2^n}\right\}$$

and using (9), Lemmas 3.1 and 3.3 we conclude

$$H(x\cdot y) + H(x\cdot y^{-1}) = 2H(x) + H(y) + H(y^{-1}).$$

Therefore, H is a Drygas function. □

Corollary 3.5. *Let G be a normed group, X be a complete non-Archimedean space and let $f : G \to X$ be a mapping satisfying the inequality*

$$\left\| f(x \cdot y) + f(x \cdot y^{-1}) - 2f(x) - f(y) - f(y^{-1}) \right\| \leq \varepsilon(\mu(\|x\|) + \mu(\|y\|)),$$

where, $\varepsilon > 0$, $x, y \in G$ and μ is defined in Corollary 3.2. Then there exists a Drygas function $H : G \to X$ such that

$$\|f(x) - H(x)\| \leq \varepsilon.$$

Theorem 3.6. *Let G and X be a normed abelian group and a complete non-Archimedean space, respectively. Assume that $\phi : G \times G \times G \to [0, \infty)$ satisfies*

$$\lim_{n\to\infty} \frac{\phi(x^{3^n}, y^{3^n}, z^{3^n})}{|3|^n} = 0, \tag{12}$$

for all x, y, $z \in G$, and let for each $x \in G$ the limit

$$\lim_{n\to\infty} \max\left\{ \frac{\phi(x^{3^i}, x^{3^i}, x^{3^i})}{|3|^i} \,\Big|\, 0 \leq i < n \right\}, \tag{13}$$

is denoted by $\Phi(x)$, exist. Assume that a mapping $f : G \to X$ satisfies the inequalities

$$\begin{aligned} &\big\| f(x.y.z) + f(x) + f(y) + f(z) \\ &\qquad - f(x.y) - f(x.z) - f(y.z) \big\| \\ &\qquad \leq \phi(x, y, z), \end{aligned} \tag{14}$$

and,

$$\|f(x^3) - 3f(x)\| \leq \phi(x, x, x), \tag{15}$$

then there exists a Deeba function $H : G \to X$ such that

$$\|f(x) - H(x)\| \leq \frac{1}{|3|}\Phi(x). \tag{16}$$

Proof. Replacing x by $x^{3^{n-1}}$ in the (15) and dividing by $|3|^n$, we get

$$\left\| \frac{1}{3^n} f(x^{3^n}) - \frac{1}{3^{n-1}} f(x^{3^{n-1}}) \right\| \leq \frac{\phi(x^{3^{n-1}}, x^{3^{n-1}}, x^{3^{n-1}})}{|3|^n},$$

for all $x \in G$. Taking the limit as $n \to \infty$ and considering (12), we can see the sequence

$$\left\{ \frac{f(x^{3^n})}{3^n} \right\}$$

is Cauchy. Since X is a complete non-Archimedean space, it implies that the sequence

$$\left\{ \frac{f(x^{3^n})}{3^n} \right\}$$

is convergent for each $x \in X$. Now we put:

$$H(x) = \lim_{n\to\infty} \left\{ \frac{f(x^{3^n})}{3^n} \right\}.$$

Applying an induction to n, we can prove that

$$\left\| \frac{1}{3^n} f(x^{3^n}) - f(x) \right\| \le \frac{1}{|3|} \max \left\{ \frac{\phi(x^{3^k}, x^{3^k}, x^{3^k})}{|3|^k} \,\Big|\, 0 \le k < n \right\},$$

for all $x \in G$. Taking the limit in the above inequality as $n \to \infty$ and using equation (13) we obtain (16). By substituting x^{3^n}, y^{3^n} and z^{3^n} for x, y and z in (14), respectively and dividing by $|3|^n$, we get

$$\begin{aligned} &\left\| \frac{f(x^{3^n}.y^{3^n}.z^{3^n})}{3^n} + \frac{f(x^{3^n})}{3^n} + \frac{f(y^{3^n})}{3^n} + \frac{f(z^{3^n})}{3^n} \right. \\ &\qquad \left. - \frac{f\left((x.y)^{3^n}\right)}{3^n} - \frac{f\left((x.z)^{3^n}\right)}{3^n} - \frac{f\left((y.z)^{3^n}\right)}{3^n} \right\| \\ &\qquad \le \frac{\phi(x^{3^n}, y^{3^n}, z^{3^n})}{|3|^n}, \quad \forall x, y, z \in G. \end{aligned}$$

