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THE CONSTRUCTION OF FRACTIONS OF Γ-MODULE OVER COMMUTATIVE Γ-RING

L. Amjadi, M. Ghadiri ✉, and S. Mirvakili

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Abstract. The aim of this paper is to construct fraction of a Γ-module over a commutative Γ-ring. There should be an appropriate set S of elements in a Γ-ring R to be used as a Γ-module of fractions. Then we study the homomorphisms of a Γ-module which can lead to related basic results. We show that for every Γ-module M, $S^{-1}(0 :_R M) = (0 :_{S^{-1}R} S^{-1}M)$. Also, if M is a finitely generated R_Γ-module, then $S^{-1}M$ is finitely generated.

Keywords: Γ-ring of fraction, Γ-module of fraction, Finitely generated Γ-module.
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1. Introduction and Basic Definitions

The formation of rings of fractions and the associated process of localization are perhaps the most important technical tools in commutative algebra. They correspond in the algebra to concentrating attention on the importance of these notions should be self evident. Atiyeh [2] gave the definition and simple properties of the formation of fractions in commutative rings and modules. Fraction rings and fraction modules have various applications in mathematics, computer science, and engineering. Some of these applications are:

1. Algebraic geometry: Fraction rings are used to study algebraic varieties and their properties. They provide a way to localize a ring at a prime ideal and study the behavior of the ring near that ideal.
2. Number theory: Fraction rings are used to study number fields and their properties. They provide a way to extend the field of rational numbers by adjoining roots of polynomials.
3. Coding theory: Fraction modules are used to construct error-correcting codes. They provide a way to encode messages using a finite-dimensional vector space over a field, and then decode them using linear algebraic techniques.
4. Cryptography: Fraction rings and modules are used to construct public-key cryptosystems. They provide a way to encrypt messages using

✉ mghadiri@yazd.ac.ir, ORCID: 0000-0002-9033-4252
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modular arithmetic, and then decrypt them using the inverse operation.

5. Control theory: Fraction modules are used to model and analyze linear control systems. They provide a way to represent the system as a set of equations involving matrices and vectors, and then analyze its stability and performance.

The notation of Γ-ring was first introduced by Nobusawa [6] as a generalization of a classical ring and afterward Barnes [3] improved the concepts of Nobusawa's Γ-ring and developed the more general Γ-ring in which all classical rings were contained [8, 9]. The concept of Γ-structures in related structures to Γ-ring such as fuzzy Γ-rings, Γ-hyperrings and Γ-hemirings is used by the researchers of the century [4, 5, 12].

Recently, Tabatabaee and Roodbarylor [10] constructed commutative Γ-rings of fractions and discussed the quotient field of commutative integral domain by used local Γ-rings. Also, Ostadhadi-Dehkordi using strongly regular relation and constructed quotient (Γ, R)-hypermodules [7].

The definition of Γ-module was given for the first time by Ameri et al [1], studying some preliminary properties of them such as: Γ-submodules, homomorphism of Γ-module and finitely generated Γ-module.

Considering the applications of rings and modules of fractions that were mentioned and considering that Γ-modules and Γ-rings are generalizations of modules and rings, therefore, construction and studying the properties of Γ-rings and Γ-modules of fractions can help to expand the previous concepts. In this paper, we extend the concept of fraction from the category of modules to that Γ-modules over Γ-rings and the researchers discussed its characteristics and relations by using Γ-rings. Further, we investigate some theorems of homomorphism of Γ-modules.

In Section 2, we construct fraction of a Γ-module by choosing appropriate equivalence relation on $M \times S$, where S is a multiplication closed subset on a Γ-ring R. Finally, finitely generated Γ-modules and homomorphism of Γ-modules are investigated.

Definition 1.1. [6] Let R and Γ be abelian groups. Then R is called a Γ-ring if there exists a mapping $(a, \gamma, b) \longrightarrow a\gamma b$ of $R \times \Gamma \times R \longrightarrow R$ satisfying the following conditions: for all $a, b, c \in R$ and $\gamma, \gamma_1, \gamma_2 \in \Gamma$,

(1) $(a+b)\gamma c = a\gamma c+b\gamma c,\ \ a\gamma(b+c) = a\gamma b+a\gamma c, ,\ \ a(\gamma_1+\gamma_2)b = a\gamma_1 b+a\gamma_2 b.$
(2) $a\gamma_1(b\gamma_2 c) = (a\gamma_1 b)\gamma_2 c.$

Definition 1.2. [11] Let R be a Γ-ring.

(1) If there exists $\gamma_0 \in \Gamma$ and $1_{R_{\gamma_0}} \in R$ such that for all $r \in R$, $1_{R_{\gamma_0}}\gamma_0 r = r\gamma_0 1_{R_{\gamma_0}} = r$, then $1 = 1_{R_{\gamma_0}}$ is called identity element of R and R is called a Γ-ring with identity.
(2) If for all $a, b \in R$ and $\gamma \in \Gamma$, $a\gamma b = b\gamma a$, then R is called a commutative Γ-ring.

If 0 is the zero element of group $(R, +)$, by using (1) of Definition 1.1 it is obtained that $0\gamma a = a\gamma 0 = 0$ and $(-a)\gamma b = -(a\gamma b)$ for $a, b \in R, \gamma \in \Gamma$.

In this paper we set $a\Gamma b = \{a\gamma b | \gamma \in \Gamma\}$.

Definition 1.3. [10] Let R be a Γ-ring.

(1) A multiplicatively closed subset $(m.c.s)$ of a Γ-ring R is a subset S of R such that $1 \in S$ and $s_1\Gamma s_2 \subseteq S$ for all $s_1, s_2 \in S$,
(2) An element $a \in R$ is said to be zero-devisor on Γ-ring R if there exists $b(\neq 0) \in R$ and $\gamma_0 \in \Gamma$ such that $a\gamma_0 b = b\gamma_0 a = 0$,
(3) A subset I of Γ-ring R is said left(right) Γ-ideal if I is an additive subgroup of R and $R\Gamma I \subseteq I$ $(I\Gamma R \subseteq I)$.

Remark 1.4. [11] We consider the following assumptions over Γ-ring R for all $a, b, c \in R, \alpha, \beta \in \Gamma$ and $s_1, s_2 \in S$,

(∗) $a\alpha b\beta c = a\beta b\alpha c$,
(∗∗) $(s_1\alpha s_2)\gamma_0(s_1\alpha s_2)\gamma_0(a\beta b) + (s_1\beta s_2)\gamma_0(s_1\beta s_2)\gamma_0(a\alpha b) = 0$.

After this, the word Γ-ring R means a commutative Γ-ring with 1 and without zero-divisor. It is modified the Proposition 2.2 of [10] as follows:

Proposition 1.5. *[10] Let R be a Γ-ring and $S = R - \{0\}$. Define the relation $\sim$ on $R \times S$ as follows: $(a, s) \sim (b, t) \iff a\gamma t - b\gamma s = 0$ for all $a, b \in R$ and $s, t \in S$ and some $\gamma \in \Gamma$. Then $\sim$ is an equivalence relation.*

Theorem 1.6. *[10] Let $[r, s]$ denote the equivalence class containing (r, s) and $S^{-1}R$ denote the set of all equivalence classes. If R satisfies the conditions (∗) and (∗∗), we define addition and multiplication of these fractions as follows:*

$$S^{-1}R \times \Gamma \times S^{-1}R \longrightarrow S^{-1}R,$$
$$[r, s]\gamma[r', s'] = [r\gamma r', s\gamma s'],$$
$$[r, s] \oplus [r', s'] = [r\gamma s' + s\gamma r', s\gamma s'].$$

Then

(1) *These operations are well-defined.*
(2) *$S^{-1}R$ is a Γ-ring with identity element $[1, 1]$.*

The next two Examples show that the condition (∗∗) is not necessary in the Theorem 1.6, but since we cannot do a proof in the general case, we have to use this condition.

