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STABILITY OF LATTICE FUNCTIONAL EQUATION IN UCBF-ALGEBRA

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Abstract. The main aim of this research is to investigate the stability of a functional equation that maintains the lattice structure in a uniformly complete unital Banach f-algebra. Through this inquiry, we can shed light on the behavior of this equation and its relationship with the algebraic properties of a Banach space. This research has both theoretical and practical implications. It contributes to the foundations of functional analysis, lattice theory, operator theory, approximation theory, and various applied mathematical disciplines. The findings from this research can have implications in diverse fields ranging from mathematics and physics to engineering and computer science, offering valuable insights and potential applications.

Keywords: Hyers-Ulam stability, Functional equation; Banach lattice, f-algebra; Fixed point method.
2020 MSC: 39B82, 46A40, 97H50, 46B422.

1. Introduction

The topic of stability in functional equations has been of great interest for more than 50 years, and in 1941, Stanislaw Ulam, a renowned Polish American mathematician, presented a lecture that raised several important unresolved mathematical questions [11]. One of these questions related to the stability of homomorphisms, which, despite its abstract nature and simplicity, had significant implications. This classic question posed by Ulam remains a fundamental topic in the field of mathematics to this day. The Ulam's question was as follows:
Let H_1 be a group and let H_2 be a metric group with a metric $d(.,.)$. Given $\gamma > 0$. Does there exists a $\lambda > 0$ such that if a function $\rho : H_1 \to H_2$ satisfies the inequality $d(\rho(ts), \rho(t)\rho(s)) < \lambda$ for all $t, s \in H_1$, then there is a unique homomorphism P from H_1 to H_2 with $d(P(t), \rho(t)) < \gamma$ for all $t \in H_1$?

In the years following Ulam's seminal lecture, several mathematicians dedicated their research to solving the question of the stability of homomorphisms. In 1942, Hyers [5] provided an initial response to Ulam's problem with the following theorem:

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Theorem 1.1. *Suppose that V_1 and V_2 are two Banach spaces and ρ is a function from V_1 to V_2 such that the following inequality holds for some $\lambda > 0$ and for each $t, s \in V_1$,*

$$\|\rho(t+s) - \rho(t) - \rho(s)\| \leq \lambda. \tag{1}$$

Then, there is only one additive mapping $P : V_1 \to V_2$ so that

$$\|P(t) - \rho(t)\| \leq \lambda$$

for any $t \in V_1$.

In the decades that followed Hyers's initial response, mathematicians continued to explore the topic of stability in functional equations. One of the most notable contributions came from Rassias, who was able to refine the conditions for linear mapping and modify the control function [9], leading to significant advancements in the field. The results of Rassias were so remarkable that the field of stability theory for functional equations is now commonly referred to as the Hyers-Ulam-Rassias stability theory.

In recent years, many mathematicians have investigated the stability of various functional, differential, and integral equations in different spaces. The results of these studies have been fascinating and have contributed greatly to our understanding of stability in mathematics([4]- [6]).

The theory of Riesz spaces is a field of study that emerged from the pioneering research of Frigyes Riesz in 1928 [10]. Riesz spaces are real vector spaces that are equipped with a partial order and satisfy certain axioms, including the requirement that they form a lattice under that partial order. These spaces have diverse applications in both economics and mathematics.

This research sheds light on how this equation affects the lattice structure within the Banach space, providing a valuable contribution to the study of lattices and their applications. Also, it can be valuable for constructing efficient approximation methods and algorithms in areas such as numerical analysis, signal processing, and data science [7]. Moreover, this article can find applications in areas like control systems, optimization, and data modelling. The insights gained from it may lead to the development of more robust and reliable mathematical models and algorithms in these fields. One of the fundamental properties of Riesz spaces includes their lattice structure. While this property has been extensively studied, the theory of Riesz spaces remains a complex and challenging field of mathematics. Interested readers are directed to [1], [8] and [14] for a comprehensive treatment of the fundamental theory of Riesz space and its associated terminology.

A partially ordered set $(V, \leq)$ is said to be a lattice if $t \vee s := \sup\{t, s\}$ and $t \wedge s := \inf\{t, s\}$ exist, for each $t, s \in V$.

Let $(V, \leq)$ be an ordered set which is also a vector space over $\mathbb{R}$. Then V is said an ordered vector space if the following hold:

1. $t \leq s \;\Rightarrow t + w \leq s + w$ for every $t, s, w \in V$.
2. $t \leq s \;\Rightarrow \lambda t \leq \lambda s$ for each $t, s \in V$ and $\lambda \geq 0$.