Letting $n \to +\infty$ and using equation (12), we conclude that

$$H(x \cdot y \cdot z) + H(x) + H(y) + H(z) = H(x \cdot y) + H(x \cdot z) + H(y \cdot z). \tag{17}$$

Therefore, H is a Deeba function. □

Example 3.7. *Let $G = \mathbb{R} - \{0\}$ be a multiplication group and X be a non-Archimedean space with $|3| \le |2| \ne 1$* (Examples 2.4 and 2.8)*. Define*

$$\phi(x^{3^n}, y^{3^n}, z^{3^n}) = |3|^{2n}. \tag{18}$$

By replacing $n = 0$ in (18)*, we get $\phi(x, y, z) = 1$. Suppose that, f satisfies in* (14) *and* (15) (i.e., $f(x) = 1$)*. It's easy to see that $\Phi(x) = 1$, and $H(x) = 0$ is a Deeba function satisfying* (16)*.*

Another view of Theorem 3.6 is provided in the following theorem. The previous example also works for this theorem.

Theorem 3.8. *Let G and X be a normed abelian group and a complete non-Archimedean space, respectively. Assume that $\phi : G \times G \times G \to [0, \infty)$ satisfies*

$$\lim_{n\to\infty} \frac{\phi(x^{3^n}, y^{3^n}, z^{3^n})}{|3|^n} = 0, \tag{19}$$

for all $x, y, z \in G$, and let for each $x \in G$ the limit

$$\lim_{n\to\infty} \max \left\{ \frac{\phi(x^{3^i}, x^{3^i}, x^{3^i})}{|3|^i} \,\Big|\, 0 \le i < n \right\}, \tag{20}$$

denote by $\Phi(x)$*, exist. If a mapping* $f: G \to X$ *satisfies the inequality*

$$\begin{aligned} &\big\|f(x\cdot y)+f(x\cdot z)+f(y\cdot z)-2f(x)-2f(y)-2f(z)\big\| \\ &\le \big\|f(x\cdot y\cdot z)-f(x)-f(y)-f(z)\big\| \\ &\le \phi(x,\,y,\,z), \end{aligned} \tag{21}$$

then there exists a Deeba function $H: G \to X$ *such that*

$$\big\|f(x)-H(x)\big\| \le \frac{1}{|3|}\Phi(x).$$

As the last corollary, we state another form of Corollary 3.2 by changing the assumptions and its functional equation.

Corollary 3.9. *Suppose* G *is a normed abelian group,* X *is a complete non-Archimedean space and* $\mu: \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *is a function satisfying*

$$\mu\left(\|x^3\|\right) \le \mu(|3|)\,\mu(\|x\|), \quad \mu(|3|) < |3|.$$

If $f: G \to X$ *is a function satisfying*

$$\begin{aligned} &\big\|f(x\cdot y)+f(x\cdot z)+f(y\cdot z)-2f(x)-2f(y)-2f(z)\big\| \\ &\le \big\|f(x\cdot y\cdot z)-f(x)-f(y)-f(z)\big\| \\ &\le \frac{\varepsilon}{3}\Big(\mu(\|x\|^p)+\mu(\|y\|^p)+\mu(\|z\|^p)\Big), \end{aligned}$$

for some $\varepsilon > 0$*, positive real number* p *and for all* $x,\,y,\,z \in G$*, then there exists a Deeba function* $H: G \to X$ *such that*

$$\|f(x)-H(x)\| \le \varepsilon\mu\left(\|x\|^p\right).$$

Proof. We define $\phi: G\times G\times G \to \mathbb{R}_0^+$ as

$$\phi(x,y,z) = \frac{\varepsilon}{3}\Big(\mu(\|x\|^p)+\mu(\|y\|^p)+\mu(\|z\|^p)\Big).$$