Example 1.7. *Let $(R, +, \cdot)$ be a commutative ring and S be a m.c.s. Put $\Gamma = \{\cdot\}$. Then R is a Γ-ring. Moreover, the fraction ring $S^{-1}R$ is a Γ-ring.*

Example 1.8. *[10] Let $(\mathbb{Z}, +)$ be the group of integer numbers and $M_{m\times n}(\mathbb{Z})$ be the set of all $m \times n$ matrices with entries in $\mathbb{Z}$. We consider $R = \{\begin{bmatrix} x & x \end{bmatrix} | x \in \mathbb{Z}\} \subseteq M_{1\times 2}(\mathbb{Z})$ and $\Gamma_1 = \{\begin{bmatrix} n \\ 0 \end{bmatrix} | n \in \mathbb{Z}\}, \Gamma_2 = \{\begin{bmatrix} 0 \\ n \end{bmatrix} | n \in \mathbb{Z}\}$ the subsets of $M_{2\times 1}(\mathbb{Z})$ and $M = \{\begin{bmatrix} y & y \end{bmatrix} | y \in \mathbb{Z}\} \subseteq M_{1\times 2}(\mathbb{Z})$. The mapping $R \times \Gamma_1 \times R \longrightarrow R$*

by $\begin{bmatrix} x & x \end{bmatrix} \begin{bmatrix} n \\ 0 \end{bmatrix} \begin{bmatrix} y & y \end{bmatrix} = \begin{bmatrix} nxy & nxy \end{bmatrix}$ *for all* $\begin{bmatrix} x & x \end{bmatrix}, \begin{bmatrix} y & y \end{bmatrix} \in R$ *and* $\begin{bmatrix} n \\ 0 \end{bmatrix} \in \Gamma_1$, *$R$ become a Γ_1-ring and similarly a Γ_2-ring. Also we can see R is an R_{Γ_1}-module with unitary elements* $[1,1],[1,0]$ *and* $\gamma_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ *and R is an R_{Γ_2}-module with unitary elements* $[1,1],[0,1]$ *and* $\gamma_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. *It is easy to consider that if* $S = R - \{[0,0]\}$, *then $S^{-1}R$ is a $\Gamma_1(\Gamma_2)$-ring.*

Example 1.9. *Let $R = \mathbb{Z}_{p^4}$, $\Gamma = \{p^2, p^3\}$ and $S = R - \{0\}$. Define the $R \times \Gamma \times R \longrightarrow R$ by $(x, \gamma, y) \mapsto x\gamma y$ for all $x, y \in R, \gamma \in \Gamma$. Conditions $(*)$ and $(**)$ hold. We see that $[p^3, p^3] = \{[r, s] | r \in R, s \in S\}$. So $S^{-1}R = \{[0,0]\}$.*

Definition 1.10. [1] Let R be a Γ-ring. A (left)R_Γ-module is an additive abelian group M together with a mapping: $R \times \Gamma \times M \longrightarrow M$ (the image of (r, γ, m) denoted by $r\gamma m$), such that for all $m, m_1, m_2 \in M$ and $\gamma, \gamma_1, \gamma_2 \in \Gamma$ and $r, r_1, r_2 \in R$ the following hold:

(M1) $r\gamma(m_1 + m_2) = r\gamma m_1 + r\gamma m_2$,
(M2) $(r_1 + r_2)\gamma m = r_1\gamma m + r_2\gamma m$,
(M3) $r(\gamma_1 + \gamma_2)m = r\gamma_1 m + r\gamma_2 m$,
(M4) $r_1\gamma_1(r_2\gamma_2 m) = (r_1\gamma_1 r_2)\gamma_2 m$.

It is easy to see that:

(1) $0_R\gamma m = r\gamma 0_m = 0_m$ (Also we ignore the indexes in 0_R and 0_m),
(2) Every abelian group M is an R_Γ-module with trivial module structure by defining $r\gamma m = 0$ for every $r \in R, \gamma \in \Gamma, m \in M$,
(3) Every Γ-ring is an R_Γ-module with $r\gamma s(r, s \in R, \gamma \in)$ being the Γ-ring structure in R, i.e., the mapping

$$\cdot : R \times \gamma \times R \to R.(r, \gamma, s) \to r \cdot \gamma \cdot s$$

Definition 1.11. [1] Let R be a Γ-ring with identity 1, a (left) R_Γ-module M is called unitary R_Γ-module, if there exists $\gamma_0 \in \Gamma$ such that $1\gamma_0 m = m$ for every $m \in M$.

In this article, γ_0 is the γ_0 stated in Definition 1.11.

Example 1.12. *If R is a Γ-ring, then every abelian group M can be made into an R_Γ-module with trivial module structure by defining*

$$r\gamma m = 0, \ \forall r \in R, \forall \gamma \in , \forall m \in M.$$

Example 1.13. *[1] Let M be an arbitrary abelian group and L be an arbitrary subring of $\mathbb{Z}$. Then M is a $\mathbb{Z}_L$-module under the mapping*

$$\cdot : \mathbb{Z} \times L \times M \to M(n, n_0, x) \to nn_0x.$$

Example 1.14. *[1] If R is a Γ-ring and I is a left ideal of R. Then I is an R_Γ-module under the mapping $\cdot : R \times \Gamma \times I \to I$ such that $(r, \gamma, a) \to r\gamma a$.*

2. Γ-module of fractions

The construction of $S^{-1}R$ can be carried through with an Γ-module M in place of the Γ-ring R. Throughout this paper, the word Γ-ring R means a commutative Γ-ring with 1 without zero-divisor. Also, $\gamma_0 \in \Gamma$ means the same γ_0 in the Definition 1.11.

Proposition 2.1. *Let M be a Γ-module and S be an m.c.s of Γ-ring R. Define the relation $\sim$ on $M \times S$ as follows: for all $m, m' \in M$ and $s, s' \in S, \gamma \in \Gamma$,*

$$(m, s) \sim (m', s') \Longleftrightarrow \exists t \in S, \;\; t\gamma(s\gamma m') = t\gamma(s'\gamma m).$$

Then $\sim$ is an equivalence relation.

Proof. It is easy to see that $\sim$ is reflexive and symmetric. For transitively, if $(m, s) \sim (m', s')$ and $(m', s') \sim (m'', s'')$, then for some $t, u \in S$ and for some $\alpha, \beta \in \Gamma$ we have,

$$t\alpha(s\alpha m') = t\alpha(s'\alpha m), \tag{1}$$

$$u\beta(s'\beta m'') = u\beta(s''\beta m'). \tag{2}$$

A multiplication by $u\beta s''\beta$ of (1) and $t\alpha s\alpha$ of (2) gives:

$$t\alpha u\beta s''\beta(s\alpha m') = t\alpha u\beta s''\beta(s'\alpha m), \tag{3}$$

$$u\beta t\alpha s\alpha(s'\beta m'') = u\beta t\alpha s\alpha(s''\beta m'). \tag{4}$$

By using of commutativity we have,

$$\begin{aligned} t\alpha u\beta s''\beta(s\alpha m') &= u\beta t\alpha s\alpha(s''\beta m') \\ &= t\alpha u\beta s''\beta(s\alpha m'). \end{aligned}$$

Hence,

$$\begin{aligned} &t\alpha u\beta s''\beta(s'\alpha m) = u\beta t\alpha s\alpha(s'\beta m'') \\ &t\alpha u\beta s'\beta(s''\alpha m) = t\alpha u\beta s'\alpha(s\beta m'') = t\alpha u\beta s'\beta(s\alpha m''), \end{aligned}$$

where $t\alpha u\beta s' \in S$. Thus $(m, s) \sim (m'', s'')$. □

Theorem 2.2. *Let M be a Γ-module and S be an m.c.s of Γ-ring R. Let $[m, s]$ denote the equivalence class containing (m, s) and $S^{-1}M$ denote the set of equivalence classes. If R satisfies the conditions $(*)$ and $(**)$, we define addition and multiplication of these fractions as follows:*

$$\begin{aligned} &S^{-1}R \times \Gamma \times S^{-1}M \longrightarrow S^{-1}M \\ &[r, t]\gamma[m, s] = [r\gamma m, t\gamma s], \\ &[m, s] \oplus [m', s'] = [s\gamma m' + s'\gamma m, s\gamma s']. \end{aligned}$$

These operations are well-defined.

