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ON BICONSERVATIVE RIEMANNIAN HYPERSURFACES OF LORENTZIAN 4-SPACE FORMS

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Abstract. In this manuscript, we consider an extended version of biconservativity condition (namely, C-biconservativity) on the Riemannian hypersurfaces of Lorentzian standard 4-space forms. This new condition is obtained by substituting the Cheng-Yau operator C instead of the Laplace operator Δ. We show that every C-biconservative Riemannian hypersurface of a Lorentzian 4-space form with constant mean curvature has constant scalar curvature.

Keywords: Cheng-Yau operator , C-biconservative, scalar curvature.
2020 MSC: 53C40, 53C42, 58G25.

1. Introduction

In the 1920s, David Hilbert showed that the stress-energy tensor associated to a given functional θ is a conservative symmetric bi-covariant tensor Θ (at the critical points of θ) such that θ is biconservative if and only if $\mathrm{div}\Theta = 0$. This fact can be considered as the starting point of the study of biconservative submanifolds. Precisely, this subject has been started by Eells and Sampson and followed by Jiang ([8,14]). The condition $\mathrm{div}\Theta_2 = 0$ on the stress bienergy tensor Θ_2 has been introduced by Jiang.

Regardless of the historical motivation to study and develop the theory of biconservative submanifolds, one can point out the important role of minimal surfaces in physics and differential geometry. There is a very close connection between minimal and biharmonic surfaces. In 1991, Bang-Yen Chen has conjectured that there are no proper biharmonic submanifolds in Euclidian spaces ([5]). It means that every biharmonic submanifolds in an Euclidian space has to be minimal. In this context, many examples and classification results have been provided in Euclidian spaces. In general the study of biharmonic maps between Riemannian manifolds is one of the interesting research topics in differential geometry. From the theory of biharmonic submanifolds, a new interesting subject in mathematical physics is the theory of biconservative submanifolds, which arose and keeps gaining ground in today s mathematical

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research as there are many interesting examples of biconservative submanifolds, even when the biharmonic ones fail to exist.

A hypersurface of an Euclidean space defined by an isometric immersion $\mathbf{x} : M^n \to \mathbb{E}^{n+1}$ is said to be biharmonic if it satisfies the equation $\Delta^2\mathbf{x} = 0$, where Δ is the Laplace operator on M^ns. Also, M^n is called biconservative if $\mathbf{x}$ satisfies the equation $(\Delta^2\mathbf{x})^\top = 0$, where $\top$ stands for the tangent component of vectors.

From the physical points of view, we deal with the bienergy functional and its critical points arisen form the tension field. In geometric context, the subject of biconservative submanifolds has received much attentions. In 1995, Hasanis and Vlachos have classified the biconservative hypersurfaces (namely, H-hypersurfaces) of 3 and 4 dimensional Euclidean spaces ([13]). The terminology "biconservative" has been introduced (firstly) in [4]. In 2015, Turgay has studied H-hypersurfaces with 3 distinct principal curvatures in the Euclidean spaces ([25]). Biconservative surfaces of constant mean curvature in $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$ has been studied in [10]. In 2019, Gupta studied the biconservative hypersurfaces in Euclidean 5-space ([12]). Also, the biconservative hypersurfaces in Riemannian 4-space forms have been classified by Turgay and Upadhyay in 2019 ([26]).

As known, the Laplace operator Δ of a hypersurface M^n in the space form $\mathbb{M}_1^4(c)$ arises as the linearized operator of the first variation of mean curvature vector field associated to the normal variations of M^n. From extension point of view, Δ is the first one of a sequence of n operators, $L_0 = \Delta$, $L_1 = \mathrm{C}, \ldots, L_{n-1}$, where L_k is the kth linearized operator of the first variation of the $(k+1)$th mean curvature arisen from the normal variations of M^n. The operator C (sometimes denoted by symbol $\Box$) was introduced in [6]. Based on this background, many researchers ([1–3, 19, 20]) have considered hypersurfaces in space forms whose position vector field satisfies the general condition $L_k x = Ax + b$, for a fixed integer $0 \leq k < n$, where A is a constant matrix and b is a constant vector.