An ordered vector space $(V, \leq)$ is said to be a Riesz space if $(V, \leq)$ is a lattice. Let V be a Riesz space. Let V be a Riesz space, we denote the positive cone of V by V^+ and define it as follows.

$$V^+ := \{t \in V \ : \ t \geq 0\}.$$

For any $t \in V$, let

$$t^+ = t \vee 0, \quad t^- = -t \vee 0, \quad |t| = t \vee -t.$$

Two elements t and s in a Riesz space V are said to be orthogonal, denoted as $t \perp s$, if $|t| \wedge |s| = 0$. Some of the properties of Riesz spaces are mentioned in below. For each $t, s, w \in V$, we can prove that

1. $(t \vee s) = -(-t \wedge -s)$,
2. $t \vee s + t \wedge s = t + s$,
3. $t + (s \vee w) = (t + s) \vee (t + w)$,
4. $t + (s \wedge w) = (t + s) \wedge (t + w)$,
3. $|t| = t^+ + t^-$, $|t + s| \leq |t| + |s|$,
4. $t \leq s$ is equivalent to $t^+ \leq s^+$ and $t^- \leq s^-$,
5. $(t \vee s) \wedge w = (t \wedge w) \vee (s \wedge w)$, $(t \wedge s) \vee w = (t \vee w) \wedge (s \vee w)$.

A Riesz space V is said to be Archimedean, if

$$\inf\{n^{-1}t : n \in \mathbb{N}\} = 0,$$

holds for each $t \in V^+$.
A norm $\|.\|$ on V is said to be a lattice norm if $\|t\| \leq \|s\|$ whenever $|t| \leq |s|$. In this case $(V, \|.\|)$ is said to be a normed Riesz space. $(V, \|.\|)$ is said to be a Banach lattice if it is complete with respect to the norm.

Definition 1.2. [8] Let V be a Riesz space. The sequence $\{t_n\}$ is said to be uniformly bounded if there are $\{s_n\} \in l^1$ and $e \in V^+$ so that $t_n \leq s_n.e$.

Definition 1.3. [8] A Riesz space V is said to be uniformly complete if for each uniformly bounded sequence t_n in V^+ the supremum of $\{\sum_{i=1}^n t_i \ : \ n \in \mathbf{N}\}$ exists.

Definition 1.4. [8] Let U, V be two Archimedean Riesz spaces. A function H from U to V is said to be positive if $H(U^+) = \{H(|t|) : t \in U\} \subset V^+$.

Theorem 1.5. *[1] For a mapping $H : U \to V$ between two vector lattices, the following are equivalent,*

1. *H is a lattice homomorphism, i.e., $H(t \vee s) = H(t) \vee H(s)$.*
2. *$H(t^+) = H(t)^+$ for all $t \in U$.*
3. *$H(t \wedge s) = H(t) \wedge H(s)$.*
4. *if $t \wedge s = 0$ in U, then $H(t) \wedge H(s) = 0$ holds in V.*
5. *$H(|t|) = |H(t)|$.*

Definition 1.6. [2] The (real) vector lattice V is said to be a lattice ordered algebra (Riesz algebra), if it is a linear algebra (not necessary associative) so that if $t, s \in V^+$, then $ts \in V^+$. The latter property is equal to every of the succeeding declaration:

(i) $|ts| \leq |t||s|$ for every $t, s \in V$;
(ii) $(ts)^+ \leq t^+s^+ + t^-s^-$ for each $t, s \in V$;
(ii) $(ts)^- \leq t^+s^- + t^-s^+$ for all $t, s \in V$;

Definition 1.7. [2] An l-algebra V is said to be

(i) an almost f-algebra if $t \wedge s = 0$ implies $ts = 0$ for every $t, s \in V$;
(ii) a d-algebra if $w(t \vee s) = wt \vee ws$ and $(t \vee s)w = tw \vee sw$ for each $t, s \in V$ and $w \in V^+$;
(iii) an f-algebra if $t \wedge s = 0$ implies $wt \wedge s = tw \wedge s = 0$ for each $t, s \in V$ and $w \in V^+$.