Proof. If $[m,s] \sim [a,u]$ and $[m',s'] \sim [a',u']$ for all $m,a,m',a' \in M$ and $s,u,s',u' \in S$, then we have for some $t,t' \in S$,

$$t\alpha(s\alpha a) = t\alpha(u\alpha m), \tag{5}$$

$$t'\beta(s'\beta a') = t'\beta(u'\beta m'). \tag{6}$$

A multiplication by $t'\beta s'\beta u'\beta$ of (5) and $t\alpha s\alpha u\alpha$ of (6) gives:

$$t\alpha t'\beta s'\beta u'\beta(s\alpha a) = t\alpha t'\beta s'\beta u'\beta(u\alpha m), \tag{7}$$

$$t'\beta t\alpha s\alpha u\alpha(s'\beta a') = t'\alpha t\alpha s\alpha u\alpha(u'\alpha m'). \tag{8}$$

Sum of (7) and (8) and by commutativity of R we obtain:

$$t'\beta t\alpha s\alpha u\alpha(s'\beta a')+t\alpha t'\beta s'\beta u'\beta(s\alpha a) = t'\alpha t\alpha s\alpha u\alpha(u'\alpha m')+t\alpha t'\beta s'\beta u'\beta(u\alpha m).$$

Therefore,

$$[s\gamma m' + s'\gamma m, s\gamma s'] = [u\gamma a' + u'\gamma a, u\gamma u'], \tag{9}$$

Now, by Definition of operation $\oplus$ we have:

$$[m,s] \oplus [m',s'] = [a,u] \oplus [a',u']. \tag{10}$$

Thus the addition is well-defined.

Now, let $[r_1,t_1] = [r_2,t_2]$, $\gamma_1 = \gamma_2$ and $[m_1,s_1] = [m_2,s_2]$, by Proposition 1.5 we have:

$$r_1\gamma_1 t_2 - r_2\gamma_1 t_1 = 0, \tag{11}$$

$$u\gamma_2(s_1\gamma_2 m_2 - s_2\gamma_2 m_1) = 0. \tag{12}$$

We prove that $[r_1\gamma_1 m_1, t_1\gamma_1 s_1] = [r_2\gamma_2 m_2, t_2\gamma_2 s_2]$, or
$v\gamma((t_1\gamma_1 s_1)\gamma(r_2\gamma_2 m_2) - (t_2\gamma_2 s_2)\gamma(r_1\gamma_1 m_1)) = 0$, for some $v \in S$.

$$\begin{aligned} v\gamma((t_1\gamma_1 s_1)\gamma(r_2\gamma_2 m_2)) \quad & -v\gamma((t_2\gamma_2 s_2)\gamma(r_1\gamma_1 m_1)) = \\ & v\gamma((t_1\gamma_1 s_1)\gamma(r_2\gamma_2 m_2)) - v\gamma((t_2\gamma_2 s_2)\gamma(r_1\gamma_1 m_1)) \\ & +v\gamma(r_2\gamma_1 t_1)(m_1\gamma_2 s_2) - v\gamma(r_2\gamma_1 t_1)(m_1\gamma_2 s_2)) \\ & = (r_2\gamma_1 t_1 - r_1\gamma_1 t_2)\gamma_1(m_1\gamma_1 s) \\ & +(s_1\gamma_1 m_2 - s_2\gamma_2 m_1)\gamma_2(r_2\gamma_1 t_1) \\ & = 0 + 0 = 0 \end{aligned}$$

Therefore the multiplication is well-defined. □

Lemma 2.3. *Let M be an R_Γ-module and S be an m.c.s of R. Then $(S^{-1}M, \oplus)$ is an abelian group.*

Proof. For all $[m_1,s_1],[m_2,s_2] \in S^{-1}M$, $[m_1,s_1] \oplus [m_2,s_2] = [s_1\gamma m_2 + s_2\gamma m_1, s_1\gamma s_2]$. Since S is an *m.c.s* of R, we have $s_1\gamma s_2 \in S$ and also M is an R_Γ-module $s_1\gamma m_2 \in M, s_2\gamma m_1 \in M$. Because $(M,+)$ is a group so $s_1\gamma m_2 + s_2\gamma m_1 \in M$ for $[m_3,s_3] \in S^{-1}M$,

$$\begin{aligned} ([m_1,s_1] \oplus [m_2,s_2]) \oplus [m_3,s_3] \quad & = [(s_1\gamma s_2)\gamma m_3 + s_3\gamma(s_1\gamma m_2 + s_2\gamma m_1), (s_1\gamma s_2)\gamma s_3], \\ & = [(s_1\gamma s_2)\gamma m_3 + s_3\gamma(s_1\gamma m_2) + s_3\gamma(s_2\gamma m_1), (s_1\gamma s_2)\gamma s_3]. \end{aligned}$$

On the other hand:

$$\begin{aligned}[m_1,s_1]\oplus([m_2,s_2]\oplus[m_3,s_3]) &= [m_1,s_1]\oplus[s_2\gamma m_3+s_3\gamma m_2, s_2\gamma s_3],\\ &= [s_1\gamma(s_2\gamma m_3+s_3\gamma m_2)+(s_2\gamma s_3)\gamma m_1, s_1\gamma(s_2\gamma s_3)],\\ &= [s_1\gamma(s_2\gamma m_3)+s_1\gamma(s_3\gamma m_2)+(s_2\gamma s_3)\gamma m_1, s_1\gamma(s_2\gamma s_3)].\end{aligned}$$

By commutativity of R, $(S^{-1}M,\oplus)$ is associative. We use $\gamma_0\in\Gamma$ where $1\gamma_0 m = m$ to show that $[0,1]$ is the zero element of $(S^{-1}M,+)$ as the following:

$$\begin{aligned}&[m_1,s_1]\oplus[0,1]=[m_1,s_1],\\ &[s_1\gamma_0 0+1\gamma_0 m_1, s_1\gamma_0 1]=[m_1,s_1],\\ &[0,1]\oplus[m_1,s_1]=[1\gamma_0 m_1+s_1\gamma_0 0, 1\gamma_0 s_1]=[m_1,s_1].\end{aligned}$$

Also $[-m_1,s_1]$ is the inverse element of $[m_1,s_1]$:

$$\begin{aligned}&[m_1,s_1]\oplus[-m_1,s_1]=[0,1],\\ &[s_1\gamma(-m_1)+s_1\gamma m_1, s_1\gamma s_1]=[0,s_1]\sim[0,1].\end{aligned}$$

□

Proposition 2.4. *Let M be an R_Γ-module and S be an m.c.s of R. Then:*

(1) *For every $[m,s]\in S^{-1}M$, $t\in S$ and $\gamma\in\Gamma$, $[m,s]=[m\gamma t, s\gamma t]$,*
(2) *If $r\alpha[m,s]=[r\alpha m,s]$, then $S^{-1}M$ becomes a construction R_Γ-module.*

Proof. (1) It is straightforward.