In this manuscript, we study the C-biconservativity condition on some hypersurfaces of Lorentzian space forms. A Riemannian hypersurface of a Lorentzian space form defined by an isometric immersion $\mathbf{x} : M^n \to \mathbb{M}_1^{n+1}(c)$ is said to be biconservative if the tangent component of $\Delta^2\mathbf{x}$ is identically zero. Inspired by this concept, we have introduced and used the C-biconservativity condition as $(\mathrm{C}^2\mathbf{x})^\top = 0$ ([21–23]). The C-biconservativity condition is obtained by substituting the Cheng-Yau operator C instead of Δ. In mentioned papers, we have studied C-biconservative (Riemannian or Lorentzian) hypersurfaces of some Minkowski spaces. In this paper, we study some C-biconservative Riemannian hypersurfaces of non-flat 4-dimensional Lorentzian space forms. It is proven that a Riemannian hypersurface M^3 of $\mathbb{M}_1^4(c)$ is C-biconservative if its first and second mean curvatures (i.e. H_1 and H_2) satisfy the condition

$$\mathrm{N}_2(\nabla H_2) - c\mathrm{N}_1(\nabla H_1) = \frac{9}{2} H_2 \nabla H_2, \tag{1}$$

where N_1 and N_2 are the first and second Newton transformations. We show that the C-biconservative hypersurfaces of $\mathbb{M}_1^4(c)$ with constant ordinary mean curvature have constant scalar curvature.

2. **Preliminaries**

We recall some notations and formulae from [2, 3, 15, 17, 28]. The semi-Euclidean 5-space $\mathbb{E}_\xi^5$ of index $\xi = 1, 2$ is equipped with the product defined by $\langle \mathbf{v}, \mathbf{w} \rangle = -\sum_{i=1}^{\xi} v_i w_i + \sum_{i=\xi+1}^{5} v_i w_i$, for each vectors $\mathbf{v} = (v_1, \ldots, v_5)$ and $\mathbf{w} = (w_1, \ldots, w_5)$ in $\mathbb{R}^5$. The 4-dimensional Lorentzian space forms are defined as:

$$\mathbb{M}_1^4(c) = \begin{cases} \mathbb{S}_1^4(r) & (\text{if } c = 1/r^2) \\ \mathbb{L}^4 = \mathbb{E}_1^4 & (\text{if } c = 0) \\ \mathbb{H}_1^4(-r) & (\text{if } c = -1/r^2), \end{cases}$$

where, for $r > 0$, $\mathbb{S}_1^4(r) = \{\mathbf{v} \in \mathbb{E}_1^5 | \langle \mathbf{v}, \mathbf{v} \rangle = r^2\}$ denotes the 4-pseudosphere of radius r and curvature $1/r^2$ and $\mathbb{H}_1^4(-r) = \{\mathbf{v} \in \mathbb{E}_2^5 | \langle \mathbf{v}, \mathbf{v} \rangle = -r^2, v_1 > 0\}$ denotes the pseudo-hyperbolic 4-space of radius $-r$ and curvature $-1/r^2$. The canonical cases $c = \pm 1$ give the de Sitter 4-space $d\mathbb{S}^4 := \mathbb{S}_1^4(1)$ and anti de Sitter 4-space $Ad\mathbb{S}^4 = \mathbb{H}_1^4(-1)$. Also, the case $c = 0$ gives the Lorentz-Minkowski 4-space $\mathbb{L}^4 := \mathbb{E}_1^4$.