Therefore,

$$\begin{aligned} \phi(x^{3^n},y^{3^n},z^{3^n}) &= \frac{\varepsilon}{3}\Big(\mu\big(\|x^{3^n}\|^p\big)+\mu\big(\|y^{3^n}\|^p\big)+\mu\big(\|z^{3^n}\|^p\big)\Big) \\ &= \frac{\varepsilon}{3}\Big(\mu(|3|)\Big)^n\Big(\|x\|^p+\|y\|^p+\|z\|^p\Big). \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{n\to\infty}\frac{\phi(x^{3^n},y^{3^n},z^{3^n})}{|3|^n} &\le \frac{\varepsilon}{3}\bigg(\mu(\|x\|^p)+\mu(\|y\|^p)+\mu(\|z\|^p)\bigg)\lim_{n\to\infty}\left(\frac{\mu(|3|)}{|3|}\right)^n \\ &= 0. \end{aligned}$$

Also, it's easy to see that

$$\max\left\{\frac{\phi(x^{3^i},\, x^{3^i},\, x^{3^i})}{|3|^i}\,\Big|\, 0\le i<n\right\}$$
$$=\varepsilon\left(\max\left\{\left(\frac{\mu(|3|)}{|3|}\right)^i\mu(\|x\|^p)\,\Big|\,0\le i<n\right\}\right)$$
$$\le \varepsilon\ \mu(\|x\|^p).$$

Now, applying Theorem 3.6, we get the desired result. □

4. Conclusion

The concept of Hyers-Ulam stability is rather significant in tackling real-world problems in such fields as economics, numerical analysis, differential equations, biology, etc. For example, in the field of differential equations, many mathematicians are studying the Hyers-Ulam stability of solutions of ordinary or partial differential equations. Some applications of the concept of Hyers-Ulam stability in differential equations are given in [1, 2, 19]. Also, a mathematical modeling for the corona virus epidemic (COVID-19) using Hyers-Ulam stability was given in [8].

In general, fixed-point theorems and the direct method are often used to evaluate the Hyers-Ulam stability of functional equations [20, 25]. In this work, we studied stability of functional equations in non-Archimedean spaces using direct method. We inspected the stability of functional equations for three types of such equations in non-Archimedean space, from a different viewpoint, by providing various definitions, examples, and theorems. For future works, we recommend obtaining stability results for functional equations in various normed spaces such as fuzzy normed spaces, multiplicative normed spaces, random normed spaces, etc. [6, 7, 10, 11].

5. Acknowledgment

The authors would like to thank the anonymous reviewer for providing insightful and helpful suggestions.

References

[1] Abdollahpour, M.R., & *et al.* (2019). *Hyers-Ulam stability of hypergeometric differential equations*, Aequationes Math., **93(4)**, 691–698. https://doi.org/10.1007/s00010-018-0602-3

[2] Abdollahpour,M.R., & *et al.* (2016). *Hyers–Ulam stability of associated Laguerre differential equations in a subclass of analytic functions*, J. Math. Anal. Appl., **437(1)**, 605–612. https://doi.org/10.1016/j.jmaa.2016.01.024

[3] Aoki, T. (1950). *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan, **2**, 64–66. https://doi.org/10.2969/jmsj/00210064

[4] Azadi Kenari, H. (2011). *Non-Archimedean stability of Cauchy–Jensen type functional equation*, Int. J. Nonlinear Anal. Appl., **2(2)**, 92–102. https://doi.org/10.22075/IJNAA.2011.104

[5] Badora R. (2006). *On approximate derivations*, Math. Inequal. Appl., **2(1)**, 167–173. https://doi.org/10.22075/IJNAA.2011.104

[6] El-Hady, El., & Brzdek, J. (2022). *On Ulam stability of functional equations in 2-normed spaces—A survey*, Symmetry, **13**, 2200–2218. https://doi.org/10.1007/s00025-023-01840-7

[7] Dutta, H., Senthil Kumar, B.V., & Sabarinathan S. (2022). *Fuzzy stabilities of a new hexic functional equation in various spaces*, An. Şţ. Univ. Ovidius Constanţa, **30(3)**, 143–171. https://doi.org/10.2478/auom-2022-0038

[8] Yavuz, M., Coşar, F., Günay, F. & Özdemir, F. (2021). *A New Mathematical Modeling of the COVID-19 Pandemic Including the Vaccination Campaign*, Open Journal of Modelling and Simulation, **9**, 299–321. https://doi.org/10.3390/math11010142