Corollary 1.8. *[2] Let V be an Archimedean f-algebra with unit member e. Then e is a weak order unit (i.e., $t \in V$ and $t \perp e$ imply $t = 0$).*

Example 1.9. *[2] The standard example of an f-algebra is the set $C(X)$ of all real continuous functions on some topological space X. Particularly, consider the vector lattice $C_b(\mathbb{R})$ of all bounded real-valued continuous functions on $\mathbb{R}$ equipped with the pointwise algebraic operations and partial order. Then $C_b(\mathbb{R})$ turns out to be an Archimedean f-algebra. It is a Banach lattice if the norm is defined by $\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|$ for $f \in C_b(\mathbb{R})$.*

Proposition 1.10. *[2] For an l-algebra V, the following are equivalent:*

(i) *V is a d-algebra;*
(ii) *$w|t| = |wt|$ and $|t|w = |tw|$ for each $t \in V$ and $w \in V^+$;*
(iii) *$w(t \vee s) = wt \vee ws$ and $(t \wedge s)w = tw \wedge sw$ for all $t, s \in V$ and $w \in V^+$.*

Definition 1.11. [12] Let V be a lattice ordered algebra which is a Banach lattice. V is said to be a Banach lattice algebra whenever $\|ts\| \leq \|t\|\|s\|$ holds for every $t, s \in V^+$. Furthermore, if V is an f-algebra, then it is said to be Banach lattice f-algebra, clearly V is then a (real) Banach algebra.

It has been proved that any Archimedean f-algebra is commutative and associative [2].

Theorem 1.12. *Let V be an l-algebra with unit member $e > 0$, the following are equivalent,*

(i) *V is an f-algebra.*
(ii) *V is a d-algebra.*
(iii) *V is an almost f-algebra.*
(iv) *e is a weak order unit (i. e., $t \perp e$ implies $t = 0$).*
(v) *$tt^+ \geq 0$ for every $t \in V$.*
(vi) *$t^2 \geq 0$ for all $t \in V$.*

Let V be an f-algebra,

(1) $|ts| = |t||s|$ for every $t, s \in V$.
(2) $t \perp s$ implies that $ts = 0$.
(3) $t^2 = (t^+)^2 + (t^-)^2 \geq 0$ for each $t \in V$.

(4) $tt^+ = (t^+)^2 \geq 0$ for each $t \in V$.
(5) $ts = (t \vee s)(t \wedge s)$ for each $t, s \in V$.
(6) If $t^2 = 0$, then $ts = 0$ for every $s \in V$.
(7) If V is semiprime (i. e., $t^2 = 0$ in V implies $t = 0$), then $t^2 \leq s^2$ iff $|t| \leq |s|$.
(8) If V is semiprime, then $t \perp s$ iff $ts = 0$.
(9) Every unital f-algebra is semiprime.

We refer to [3] and [13] for consider more properties of f-algebras.

Theorem 1.13. *[3] Let V be an f-algebra. Then*

$$V \text{ semiprime } \Longleftrightarrow [t, s \in V^+; t^2 = s^2, \text{ then } t = s].$$

Corollary 1.14. *[3] Let V be a uniformly complete unital f-algebra and $t \in V^+$. Then $\sqrt{t}$ exists in V^+.*

Theorem 1.15. *[3] Let V be a uniformly complete semiprime f-algebra. Then $\sqrt{ts}$ exists in V^+ for every $t, s \in V^+$.*

Example 1.16. *[2] Let $V = C([0,\infty))$ be the vector lattice of all continuous functions on the interval $[0,\infty)$ and equip V with the pointwise order and the algebraic operations. Then V is an Archimedean unital semiprime uniformly complete f-algebra.*

Definition 1.17. Let U and V be uniformly complete unital Banach f-algebras and $H : U \to V$ be a function. We define

(H_1) semi-lattice homomorphism:

$$\|H(t \vee s)\| = \|H(t) \vee H(s)\| \text{ or } \|H(t \vee s)\| = \|H(t) \vee -H(s)\|$$

for every $t, s \in V$.

(H_2) semi-homogeneity:

$$H(\lambda t) = \lambda H(t)$$

for every $t \in U^+$ and each number $\lambda \in \mathbb{R}^+$.

2. Stability in Uniformly Complete Banach F-Algebras

In this part, we will demonstrate the stability of lattice functional equation (2) in uniformly complete Banach f-algebra by the direct method.