(2) By defining the multiplication $R\times\Gamma\times S^{-1}M\longrightarrow S^{-1}M$ where $(r,\alpha,[m,s])\mapsto [r\alpha m,s]$, let $r=r',\gamma=\gamma'$ and $[m,s]=[m',s']$ so $t\alpha(s\alpha m')=t\alpha(s'\alpha m)$ for some $t\in S,\alpha\in\Gamma$. By multiplication this equality in $r'\gamma$ we have $t\alpha s\alpha(r'\gamma m')=t\alpha s'\alpha(r'\gamma m)$ since $r=r',\gamma=\gamma'$, $t\alpha s\alpha(r'\gamma' m')=t\alpha s'\alpha(r\gamma m)$, then $[r\gamma m,s]=[r'\gamma' m',s']$. We show that it is well-defined.
(M_1)

$$\begin{aligned}(r_1+r_2,\gamma,[m,s]) &= [(r_1+r_2)\gamma m,s],\\ &= [r_1\gamma m+r_2\gamma m,s].\end{aligned}$$

On the other hand,

$$\begin{aligned}[r_1\gamma m,s]\oplus[r_2\gamma m,s] &= [s\alpha(r_2\gamma m)+s\alpha(r_1\gamma m), s\alpha s].\\ &= [r_2\gamma m+r_1\gamma m,s].\end{aligned}$$

□

We consider the following assumptions on R_Γ-module M, $\forall a,b,r\in R, m\in M$ and $\alpha,\beta\in\Gamma$,

(1) $a\alpha(b\beta m)=a\beta(b\alpha m)$,
(2) $a\alpha b\gamma(r\beta m)+a\beta b\gamma(r\alpha m)=0$.

Theorem 2.5. *Let M be an R_Γ-module and S be an m.c.s of R, then $S^{-1}M$ is an $S^{-1}R_\Gamma$-module.*

Proof. Define the $S^{-1}R \times \Gamma \times S^{-1}M \longrightarrow S^{-1}M$ where $([r,s],\gamma,[m,t]) \mapsto [r\gamma m, s\gamma t]$. For $[r_1,s_1],[r_2,s_2] \in S^{-1}R$, $[m_1,t_1],[m_2,t_2] \in S^{-1}M$ and $\alpha,\beta,\gamma \in \Gamma$ we have,

$(M1)$

$$[r_1,s_1]\alpha([m_1,t_1] \oplus [m_2,t_2]) = [r_1\alpha(t_1\gamma m_2) + r_1\alpha(t_2\gamma m_1), s_1\alpha(t_1\gamma t_2)].$$

On the other hand,

$$\begin{aligned} [r_1,s_1]\alpha[m_1,t_1] \oplus [r_1,s_1]\alpha[m_2,t_2] &= [r_1\alpha m_1, s_1\alpha t_1] \oplus [r_1\alpha m_2, s_1\alpha t_2], \\ &= [s_1\alpha t_1\gamma(r_1\alpha m_2) + s_1\alpha t_2\gamma(r_1\alpha m_1), s_1\alpha t_1\gamma(s_1\alpha t_2)]. \end{aligned}$$

By using commutativity R and conditions $(*)$ and (1), the equality is valid.

(M_2)

$$\begin{aligned} ([r_1,s_1] \oplus [r_2,s_2])\alpha[m_1,t_1] &= [r_1\gamma s_2 + s_1\gamma r_2, s_1\gamma s_2]\alpha[m_1,t_1], \\ &= [(r_1\gamma s_2)\alpha m_1 + (s_1\gamma r_2)\alpha m_1, (s_1\gamma s_2)\alpha t_1]. \end{aligned}$$

On the other hand,

$$\begin{aligned} [r_1,s_1]\alpha[m_1,t_1] \oplus [r_2,s_2]\alpha[m_1,t_1] &= [r_1\alpha m_1, s_1\alpha t_1] \oplus [r_2\alpha m_1, s_2\alpha t_1], \\ &= [s_1\alpha t_1\gamma(r_2\alpha m_1) + s_2\alpha t_1\gamma(r_1\alpha m_1), s_1\alpha t_1\gamma(s_2\alpha t_1)], \\ &= [(r_1\gamma s_2)\alpha m_1 + (s_1\gamma r_2)\alpha m_1, (s_1\gamma s_2)\alpha t_1]. \end{aligned}$$

$(M3)$

$$\begin{aligned} [r_1,s_1](\alpha+\beta)[m_1,t_1] &= [r_1(\alpha+\beta)m_1, s_1(\alpha+\beta)t_1], \\ &= [r_1\alpha m_1 + r_1\beta m_1, s_1\alpha t_1 + s_1\beta t_1]. \end{aligned}$$

On the other hand,

$$\begin{aligned} [r_1,s_1]\alpha[m_1,t_1] \oplus [r_1,s_1]\beta[m_1,t_1] &= [r_1\alpha m_1, s_1\alpha t_1] \oplus [r_1\beta m_1, s_1\beta t_1], \\ &= [s_1\alpha t_1\gamma(r_1\beta m_1) + s_1\beta t_1\gamma(r_1\alpha m_1), s_1\alpha t_1\gamma(s_1\beta t_1)]. \end{aligned}$$

Now, we need to have the following equality for some $u \in S$.

$$u\gamma(s_1\alpha t_1+s_1\beta t_1)(s_1\alpha t_1\gamma(r_1\beta m_1)+s_1\beta t_1\gamma(r_1\alpha m_1)) = u\gamma s_1\alpha t_1\gamma(s_1\beta t_1)\gamma(r_1\alpha m_1+r_1\beta m_1).$$

But

$$\begin{aligned} &u\gamma s_1\alpha t_1\gamma s_1\alpha t_1\gamma(r_1\beta m_1) + u\gamma s_1\beta t_1\gamma s_1\alpha t_1\gamma(r_1\beta m_1) \\ &+u\gamma s_1\alpha t_1\gamma s_1\beta t_1\gamma(r_1\alpha m_1) + u\gamma s_1\alpha t_1\gamma s_1\beta t_1\gamma(r_1\alpha m_1) \\ &= u\gamma s_1\alpha t_1\gamma(s_1\beta t_1)\gamma r_1\alpha m_1 + u\gamma s_1\alpha t_1\gamma(s_1\beta t_1)\gamma r_1\beta m_1. \end{aligned}$$

By using the conditions $(*)$ and (ii),

$$u\gamma s_1\alpha t_1\gamma s_1\alpha t_1\gamma(r_1\beta m_1) + u\gamma s_1\alpha t_1\gamma(s_1\beta t_1)\gamma r_1\alpha m_1 = 0$$

Hence the above relation satisfies.