[9] Kannappan, Pl. (1995). *Quadratic functional equation and inner product spaces*, Results Math. **27**, 368–372. https://doi.org/10.1007/BF03322841

[10] Kumar, B.V.S., Al-Shaqsi, K., & Dutta, H. (2021). *Approximate reciprocal Lie ⋆-derivations*, Soft Comput., **25**, 14969–14977. https://doi.org/10.1007/s00500-021-06395-9

[11] Kumar, B.V.S., Dutta H., & Sabarinathan S. (2021). *Stabilities and instabilities of an advanced quartic functional equation in various normed spaces*, Math. Methods Appl. Sci. **44(14)**, 11469–11481. https://doi.org/10.1002/mma.7506

[12] Eshaghi Gordji, M., & Khodaei, H. (2010). *Stability of Functional Equations*, LAP Lambert Academic Publishing, Saarbrucken. ISBN:9783843381918

[13] Forti, G.L., Shulman, E. (2020). *A comparison among methods for proving stability*, Aequationes Math., **94**, 547–574. https://doi.org/10.1007/s00010-019-00679-0

[14] Gilanyi, A. (2001). *Eine zur parallelogrammgleichung äquivalente ungleichung*, Aequationes Math., **62(3)**, 303–309. https://doi.org/10.1007/PL00000156

[15] Goss, D. (2012). *Basic Structures of Function Field Arithmetic*, Springer Science & Business Media, **35**, ISBN:9783642614804, 3642614809.

[16] Hyers, D.H. (1941). *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A. **27(4)**, 222–224. https://doi.org/10.1073/pnas.27.4.222

[17] Jung, S.-M. (2011). *Hyers–Ulam–Rassias stability of functional equations in nonlinear analysis*, Springer Optimization and Its Applications. https://doi.org/10.1007/978-1-4419-9637-4

[18] Moslehian, M.S. (2007). *Stability of functional equations in non-Archimedean spaces*, Appl. Anal. Discrete Math., **1(2)**, 325–334. https://doi.org/10.2298/AADM0702325M

[19] Nie, D., Niazi, A.U.K., & Ahmed, B. (2020). *On local generalized Ulam–Hyers stability for nonlinear fractional functional differential equation*, Math. Prob. Eng. https://doi.org/10.1155/2020/3276873

[20] Park, C., & Rassias, J.M. (2009). *Stability of the Jensen-type functional equation in C^*-algebras: a fixed point approach*, Abstr. Appl. Anal. https://doi.org/10.1155/2009/360432.

[21] Perez-Garcia C. & Schikhof, W.H. (2012). *New examples of non-Archimedean Banach spaces and applications*, Canad. Math. Bull. **55(4)**, 821–829. https://doi.org/10.4153/CMB-2011-133-2

[22] Rassias, Th.M. (1978). *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., **72(2)**, 297–300. https://doi.org/10.2307/2042795

[23] Sahoo P.K., & Kannappan, P.L. (2011), *Introduction to Functional Equations*, CRC Press, Boca Raton, Florida. ISBN: 1138114553

[24] Schneider, P. (2002). *Nonarchimedean Functional Analysis*, Springer Berlin, Heidelberg. ISBN: 978-3-642-07640-4

[25] Senthil Kumar B.V. & Sabarinathan, S. (2022). *A fixed point approach to non-Archimedean stabilities of IQD and IQA functional equations*, Thai J. Math., **20(1)**, 69–78. ISSN: 1686-0209

[26] Ulam, S.M. (1964). *Problems in Modern Mathematics, Science Editions*, John Wiley Sons, Inc., New York.

[27] Vladimirov, V.S., Volovichmm I.V., & Zelenov, E.I. (1994). *p-Adic Analysis and Mathematical Physics*, Nauka, Moscow. https://doi.org/10.1142/1581

Davood Khatibi Aghda
Orcid number: 0000-0002-9374-6164
Department of Mathematical Sciences
Yazd University
Yazd, Iran
Email address: khatibi@stu.yazd.ac.ir

Seyed Mohammad Sadegh Modarres Mosaddegh
Orcid number: 0000-0002-1371-3769
Department of Mathematical Sciences
Yazd University
Yazd, Iran
Email address: smodarres@yazd.ac.ir