Theorem 2.1. *Let U be a uniformly complete unital f-algebra and V be a uniformly complete unital Banach f-algebra. Let $h : U \to V$ be a positive function with $h(0) = 0$, so that*

$$\|h(\alpha u \vee \beta v)^2 - (\alpha h(u) \vee \beta h(v))^2\| \leq \Upsilon(\alpha u \vee \beta v, \alpha u \wedge \beta v) \tag{2}$$

for every $u, v \in U$ and $\alpha, \beta \in \mathbb{R}^+$ with $\alpha, \beta > 1$. Let $\Upsilon : U \times U \to [0,\infty)$ be a function, which satisfies

$$\Upsilon(u, v) \leq \Upsilon\left(\frac{u}{\alpha}, \frac{v}{\beta}\right) \tag{3}$$

for all $u, v \in U$ and $\alpha, \beta > 1$. Then there is only one mapping $H : U \to V$, which satisfies property (H_1) and (H_2) and inequality

$$\|H(u)^2 - h(u)^2\| \leq \frac{1}{\alpha^2 - 1}\Upsilon(u, u) \tag{4}$$

for any $u \in U$ and $\alpha > 1$.

Proof. Placing $\alpha = \beta$ and $u = v$ in (2) and by using (3), we achieve

$$\begin{aligned} \|h(\alpha u)^2 - (\alpha h(u))^2\| &\leq \Upsilon(\alpha u, \alpha u) \\ &\leq \Upsilon(u, u). \end{aligned} \tag{5}$$

By substituting αu for u in (5), we have

$$\|h(\alpha^2 u)^2 - (\alpha h(\alpha u))^2\| \leq \Upsilon(u, u). \tag{6}$$

With multiplying (5) in α^2, we attain

$$\|\alpha^2 h(\alpha u)^2 - (\alpha^2 h(u))^2\| \leq \alpha^2 \Upsilon(u, u). \tag{7}$$

With comparing (6) and (7), we achieve

$$\|h(\alpha^2 u)^2 - (\alpha^2 h(u))^2\| \leq (1 + \alpha^2)\Upsilon(u, u), \tag{8}$$

replacing u by αu in above inequality, so we have

$$\begin{aligned} \|h(\alpha^3 u)^2 - (\alpha^2 h(\alpha u))^2\| &\leq (1 + \alpha^2)\Upsilon(\alpha u, \alpha u) \\ &\leq (1 + \alpha^2)\Upsilon(u, u). \end{aligned} \tag{9}$$

Again, by multiplying (5) in α^4 and by comparing (9), we have

$$\|h(\alpha^3 u)^2 - (\alpha^3 h(u))^2\| \leq (1 + \alpha^2 + \alpha^4)\Upsilon(u, u), \tag{10}$$

we continue this trend, so we achieve the next inequality.

$$\|h(\alpha^n u)^2 - (\alpha^n h(u))^2\| \leq \sum_{i=0}^{n-1} \alpha^{2i}\Upsilon(u, u). \tag{11}$$

Therefore,

$$\begin{aligned} \|\alpha^{-2n} h(\alpha^n u)^2 - (h(u))^2\| &\leq \frac{1}{\alpha^{2n}} \sum_{i=0}^{n-1} \alpha^{2i}\Upsilon(u, u) \\ &= \frac{1}{\alpha^{2n}} . \frac{(1 - \alpha^{2n})}{1 - \alpha^2}\Upsilon(u, u) \\ &\leq \frac{1}{\alpha^2 - 1}\Upsilon(u, u) \end{aligned} \tag{12}$$

for each $u \in U$ and $n \in \mathbb{N}$. It pursues from (12), that for $n > m > 0$,

$$\begin{aligned} &\left\| \left(\frac{1}{\alpha^n} h(\alpha^n u)\right)^2 - \left(\frac{1}{\alpha^m} h(\alpha^m u)\right)^2 \right\| \\ &= \alpha^{-4m} \left\| \left(\frac{1}{\alpha^{n-m}} h(\alpha^{n-m}(\alpha^m u))\right)^2 - h(\alpha^m u)^2 \right\| \\ &\leq \alpha^{-4m} \frac{1}{\alpha^2 - 1} \Upsilon(\alpha^m u, \alpha^m u) \leq \frac{1}{\alpha^{4m}(\alpha^2 - 1)} \Upsilon(u, u). \end{aligned} \tag{13}$$

The term on right-hand desires to zero as $m \to \infty$, hence $\left\{ (\alpha^{-n} h(\alpha^n u))^2 \right\}$ is a Cauchy sequence in V. Note that V is a uniformly complete unital Banach f-algebra, so we have

(1) $\left\{ (\alpha^{-n} h(\alpha^n u))^2 \right\} \geq 0$, by Theorem 1.12.