$(M4)$

$$\begin{aligned} [r_1,s_1]\alpha([r_2,s_2]\beta[m_1,t_1]) &= [r_1,s_1]\alpha([r_2\beta m_1, s_2\beta t_1]), \\ &= [r_1\alpha(r_2\beta m_1), s_1\alpha(s_2\beta t_1)]. \end{aligned}$$

On the other hand,

$$\begin{aligned} ([r_1,s_1]\alpha[r_2,s_2])\beta[m_1,t_1] &= [r_1\alpha r_2, s_1\alpha s_2]\beta[m_1,t_1], \\ &= [(r_1\alpha r_2)\beta m_1, (s_1\alpha s_2)\beta t_1]. \end{aligned}$$

Since M is an R_Γ-module and R is commutative, the proof is completed. □

Example 2.6. *Let $R = \mathbb{Z}_4, \Gamma = \{1,3\}$ and $S = \{1,3\} \subseteq \mathbb{Z}_4$. Then $S^{-1}R = \{[0,1],[1,1]\}$ is a Γ-ring. Let $M = \mathbb{Z}_2$ and we define $R \times \Gamma \times M \longrightarrow M$ by $(r,\gamma,m) \mapsto r\gamma m$ for all $r \in R, \gamma \in \Gamma, m \in M$. With simple calculations, we get the equivalence class of $S^{-1}M = \{[0,1],[1,1]\}$ and $(S^{-1}M,+,\Gamma)$ is a Γ-module.*

Example 2.7. *Let $R = \mathbb{Z}_6$ and $\Gamma = \{\gamma_0, \gamma_1\}$, when $\gamma_i : R \times R \to R$ defined by $x\gamma_i y = (5i)xy$, when $i = 0,1$. Put $S = \{1,3\}$. Then $S^{-1}A = \{[0,1],[1,1]\}$.*

Example 2.8. *Let $R = \mathbb{Z}_{p^n}$ and $\Gamma = \{\gamma_k | k \in U(p^n)\}$, when $U(p^n) = \{k \in R | (k,p) = 1\}$. Then R is a Γ-ring, when $\gamma_k : R \times R \to R$ define by $x\gamma_k y = kxy$, when $k \in U(p^n)$. Put $S = \{p^m | 0 \leq m \leq n-1\}$. Then $S^{-1}A = \{[0,1],[1,1],\ldots,[p-1,1]\}$.*

3. Some properties of the R_Γ-module of fractions

In this section, we propose some theorems about homomorphism and finitely generated R_Γ- modules.

Definition 3.1. [1] Let M and N be R_Γ-modules. A mapping $f : M \longrightarrow N$ is a homomorphism of R_Γ-modules if for all $x, y \in M$ and $r \in R, \gamma \in \Gamma$ we have,

(1) $f(x+y) = f(x) + f(y)$,
(2) $f(r\gamma x) = r\gamma f(x)$.

A homomorphism f is isomorphism if f is one-to-one and onto.

Definition 3.2. [1] Let M be an R_Γ-module. A nonempty subset N of M is said to be R_Γ-submodule of M if N is a subgroup of M and $R\Gamma N \subseteq N$, where $R\Gamma N = \{r\gamma n | \gamma \in \Gamma, r \in R, n \in N\}$, that is, for all $n, n' \in N$ and $\gamma \in \Gamma, r \in R$; $n - n' \in N$ and $r\gamma n \in N$.

Remark 3.3. It is easy to see that $\ker f = \{x \in M | f(x) = 0\}$ is an R_Γ-submodule of M.

Definition 3.4. [1] Let M be an R_Γ-module and $0 \neq X \subseteq M$. The generated R_Γ-submodule of M, denoted by $\langle X \rangle$, is the smallest R_Γ-submodule of M containing X, i.e., $\langle X \rangle = \cap\{N | X \subseteq N \leq M\}$, X is called the generator of $\langle X \rangle$ and $\langle X \rangle$ is finitely generated if $|X| < \infty$. If $X = \{x_1, \ldots, x_n\}$ we write $\langle x_1, \ldots, x_n \rangle$ instead $\langle \{x_1, \ldots, x_n\} \rangle$.

Lemma 3.5. *Let $f : M \longrightarrow N$ be an R_Γ-homomorphism. Then we have,*

(1) *For every R_Γ-submodule M' of M, $f(M')$ is an R_Γ-submodule on N.*
(2) *For every R_Γ-submodule N' of N, $f^{-1}(N')$ is an R_Γ-submodule on M.*

Proof. It is straightforward. □

Lemma 3.6. *Let M be an R_Γ-module. Then the mapping $\psi : M \longrightarrow S^{-1}M$ by $\psi(m) = [m,1]$ is a natural R_Γ-homomorphism. Also,*

$$\ker \psi = \{m \in M | s\gamma_0 m = 0, \ for\ some\ s \in S\}.$$

Proof. It is not difficult to see that ψ is an R_Γ-homomorphism. Moreover,

$$\begin{aligned}\ker\psi &= \{m \in M|[m,1] = [0,1]\}\\ &= \{m \in M|\exists s \in S, s\gamma_0(1\gamma_0 0) = s\gamma_0(1\gamma_0 m)\}\\ &= \{m \in M|s\gamma_0 m = 0,\ for\ some\ s \in S\}.\end{aligned}$$

□

Theorem 3.7. *Let M and N be R_Γ-modules, $f : M \longrightarrow N$ be an R_Γ-homomorphism and S be an m.c.s of Γ-ring R. Then the map $S^{-1}f : S^{-1}M \longrightarrow S^{-1}N$ where $S^{-1}f([m,s]) = [f(m),s]$ is an $S^{-1}R_\Gamma$-homomorphism.*

Proof. Suppose that $[m_1,s_1],[m_2,s_2] \in S^{-1}M$. First, we show that $S^{-1}f$ is well-defined. If $[m_1,s_1] = [m_2,s_2]$, then there exists $u \in S$ such that

$$u\gamma(s_1\gamma m_2) = u\gamma(s_2\gamma m_1)$$

which implies $f(u\gamma(s_1\gamma m_2)) = f(u\gamma(s_2\gamma m_1))$, and so $u\gamma f(s_1\gamma m_2) = u\gamma f(s_2\gamma m_1)$ or $u\gamma(s_1\gamma f(m_2)) = u\gamma(s_2\gamma f(m_1))$. Therefore, $[f(m_1),s] = [f(m_2),s]$.
Now, we prove that $S^{-1}f([m_1,s_1] \oplus [m_2,s_2]) = S^{-1}f[m_1,s_1] \oplus S^{-1}f[m_2,s_2]$.

$$\begin{aligned}S^{-1}f([m_1,s_1] \oplus [m_2,s_2]) &= S^{-1}f([s_1\gamma m_2 + s_2\gamma m_1, s_1\gamma s_2]),\\ &= [f(s_1\gamma m_2 + s_2\gamma m_1), s_1\gamma s_2],\\ &= [f(s_1\gamma m_2) + f(s_2\gamma m_1), s_1\gamma s_2],\\ &= [s_1\gamma f(m_2) + s_2\gamma f(m_1), s_1\gamma s_2],\\ &= S^{-1}f[m_1,s_1] \oplus S^{-1}f[m_2,s_2].\end{aligned}$$

For all $[r,t] \in S^{-1}R$ and $\gamma \in \Gamma$,

$$\begin{aligned}[r,t]\gamma S^{-1}f[m_1,s_1] &= [r,t]\gamma[f(m_1),s_1],\\ &= [r\gamma f(m_1), t\gamma s_1],\\ &= [f(r\gamma m_1), t\gamma s_1],\\ &= S^{-1}f([r,t]\gamma[m_1,s_1]).\end{aligned}$$

□

Theorem 3.8. *Let M, N and L be unitary R_Γ-modules and S be an m.c.s of Γ-ring R. Suppose that $f, f' : M \longrightarrow N$ and $g : N \longrightarrow L$ are R_Γ-homomorphisms. Then for all $\alpha \in \Gamma$ we have:*

(1) $S^{-1}(f + f') = S^{-1}f + S^{-1}f'$,
(2) $S^{-1}(g\alpha f) = (S^{-1}g)\alpha(S^{-1}f)$,
(3) $S^{-1}(id_M) = id_{S^{-1}M}$.

Proof. (1) For all $[m,s] \in S^{-1}M$, we have $S^{-1}(f + f')[m,s] = [(f + f')m, s] = [f(m) + f'(m), s]$.