In this paper, we study some Riemannian hypersurfaces of $\mathbb{M}_1^4(c)$ for $c = 0, \pm 1$ (i.e. $\mathbb{L}^4$, $d\mathbb{S}^4$, $Ad\mathbb{S}^4$). Let $\mathbf{x} : M^3 \to \mathbb{M}_1^4(c)$ be a Riemannian hypersurface isometrically immersed into $\mathbb{M}_1^4(c)$. As usual, $\chi(M^3)$ denotes the set of smooth tangent vector fields on M^3. The Levi-Civita connections on M^3 and $\mathbb{M}_1^4(c)$ are denoted by ∇ and $\bar{\nabla}$, respectively. Also, ∇^0 denotes the Levi-Civita connection on $\mathbb{L}^5 = \mathbb{E}_1^5$ and $\mathbb{E}_2^5$. The Weingarten formula on M^3 is $\bar{\nabla}_V W = \nabla_V W - \langle SV, W \rangle \mathbf{n}$, for each $V, W \in \chi(M^3)$, where S is the shape operator of M^3 associated to a unit normal timelike vector field $\mathbf{n}$ on M^3. Furthermore, in the case $|c| = 1$, $\mathbb{M}^4(c)$ is a 4-hyperquadric with the unit normal vector field $\mathbf{x}$ and the Gauss formula $\nabla_V^0 W = \bar{\nabla}_V W - c\langle V, W \rangle \mathbf{x}$.

The shape operator of M^3 can be assumed diagonal because it is spacelike (see [17, 18]). Denoting the eigenvalues of S (i.e. the principal curvatures of M) by $\kappa_1, \kappa_2, \kappa_3$ on M, the jth elementary symmetric function is defined as

$$s_j := \sum_{1 \le i_1 < \ldots < i_j \le n} \kappa_{i_1} \ldots \kappa_{i_j},$$

and the jth mean curvature of M as $\binom{3}{j} H_j = (-1)^j s_j$ (for instance, see [2] and [3]). In special case, the second mean curvature $H_2 = \frac{1}{3} s_2$ and the normalized scalar curvature R satisfy the equality $H_2 := n(n-1)(1-R)$. The hypersurface M^3 is said to be *j-maximal* if $H_{j+1} \equiv 0$.

The Newton maps on M^3 are defined (inductively) as follow:

$$\mathrm{N}_0 = I, \ \mathrm{N}_1 = -s_1 I + S, \ \mathrm{N}_2 = s_2 I - s_1 S + S^2, \tag{2}$$

where I is the identity map. Using the spacelike unit vectors e_1, e_2 and e_3 as the eigenvectors of S with eigenvalues κ_1, κ_2 and κ_3 (respectively), the eigenvalues of N_j are given by $\mathrm{N}_j e_i = \mu_{i,j} e_i$, where $\mu_{1,1} = -\kappa_2 - \kappa_3$, $\mu_{2,1} = -\kappa_1 - \kappa_3$, $\mu_{3,1} = -\kappa_1 - \kappa_2$, $\mu_{1,2} = \kappa_2\kappa_3$, $\mu_{2,2} = \kappa_1\kappa_3$, $\mu_{3,2} = \kappa_1\kappa_2$.

Here are some useful formulae about the Newton maps:

$$\begin{aligned} &tr(\mathrm{N}_j) = c_j H_j, \; tr(S \circ \mathrm{N}_j) = -c_j H_{j+1}, \\ &tr(S^2 \circ \mathrm{N}_1) = 9H_1H_2 - 3H_3, \; tr(S^2 \circ \mathrm{N}_2) = 3H_1H_3, \end{aligned} \tag{3}$$

where $j = 0, 1, 2$, $c_0 = c_2 = 3$ and $c_1 = 6$.

The Chang-Yau operator $\mathrm{C} : \mathcal{C}^\infty(M^3) \to \mathcal{C}^\infty(M^3)$ is introduced by rule $\mathrm{C}(f) = tr(\mathrm{N}_1 \circ \nabla^2 f)$, where, $\nabla^2 f : \chi(M) \to \chi(M)$ is (equivalently) the Hessian of f by rule $\langle \nabla^2 f(X), Y\rangle = \langle \nabla_X(\nabla f), Y\rangle$ for each $X, Y \in \chi(M^3)$.