(2) $|\alpha^{-n} h(\alpha^n u)| = \sqrt{(\alpha^{-n} h(\alpha^n u))^2}$, by Corollary 1.14.

We define

$$J_u^+ = \{ n \in \mathbb{N} \mid h(\alpha^n u) \in V^+ \}$$

and

$$J_u^- = \{ n \in \mathbb{N} \mid -h(\alpha^n u) \in V^+ - \{0\} \}.$$

In view of (13) and above items, we know that if J_u^+ or J_u^- is an infinite set, then sequences $\{\alpha^{-n} h(\alpha^n u)\}_{n \in I_u^+}$ or $\{\alpha^{-n} h(\alpha^n u)\}_{n \in I_u^-}$ is a Cauchy sequence, respectively. Now, let's define

$$H(u) := \begin{cases} \lim_{\substack{n\to\infty \\ n \in J_u^+}} \alpha^{-n} h(\alpha^n u) & \text{if } J_u^+ \text{ is infinite} \\ \\ \lim_{\substack{n\to\infty \\ n \in J_u^-}} \alpha^{-n} h(\alpha^n u) & \text{otherwise} \end{cases}. \tag{14}$$

It is explicit that if both J_u^+ and J_u^- are infinite sets, then

$$H(u) = -\lim_{\substack{n\to\infty \\ n \in J_u^-}} \alpha^{-n} h(\alpha^n u). \tag{15}$$

The inequality (4) holds, by the definition of H and (12). Let $u, v \in U$ be given. We can demonstrate that there is leastways one infinite set among the sets

$$\begin{array}{lll} J_u^+ \cap J_v^+ \cap J_{u\vee v}^+ & J_u^+ \cap J_v^+ \cap J_{u\vee v}^- & J_u^+ \cap J_v^- \cap J_{u\vee v}^+ \\ J_u^- \cap J_v^+ \cap J_{u\vee v}^+ & J_u^- \cap J_v^- \cap J_{u\vee v}^+ & J_u^- \cap J_v^+ \cap J_{u\vee v}^- \\ J_u^+ \cap J_v^- \cap J_{u\vee v}^- & J_u^- \cap J_v^- \cap J_{u\vee v}^-. & \end{array}$$

It is possible to choose such an infinite set and indicate it by J. Let $n \in J$ be given. Putting $\alpha = \beta = \alpha^n$ in (2), we have

$$\left\| h(\alpha^n(u \vee v))^2 - (\alpha^n(h(u) \vee h(v)))^2 \right\| \leq \Upsilon(\alpha^n(u \vee v), \alpha^n(u \wedge v)). \tag{16}$$

Substituting u with $\alpha^n u$ and v with $\alpha^n v$ in above inequality, we achieve

$$\begin{aligned}\left\|h(\alpha^{2n}(u\vee v))^2-(\alpha^n(h(\alpha^n u)\vee h(\alpha^n v)))^2\right\| &\leq \Upsilon(\alpha^{2n}(u\vee v),\alpha^{2n}(u\wedge v))\\ &\leq \Upsilon(u\vee v,u\wedge v), \quad (17)\end{aligned}$$

by dividing the resulting inequality (17) by α^{4n}, we obtain

$$\left\|\left(\frac{1}{\alpha^{2n}}h(\alpha^{2n}(u\vee v))\right)^2-\left(\frac{h(\alpha^n u)}{\alpha^n}\vee\frac{h(\alpha^n v)}{\alpha^n}\right)^2\right\|\leq\frac{1}{\alpha^{4n}}\Upsilon(u\vee v,u\wedge v) \quad (18)$$

as $n\to\infty$, through (J) in (15) and (18) into consideration, we attain

$$H(u\vee v)^2=(H(u)\vee H(v))^2 \quad \text{or} \quad H(u\vee v)^2=(-H(u)\wedge H(v))^2 \quad (19)$$

for every $u,v\in U$. The right-hand equality (19) can happen, for example, when both J_v^+ and J_v^- are infinite sets and $J=J_u^+\cap J_v^-\cap J_{u\vee v}^+$ for some $u,v\in U$, because (18) and $J=J_u^+\cap J_v^-\cap J_{u\vee v}^+$ lead to