On the other hand,

$$\begin{aligned}(S^{-1}f + S^{-1}f')[m,s] &= S^{-1}f[m,s] \oplus S^{-1}f'[m,s],\\ &= [f(m),s] \oplus [f'(m),s],\\ &= [s\alpha f'(m) + s\alpha f(m), s\alpha s],\\ &= [f'(m) + f(m), s].\end{aligned}$$

(2) $S^{-1}(g\alpha f)[m,s]=[(g\alpha f)(m),s]=[g\alpha(f(m)),s]=S^{-1}g[f(m),s]$
$=S^{-1}g\alpha(S^{-1}f[m,s])$.
(3) $S^{-1}(id_M)[m,s]=[id_M(m),s]=[m,s]=id_{S^{-1}M}[m,s]$.

□

Theorem 3.9. *Let M and N be R_Γ-modules and $f: M\longrightarrow N$ be an R_Γ-homomorphism. If S is an m.c.s of Γ-ring R, then*

$$S^{-1}\ker f=\ker S^{-1}f.$$

Proof. For all $[m,s]\in S^{-1}\ker f$, $m\in\ker f$ and also $f(m)=0$.

$$S^{-1}f[m,s]=[f(m),s]=[0,s]=[0,1].$$

It implies that $[m,s]\in\ker S^{-1}f$, that is, $S^{-1}\ker f\subseteq\ker S^{-1}f$.
On the other hand, suppose that $[a,t]\in\ker S^{-1}f$ we have:

$$\begin{aligned}
S^{-1}f[a,t]=[0,1] \quad &\Rightarrow \quad [f(a),t]=[0,1]\\
&\Rightarrow \quad u\gamma_0(t\gamma_0 0)=u\gamma_0(1\gamma_0 f(a)),\ \textit{for some } u\in S,\\
&\Rightarrow \quad 0=u\gamma_0 f(a)\\
&\Rightarrow \quad [f(a),t]=0.
\end{aligned}$$

So $[a,t]\in S^{-1}\ker f$. □

Theorem 3.10. *Let $f: A\longrightarrow B$ be a Γ-ring homomorphism. Suppose that S is an m.c.s of A and $T=f(S)$. Then $S^{-1}B$ and $T^{-1}B$ are isomorphic as $S^{-1}A$-modules.*

Proof. First, it is clear that T is an *m.c.s* of B, since $1_A\in S$, so $f(1_A)\in f(S)=T$ and $s_1\gamma s_2\in S$. Moreover, B is an A-module by $a\gamma b=f(a)\gamma b$ for every $a\in a$, $b\in B$ and $\gamma\in\Gamma$. Hence, $S^{-1}B$ is an $S^{-1}A$-module. In other hand $S^{-1}f: S^{-1}A\longrightarrow T^{-1}B$ by $S^{-1}f([a,s])=[f(a),f(s)]$ is an $S^{-1}A$-ring homomorphism and so $T^{-1}B$ is an $S^{-1}A$-module. Now, $f(s)\gamma f(s')=f(s\gamma s')\in f(S)$. We make $T^{-1}B$ into $S^{-1}A$-module by defining $[a,s]\gamma[b,f(s)]=[f(a)\gamma b,f(s)\gamma f(s')]$. Now define $\phi: S^{-1}B\longrightarrow T^{-1}B$ by $\phi[b,s]=[b,f(s)]$. We claim that ϕ is an isomorphic. First, suppose that $[b,s]=[b',s']$ in $S^{-1}B$. Then for some $s''\in S$ we have:

$$\begin{aligned}
&s''\gamma(s\gamma b')=s''\gamma(s'\gamma b),\\
&f(s'')\gamma f(s\gamma b')=f(s'')\gamma f(s'\gamma b),\\
&f(s'')\gamma(f(s)\gamma b')=f(s'')\gamma(f(s')\gamma b).
\end{aligned}$$

So that $[b,f(s)]=[b',f(s')]$ in $T^{-1}B$. Hence, ϕ is well-defined. Notice that:

$$\begin{aligned}
\phi([b,s]\oplus[b',s']) \quad &=\phi[s\gamma b'+s'\gamma b,s\gamma s'],\\
&=[f(s)\gamma b'+f(s')\gamma b,f(s\gamma s')],\\
&=[f(s)\gamma b'+f(s')\gamma b,f(s)\gamma f(s')],\\
&=[b,f(s)]\oplus[b',f(s')],\\
&=\phi[b,s]\oplus\phi[b',s'].
\end{aligned}$$

We also have the relation,

$$\begin{aligned}\phi([a,s]\gamma[b,s']) &= \phi([a\gamma b, s\gamma s']),\\ &= [f(a)\gamma b, f(s\gamma s')],\\ &= [f(a)\gamma b, f(s)\gamma f(s')],\\ &= [a,s]\gamma[b, f(s')],\\ &= [a,s]\gamma\phi[b,s'].\end{aligned}$$

So ϕ is an $S^{-1}A_\Gamma$-homomorphism. Clearly, ϕ is surjective. Now if $\phi[b,s] = \phi[b',s']$, then for some $t \in T$ we have:

$$t\gamma(f(s)\gamma b') = t\gamma(f(s')\gamma b).$$

Choose $s'' \in S$ satisfying $t = f(s'')$. Then

$$s''\gamma(s\gamma b) = s''\gamma(s'\gamma b).$$

This means that $[b,s] = [b',s']$ in $S^{-1}A$. So ϕ is injective as well. □

Proposition 3.11. *Let M be an R_Γ-module over R and S be an m.c.s of R. Suppose that I is an ideal on R for $r \in R$. Then we have:*

(1) $S^{-1}(I\gamma_0 M) = I\gamma_0 S^{-1}M$,
(2) $S^{-1}(r\gamma_0 M) = [r,1]\gamma_0 S^{-1}M$, *for each* $r \in R$.

Proof. (1) Let $[a,s] \in S^{-1}(I\gamma_0 M)$, where $a \in I\gamma_0 M$ and $s \in S$. There exist $r_1,\dots,r_n \in I$ and $m_1,\dots,m_n \in M$ such that $a = r_1\gamma_0 m_1 + \dots + r_n\gamma_0 m_n$. It implies that $[r_i,1] \in I$ and $[m_i,s] \in S^{-1}M$ for $1 \le i \le n$. We have $[a,s] = [r_1\gamma_0 m_1, s] \oplus \dots \oplus [r_n\gamma_0 m_n, s] = [r_1,1]\gamma_0[m_1,s] \oplus \dots \oplus [r_n,1]\gamma_0[m_n,s]$. So $[a,s] \in I\gamma_0 S^{-1}M$.

Conversely, let $[a,s] \in I\gamma_0 S^{-1}M$, then there exist $r_1,\dots,r_n \in I$ and $[a_1,s_1],\dots,[a_n,s_n] \in S^{-1}M$ such that, $[a,s] = [r_1,1]\gamma_0[a_1,s_1] \oplus \dots \oplus [r_n,1]\gamma_0[a_n,s_n] = [r_1\gamma_0 a_1, s_1] \oplus \dots \oplus [r_n\gamma_0 a_n, s_n]$. For all $1 \le i \le n$, $r_i \in I$ and $a_i \in M$, we have, $r_i\gamma_0 a_i \in I\gamma_0 M$ so $[r_i\gamma_0 a_i, s_i] \in S^{-1}(I\gamma_0 M)$. Hence $[a,s] \in S^{-1}(I\gamma_0 M)$.

(2) If $[a,s] \in S^{-1}(r\gamma_0 M)$, where $a \in r\gamma_0 M$ and $s \in S$, then there exists $b \in M$ such that $a = r\gamma_0 b$. Hence,

$$[a,s] = [r\gamma_0 b, s] = [r\gamma_0 b, 1\gamma_0 s] = [r,1]\gamma_0[b,s] \in [r,1]\gamma_0 S^{-1}M.$$

Conversely, let $[a,u] \in [r,1]\gamma_0 S^{-1}M$. There exists $[m,s] \in S^{-1}M$ such that $[a,u] = [r,1]\gamma_0[m,s] = [r\gamma_0 m, 1\gamma_0 s] = [r\gamma_0 m, s]$. Hence, $[a,u] = [r\gamma_0 m, s] \in S^{-1}(r\gamma_0 M)$.