$$\begin{aligned}\lim_{\substack{n\to\infty\\ n\in J}}\alpha^{-n}h(\alpha^n(u\vee v)) &= \lim_{\substack{n\to\infty\\ n\in J_{u\vee v}^+}}\alpha^{-n}h(\alpha^n(u\vee v)) &&= H(u\vee v)\\ \lim_{\substack{n\to\infty\\ n\in J}}\alpha^{-n}h(\alpha^n u) &= \lim_{\substack{n\to\infty\\ n\in J_u^+}}\alpha^{-n}h(\alpha^n u) &&= H(u)\\ \lim_{\substack{n\to\infty\\ n\in J}}\alpha^{-n}h(\alpha^n v) &= \lim_{\substack{n\to\infty\\ n\in J_v^-}}\alpha^{-n}h(\alpha^n v). &&= -H(v).\end{aligned} \quad (20)$$

Not that, V is a unital f-algebra, then V is semi-prime so by Theorem (1.13), we attain

$$\|H(u\vee v)\|=\|H(u)\vee H(v)\| \quad (21)$$

or

$$\|H(u\vee v)\|=\|H(u)\vee -H(v)\|. \quad (22)$$

It means that H satisfies (H_1). Next, we demonstrate that $H(\alpha u)=\alpha H(u)$ for each $u\in U^+$ and $\alpha>1$. Choosing $\alpha=\beta$ and $v=0$ in (2) and substituting $2^n\alpha$ for α, we obtain

$$\left\|h(2^n\alpha u)^2-(2^n\alpha)^2h(u)^2\right\|\leq\Upsilon(2^n\alpha u,0) \quad (23)$$

for every $u\in U^+$. Now we substitute u with $2^n u$ in above inequality, therefore

$$\left\|h(2^{2n}\alpha u)^2-(2^n\alpha)^2h(2^n u)^2\right\|\leq\Upsilon(2^{2n}\alpha u,0), \quad (24)$$

by dividing the inequality (24) by 4^{2n}, we obtain

$$\left\|\left(\frac{h(4^n\alpha u)}{4^n}\right)^2-\alpha^2\left(\frac{h(2^n u)}{2^n}\right)^2\right\|\leq\frac{1}{4^{2n}}\Upsilon\left(2^{2n}\alpha u,0\right), \quad (25)$$

as $n\to\infty$, the right term tends to zero, we have

$$H(\alpha u)^2=\alpha^2H(u)^2, \quad (26)$$

since V is semi-prime then,

$$H(\alpha u) = \alpha H(u)$$

for every $u \in U^+$. Hence, H satisfies (H_2). Let us demonstrate that the function H is unique. Assume that $H' : U \to V$ is another function which satisfies (H_1), (H_2) and (4). So, we have

$$H(\alpha u) = \alpha H(u) \text{ and } H'(\alpha u) = \alpha H'(u) \tag{27}$$

for all $u \in U^+$ and real number $\alpha > 1$. Therefore, by (4), we obtain

$$\begin{aligned} \|H(u)^2 - H'(u)^2\| &= \alpha^{-2}\|H(\alpha u)^2 - H'(\alpha u)^2\| \\ &\leq \alpha^{-2}\big(\|H(\alpha u)^2 - h(\alpha u)^2\| + \|h(\alpha u)^2 - H'(\alpha u)^2\|\big) \\ &\leq \alpha^{-2}.\frac{2}{\alpha^2 - 1}.\Upsilon(u, u) \\ &\to 0 \text{ as } \alpha \to \infty, \end{aligned} \tag{28}$$

then

$$H(u)^2 = H'(u)^2,$$

since V is semi-prime, so we get

$$H(u) = H'(u)$$

which completes the proof. □

Example 2.2. *Let U, V be the uniformly complete unital Banach f-algebras of real-valued continuous functions defined on the closed interval $[0, 1]$. So, $U = V = C([0, 1])$. We define $h(u)(x) = u(x)$ for all $u \in C([0, 1])$, with $u(0) = 0$. It is clear that*

$$h(\alpha u \vee \beta v)^2 = (\alpha h(u) \vee \beta h(v))^2$$

for every $u, v \in U$ and $\alpha, \beta \in \mathbb{R}^+$. Note that $\alpha u \vee \beta v = \max\{\alpha u, \beta v\}$. It means that

$$|h(\alpha u \vee \beta v)^2 - (\alpha h(u) \vee \beta h(v))^2| = 0.$$

Therefore, the conditions of Theorem (2.1) hold, and thus there exists a unique function such as H that satisfies the conditions H_1 and H_2.

Theorem 2.3. *Assume that U be a uniformly complete unital f-algebra and V be a uniformly complete unital Banach f-algebra and $\vartheta : [0, \infty) \to (0, \infty)$ be a continuous function. Consider a positive function $h : U \to V$ with $h(0) = 0$ satisfying*

$$\left\| h(\alpha u \vee \beta v)^2 - \left(\frac{\alpha\vartheta(\alpha)h(u) \vee \beta\vartheta(\beta)h(v)}{\vartheta(\alpha) \vee \vartheta(\beta)}\right)^2 \right\| \leq \Upsilon(u \vee v, u \wedge v), \tag{29}$$

where $\Upsilon : U \times U \to [0, \infty)$ be a function such that

$$\Phi(u, v) = \sum_{i=0}^{\infty} \frac{1}{\alpha^{2i+2}} \Upsilon(\alpha^i u, \alpha^i v) < \infty \tag{30}$$

for all $u, v \in U$ and $\alpha, \beta \in \mathbb{R}$, $\alpha, \beta > 1$. Then there is only one function $H : U \to V$, which satisfies (H_1) and (H_2) and inequality

$$\left\|H(u)^2 - h(u)^2\right\| \leq \Phi(u, u) \tag{31}$$

for every $u \in U$.

Proof. For all $n \in \mathbb{N}$, the subsequent inequality is true (by induction on n),

$$\left\|\alpha^{-2n}(h(\alpha^n u))^2 - h(u)^2\right\| \leq \sum_{i=0}^{n-1} \frac{1}{\alpha^{2i+2}} \Upsilon(\alpha^i u, \alpha^i u)0 \tag{32}$$

The above inequality holds for $n = 1$, by substituting u and α for v and β in (29), respectively. Assume that (32) holds, for some $n > 0$, then it pursues from (29) and (30)

$$\begin{aligned} \left\|\frac{1}{\alpha^{2n+2}} h(\alpha^{n+1}u)^2 - h(u)^2\right\| &\leq \left\|\frac{1}{\alpha^{2n+2}} h(\alpha^{n+1}u)^2 - \frac{1}{\alpha^2} h(\alpha u)^2\right\| \\ &+ \left\|\frac{1}{\alpha^2} h(\alpha u)^2 - h(u)^2\right\| \\ &\leq \frac{1}{\alpha^2} \sum_{i=0}^{n-1} \frac{1}{\alpha^{2i+2}} \Upsilon(\alpha^{i+1}u, \alpha^{i+1}u) + \frac{1}{\alpha^2} \Upsilon(u, u) \\ &= \sum_{i=0}^{n} \frac{1}{\alpha^{2i+2}} \Upsilon(\alpha^i u, \alpha^i u), \end{aligned}$$

which implies the validity (32) for all $n \in \mathbb{N}$. Let $n > m > 0$. It pursues from (32) and (30) that

$$\begin{aligned} \left\|\frac{1}{\alpha^{2n}} h(\alpha^n u)^2 - \frac{1}{\alpha^{2m}} h(\alpha^m u)^2\right\| &= \frac{1}{\alpha^{2m}} \left\| \left(\frac{1}{\alpha^{n-m}} h(\alpha^{n-m}(\alpha^m u)\right)^2 - h(\alpha^m u)^2\right\| \\ &\leq \frac{1}{\alpha^{2+2m}} \sum_{i=0}^{n-m-1} \frac{1}{\alpha^{2i}} \Upsilon\left(\alpha^{i+m}u, \alpha^{i+m}u\right) \\ &= \frac{1}{\alpha^2} \sum_{i=m+1}^{n-1} \frac{1}{\alpha^{2i}} \Upsilon\left(\alpha^i u, \alpha^i u\right) \\ &\to 0 \quad \text{as } m \to \infty. \end{aligned}$$

for all $u \in U$. Hence, $\{\alpha^{-n} h(\alpha^n u)\}^2$ is a Cauchy sequence. Since V is a uniformly complete unital Banach f-algebra, similar the proof of the theorem mentioned above, we can define

$$H(u) = \lim_{n\to\infty} \frac{1}{\alpha^n} h(\alpha^n u). \tag{33}$$

The continuation of proof is similar to the previous one. □

Corollary 2.4. *Let U and V be uniformly complete unital Banach f-algebras and $\vartheta : [0,\infty) \to (0,\infty)$ be a continuous function. Consider a positive function $h : U \to V$ with $h(0) = 0$ satisfying*