□

Theorem 3.12. *Let M be a finitely generated R_Γ-module and $(0 :_R M) = \{x \in R \mid x\gamma m = 0, \forall m \in M, \gamma \in \Gamma\}$. Then*

$$S^{-1}(0 :_R M) = (0 :_{S^{-1}R} S^{-1}M).$$

Proof. For every $[x,s] \in S^{-1}(0:_R M)$, we have $x \in (0:_R M)$ and $s \in S$. So for every $m \in M$, $x\gamma m = 0$. If $[m,t] \in S^{-1}M$, then $[x,s]\gamma[m,t] = [x\gamma m, s\gamma t] = [0, s\gamma t] = [0,1]$. Hence, $[x,s] \in (0:_{S^{-1}R} S^{-1}M)$.

Conversely, $[x,s] \in (0:_{S^{-1}R} S^{-1}M)$ and $M = \langle x_1,\ldots,x_n\rangle$. Hence $[x,s]\gamma_0[m_1,1] = [0,1],\ldots,[x,s]\gamma_0[m_n,1] = [0,1]$ and $[x\gamma_0 m_1, s\gamma_0 1] = [0,1],\ldots,[x\gamma_0 m_n, s\gamma_0 1] = [0,1]$. Hence there exist $t_1 \in S, \gamma_i \in \Gamma$, $i \in \{1,2,\ldots,n\}$, such that $t_1\gamma_1(x\gamma_0 m_1) = 0,\ldots,t_n\gamma_n(x\gamma_0 m_n) = 0$. Apply $t = t_1\gamma_1\ldots\gamma_n t_n \in S$ and $t\gamma_0 x\gamma_0 m_i = 0$, $\forall i = 1,2,\ldots,n$. Hence $t\gamma_0 x \in (0:_R M)$. Therefore, $[x,s] = [t\gamma_0 x, t\gamma_0 s] \in S^{-1}(0:_R M)$. □

Proposition 3.13. *Let S be an m.c.s over R, and M be a finitely generated R_Γ-module. Then $S^{-1}M = 0$ if and only if there is a $s \in S$ such that $s\gamma M = 0$.*

Proof. If there exists $s \in S$ such that $s\gamma M = 0$, then obviously $S^{-1}M = 0$, because $[m,t] = [s\gamma m, s\gamma t] = [0,1]$, for any $[m,t] \in S^{-1}M$. Conversely, if $S^{-1}M = 0$ and $M = \langle x_1,\ldots,x_n\rangle$, then for every $[x_i,s] \in S^{-1}M$, $i = 1,\ldots,n$, we have $[x_i,s] = [0,1]$, so there exist $s_1,\ldots,s_n \in S$ such that $s_1\gamma_0 x_1 = 0,\ldots,s_n\gamma_0 x_n = 0$. Put $S = s_1\gamma_0 s_2\gamma_0\cdots\gamma_0 s_n \in S$, hence $s\gamma x_1\gamma\cdots\gamma x_n = 0$. Choice will clearly do. □

Definition 3.14. Let M be an R_Γ-module and R be a Γ-ring without zero divisor. Define the

$$T(M) \quad := \{x \in M | (0:_R x) \neq 0\}.$$

Remark 3.15. Let M be an R_Γ-module and R be a Γ-ring without zero divisor. We have:

$$T(M) \quad = \{x \in M \mid \exists a (\neq 0) \in R; a\gamma m = 0, \forall \gamma \in \Gamma\}.$$

Example 3.16. *Let $M = M_2(\mathbb{R})$ be the matrices 2×2 on $\mathbb{R}$. Put $\Gamma = \{\begin{pmatrix} 0 & 0 \\ 0 & t \end{pmatrix} \mid t \in \mathbb{N}\}$. Then M is an R_Γ-module with $R\times\Gamma\times M \longrightarrow M$ by $(r,\gamma,m) \mapsto r\gamma m$. for all $r \in R, \gamma \in \Gamma, m \in M$. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M$ and $\begin{pmatrix} 0 & 0 \\ 0 & t \end{pmatrix} \in \Gamma$. For every $0 \neq r \in R$ we have*

$$r\begin{pmatrix} 0 & 0 \\ 0 & t \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & rtc \\ 0 & rtd \end{pmatrix}.$$

Therefor $T(M) = \{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a,b \in \mathbb{R}\}$.

Theorem 3.17. *Let M be an R_Γ-module, R be a Γ-ring without zero divisor and S be an m.c.s of R. Then we prove:*

(1) *$T(M)$ is an R_Γ-submodule of M.*
(2) *$T(S^{-1}M) = S^{-1}(T(M))$.*

Proof. (1) Suppose that $x, y \in T(M)$. Then there exist $a, a'(\neq 0) \in R$ satisfy $a\gamma x = 0$, $a'\gamma y = 0$ for every $\gamma \in \Gamma$. Then $a\gamma a'\gamma(x+y) = 0$ and $a\gamma a' \neq 0$ since R is without zero divisor, so $x + y \in T(M)$. For $(0 \neq a'') \in R$ and $x \in T(M)$, we have $a'\gamma x = 0$, for some $a' \in R$ and every $\gamma \in \Gamma, (a'\gamma' a'')\gamma x = a''\gamma'(a'\gamma x) = 0$, since $(a', a'' \neq 0) \in R$, we obtain $a'\gamma'(a''\gamma x) = 0$ and $a''\gamma x \in T(M)$. Therefore, $T(M)$ is an R_Γ-submodule of M.

(2) If $0 \in S$, then $S^{-1}M = S^{-1}(T(M)) = 0$. Now suppose that $0 \neq S$, $[m, s] \in T(S^{-1}M)$, so there is $[a, s'] \neq [0, 1]$ in $S^{-1}R$,

$$[0, 1] = [a, s']\gamma[m, s] = [a\gamma m, s'\gamma s].$$

There exist $s'' \in S$ and $\gamma_0 \in \Gamma$, for which $s''\gamma_0 a\gamma m = 0$. Now $s''\gamma_0 a \neq 0$ since $s'', a \neq 0$ and R is without zero divisor, So $m \in T(M)$, and hence $[m, s] \in S^{-1}(T(M))$ and $T(S^{-1}M) \subseteq S^{-1}(T(M))$.

On the other hand, if $m \in T(M)$, then there is $a(\neq 0) \in R$ for which $a\gamma m = 0$. It is claimed that $[a, 1] \neq [0, 1]$, because if $[a, 1] = [0, 1]$, then $s\gamma_0 1 = 0\gamma_0 1$ and $a = 0$, that is, construction by R is non-zero devisor. Now $[a, 1]\gamma[m, s] = [0, 1]$ for any $s \in S$, we see that $[m, s] \in T(S^{-1}M)$. Then $S^{-1}(T(M)) \subseteq T(S^{-1}M)$.

□

Theorem 3.18. *Let M be an R_Γ-module and S be an m.c.s. Then there exists one to one corresponding between the set of R_Γ-submodules of M and $S^{-1}R_\Gamma$-submodules of $S^{-1}M$.*

Proof. Let N be an R_Γ-submodule of M and $S^{-1}N = \{[a, s] \mid a \in N, s \in S\}$ be a subset of $S^{-1}M$. We prove that $S^{-1}N$ is a Γ-submodule of $S^{-1}M$. Let $[a, s], [b, t] \in S^{-1}N$, where $a, b \in N$ and $s, t \in S$. So $[a, s] + [b, t] = [s\gamma b + t\gamma a, s\gamma t] \in S^{-1}N$, since N is a Γ-submodule and S is an *m.c.s* of R. Similarly, for all $[r, u] \in S^{-1}R$, we have, $[r, u]\gamma[a, s] = [r\gamma a, t\gamma u] \in S^{-1}N$. Hence, $S^{-1}N$ is a Γ-submodule of $S^{-1}M$.