$$\left\| h(\alpha u \vee \beta v)^2 - \left(\frac{\alpha\vartheta(\alpha)h(u) \vee \beta\vartheta(\beta)h(v)}{\vartheta(\alpha)+\vartheta(\beta)} \right)^2 \right\| \leq \Upsilon(u \vee v, u \wedge v), \tag{34}$$

where $\Upsilon : U \times U \to [0,\infty)$ is a function satisfying (30) for every $u, v \in U$ and $\alpha, \beta \in \mathbb{R}$, $\alpha, \beta > 1$. Then there is only one function $H : U \to V$ which satisfies (H_1), (H_2) and inequality (31) for each $u \in U$.

Corollary 2.5. *Let U and V be uniformly complete unital Banach f-algebras and $\vartheta : [0,\infty) \to (0,\infty)$ be a continuous function. Consider a positive function $h : U \to V$ with $h(0) = 0$ satisfying*

$$\left\| h\left(\frac{\alpha\vartheta(\alpha)u \vee \beta\vartheta(\beta)v}{\vartheta(\alpha)\vee\vartheta(\beta)} \right)^2 - \left(\frac{\alpha\vartheta(\alpha)h(u) \vee \beta\vartheta(\beta)h(v)}{\vartheta(\alpha)+\vartheta(\beta)} \right)^2 \right\| \leq \Upsilon(u \vee v, u \wedge v), \tag{35}$$

where $\Upsilon : U \times U \to [0,\infty)$ is a function satisfying (30) for every $u, v \in U$ and $\alpha, \beta \in \mathbb{R}$, $\alpha, \beta > 1$. Then there exists a unique function $H : U \to V$ which satisfies properties (H_1), (H_2) and inequality (31) for each $u \in U$.

Corollary 2.6. *Assume that U and V are uniformly complete unital Banach f-algebras and $h : U \to V$ is a positive function with $h(0) = 0$ so that*

$$\left\| h\left(\alpha^q u \Delta_U \beta^q v\right)^2 - \left(\alpha^p h(u) \Delta_V \beta^p h(v)\right)^2 \right\| \leq \Upsilon(u\Delta_U v, u\Delta_U v) \tag{36}$$

for every $u, v \in U$ and $\alpha, \beta \in (1,\infty)$ and $(p,q) \in (0,\infty)\times(0,\infty)$, where

$$\Delta_U \in (\wedge_U, \vee_U)\ ,\ \Delta_V \in (\wedge_V, \vee_V) \tag{37}$$

are lattice operations and $\Upsilon : U \times U \to [0,\infty)$ is a function so that

$$\Phi(u,v) = \sum_{i=0}^{\infty} \frac{1}{\alpha^{2p(i+1)}} \Upsilon(\alpha^{iq}u, \alpha^{iq}v) < \infty. \tag{38}$$

Then the sequence $\left\{\alpha^{-2np} h\left(\alpha^{nq}u\right)\right\}$ is a Cauchy sequence for every $u \in U$. Let $H : U \to V$ be define by

$$H(u) = \lim_{n\to\infty} \frac{1}{\alpha^{2np}} h\left(\alpha^{nq}u\right) \tag{39}$$

for each $u \in U$. Then H satisfies (H_1), (H_2) and inequality

$$\|H(u) - h(u)\| \leq \Phi(u,u) \tag{40}$$

for each $u \in U$.

Proof. The proof of all the above corollaries is similar to Theorem 2.1. □

3. Conclusion

This research paper is focused on investigating the stability of a functional equation that preserves the lattice structure in uniformly complete Banach f-algebra. The main goal was to clarify the behavior of this equation and its connection with the algebraic properties of the Banach $f-$algebra. For this purpose, several theorems and corollaries were proved and other corollaries can be added considering the conditions of the theorem. This research has provided a deeper understanding of the stability of the functional equation and its effect on the lattice structure in the Banach f- algebra. By studying the behavior of the equation, valuable insights into its relationship with the algebraic properties of the Banach f- algebra have been obtained, contributing to the broader field of functional analysis. The results of this research can have consequences for different areas of mathematics and its applications. Investigating the stability of the functional equation in the Banach f-algebra setting contributes to the basics of functional analysis and lattice theory. These findings may also be used in operator theory, approximation theory, and various fields of applied mathematics.

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