Conversely, suppose that W is a Γ-submodule of $S^{-1}M$. Consider the natural homomorphism $f : M \longrightarrow S^{-1}M$ and $f^{-1}(W) = \{a \in M \mid [a, 1] \in W\}$. By Lemma 3.5, $f^{-1}(W)$ is a Γ-submodule of M. Therefore, we obtain two mappings ϕ and ψ, where $\phi : \{\Gamma - \textit{submodule of } M\} \longrightarrow \{\Gamma - \textit{submodule of } S^{-1}M\}$, with $\phi(N) \mapsto S^{-1}N$. Also, define the map $\psi : \{\Gamma - \textit{submodule of } S^{-1}M\} \longrightarrow \{\Gamma - \textit{submodule of } M\}$, which $W \mapsto f^{-1}(W)$. Clearly, for every Γ-submodule of $S^{-1}M$, we have $\phi(\psi(W)) = W$. Set $N = \psi(W)$, that is, $N = \{a \in M \mid [a, 1] \in W\}$. Since $\phi(N) = S^{-1}N$ it is enough to prove that $S^{-1}N = W$. First, we suppose that $[a, t] \in S^{-1}N$ where $a \in N$ and $s \in S$. Since $[1, s] \in S^{-1}R$, it follows that $[a, s] = [1, s]\gamma_0[a, 1] \in W$, hence $S^{-1}N \subseteq W$. On the other hand, let $[a, s] \in W$, where $a \in M$ and $s \in S$. Since, $[s, 1] \in S^{-1}R$ and W is a Γ-submodule of $S^{-1}M$, we get $[a, 1] = [1, 1]\gamma_0[a, s] \in W$, which implies that $a \in N$ and $[a, s] \in S^{-1}N$ and so $W \subseteq S^{-1}N$. □

Corollary 3.19. *Let M be an R_Γ-module and S be an m.c.s on R. Then every $S^{-1}R_\Gamma$-submodule of $S^{-1}M$ is formed of $S^{-1}N$ such that N is an R_Γ-submodule of M.*

Corollary 3.20. *Let M be an R_Γ-module with identity, S be an m.c.s of R and X be the set of generators for R_Γ-submodule N. If $f : M \longrightarrow S^{-1}M$ is a natural homomorphism, then $f(X)$ is the set of generators for $S^{-1}R_\Gamma$-submodule $S^{-1}N$.*

Proof. Clearly, $f(X) \subseteq S^{-1}N$. Conversely, for $\lambda \in S^{-1}N$, we have $\lambda \in [r_1\gamma_0 a_1 + \cdots + r_n\gamma_0 a_n, s]$, where $r_1, \ldots, r_n \in R$ and $a_1, \ldots, a_n \in X, s \in S$.

On the other hand we can write

$$\lambda = [r_1, s]\gamma_0[a_1, 1] + [r_2, s]\gamma_0[a_2, 1] + \cdots + [r_n, s]\gamma_0[a_n, 1].$$

Since $[a_i, 1] \in f(X)$ for every $1 \leq i \leq n$, we obtain $S^{-1}N \subseteq (f(X))$. □

Theorem 3.21. *Let M be an R_Γ-module S be an m.c.s of R. Suppose that N and P are R_Γ-submodules. Then we have:*

(1) $S^{-1}(N + P) = S^{-1}N + S^{-1}P$,
(2) $S^{-1}(N \cap P) = S^{-1}N \cap S^{-1}P$.

Proof. (1) Clearly, $S^{-1}N, S^{-1}P \subseteq S^{-1}(N + P)$, so $S^{-1}N + S^{-1}P \subseteq S^{-1}(N + P)$, since $S^{-1}(N + P)$ is a sub-Γ-module of $S^{-1}M$. Also,

$$\begin{aligned} S^{-1}(N + P) &= \{[x + y, s] | x \in N, y \in P, s \in S\} \\ &= \{[x, s] \oplus [y, s] | x \in N, y \in P, s \in S\} \\ &\subseteq S^{-1}N + S^{-1}P. \end{aligned}$$

(2) Clearly, $N \cap P \subseteq N$ and $N \cap P \subseteq P$, so $S^{-1}(N \cap P) \subseteq S^{-1}N$ and $S^{-1}(N \cap P) \subseteq S^{-1}P$. Hence, $S^{-1}(N \cap P) \subseteq S^{-1}N \cap S^{-1}P$.
Suppose $\alpha \in (S^{-1}N) \cap (S^{-1}P)$, so $\alpha = [x, s] = [y, t]$ for some $x \in N, y \in P, s, t \in S$. Then $u\gamma(s\gamma y) = u\gamma(t\gamma x)$ for some $u \in S$. So $u\gamma(s\gamma y) = u\gamma(t\gamma x) \in N \cap P$. Hence, $[x, s] = [u\gamma(t\gamma x), u\gamma(t\gamma s)] \in S^{-1}(N \cap P)$. Thus $S^{-1}N \cap S^{-1}P \subseteq S^{-1}(N \cap P)$, whence equality holds.

□

Theorem 3.22. *Let M be a finitely generated R_Γ-module with identity and S be an m.c.s of R. Then $S^{-1}M$ is a finitely generated with identity.*

Proof. For all $a \in M$ and $s \in S$ we have $[a, s] = [1, 1]\gamma_0[a, s]$, where $[1, 1]$ is an identity element in $S^{-1}R$. So $S^{-1}M$ has an identity element. By hypothesis there exist $a_1, \ldots, a_n \in M$ such that $M = \langle a_1, \ldots, a_n \rangle = R\alpha a_1 + \cdots + R\alpha a_n$, by Theorem 3.21, $S^{-1}M = S^{-1}(R\alpha a_1) + \cdots + S^{-1}(R\alpha a_n)$, Hence $S^{-1}M$ is finitely generated. □

4. **Conclusion**

The current paper has defined and considered the notion of fraction of Γ-module over commutative Γ-ring. We investigated some theorems of homomorphism of fraction Γ-module and proved that if M is a R_Γ-module then $S^{-1}M$ is a Γ-module over Γ-ring $S^{-1}R$. Moreover, if M is a finitely generated R_Γ-module, then $S^{-1}M$ is finitely generated.

In our future studies, we hope to obtain more results regarding the fraction Γ-module over ring and their applications in other research. Fraction rings and fraction modules have various applications in algebraic geometry, number theory, cryptography, coding theory and control theory. Therefore, examining the fraction Γ-modules and fraction Γ-rings , which are a kind of extension of the fraction modules and fraction rings, seems like a good idea in these areas. Moreover, it would be interesting to continue this article in the areas of commutative and non-commutative Γ-algebra, such as the exact sequences, tensor products and etc.

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Leila Amjadi
Orcid number: 0009-0002-5246-992X
Department of Mathematical Sciences
Yazd University
Yazd, Iran
Email address: s.amjadi62@yahoo.com

Mansour Ghadiri
Orcid number: 0000-0002-9033-4252
Department of Mathematical Sciences
Yazd University
Yazd, Iran
Email address: mghadiri@yazd.ac.ir

Saeed Mirvakili
Orcid number: 0000-0002-9474-0709
Department of Mathematics
Payame Noor University,
Tehran, Iran
Email address: saeed_mirvakili@pnu.ac.ir