In other words, $\mathrm{C}(f)$ is given by $\mathrm{C}(f) = \sum_{i=1}^3 \mu_{i,1}(e_ie_if - \nabla_{e_i}e_if)$. The key formulae in this paper are

$$\mathrm{C}\mathbf{x} = 6(H_2\mathbf{n} + cH_1\mathbf{x}),$$

$$\mathrm{C}\mathbf{n} = 3\nabla H_2(9H_1H_2 - 3H_3)\mathbf{n} - 6cH_2\mathbf{x}$$

and

$$\begin{aligned} \mathrm{C}^2\mathbf{x} &= -54H_2\nabla H_2 + 12\mathrm{N}_2\nabla H_2 - 12c\mathrm{N}_1\nabla H_1 \\ &+ 6\left(\mathrm{C}(H_2) - 9H_1H_2^2 + 3H_2H_3 - 6cH_1H_2\right)\mathbf{n} \\ &- 6c\left(\mathrm{C}(H_1) - 6H_2^2 - 6cH_1^2\right)\mathbf{x}. \end{aligned} \tag{4}$$

By definition, M^3 is called *C-biconservative* if $\mathbf{x}$ satisfies $(\mathrm{C}^2\mathbf{x})^\top = 0$ (i.e the condition (1)).

According to (local) orthonormal tangent frame $\{e_m\}_{1\le m\le 4}$ and associated co-frame $\{\omega_m\}_{1\le m\le 4}$ on $\mathbb{M}_1^4(c)$, where e_1, e_2, e_3 are tangent to M^3 and e_4 is positively normal to M^3, the structure equations of $\mathbb{M}_1^4(c)$ are

$$d\omega_A = \sum_{B=1}^4 \omega_{AB} \wedge \omega_B, \; \omega_{AB} + \omega_{BA} = 0, \; d\omega_{AB} = \sum_{C=1}^4 \omega_{AC} \wedge \omega_{CB}.$$

Of course, clearly $\omega_4 = 0$ and $0 = d\omega_4 = \sum\limits_{i=1}^3 \omega_{4i} \wedge \omega_i$ on M^3.

Using the well-known Cartan's Lemma, there are smooth functions h_{ij} such that $h_{ij} = h_{ji}$ and

$$\omega_{4i} = \sum_{j=1}^3 h_{ij}\omega_j. \tag{5}$$

Since the second fundamental form of M^3 is $B = \sum\limits_{i,j=1}^4 h_{ij}\omega_i\omega_j e_4$, the mean curvature H has the simple form $H = \frac{1}{3}\sum\limits_{i=1}^3 h_{ii}$. Hence, equation (5) gives the structure equations as follow (see [28]).

$$d\omega_i = \sum_{j=1}^{3} \omega_{ij} \wedge \omega_j, \ \omega_{ij} + \omega_{ji} = 0,$$

$$d\omega_{ij} = \sum_{k=1}^{3} \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l=1}^{3} R_{ijkl} \omega_k \wedge \omega_l.$$

Also, the Gauss equation on M^3 is $R_{ijkl} = (h_{ik}h_{jl} - h_{il}h_{jk})$, where R_{ijkl} stand for the components of the tensor of Riemannian curvature on M^3. Finally, we have

$$\sum_k h_{ijk}\omega_k = dh_{ij} + \sum_k h_{kj}\omega_{ki} + \sum_k h_{ik}\omega_{kj}, \tag{6}$$

where h_{ijk} is the covariant derivative of h_{ij}. Thus, by exterior differentiation of (5), the Codazzi equation is obtained as $h_{ijk} = h_{ikj}$. One can choose e_1, e_2, e_3 such that $h_{ij} = \kappa_i \delta_{ij}$. On the other hand, the Levi-Civita connection of M^3 satisfies

$$\nabla_{e_i} e_j = \sum_k \omega_{jk}(e_i) e_k,$$

and therefore

$$\begin{aligned} &e_i(k_j) = \omega_{ij}(e_j)(\kappa_i - \kappa_j), \\ &\omega_{ij}(e_l)(\kappa_i - \kappa_j) = \omega_{il}(e_j)(\kappa_i - \kappa_l), \end{aligned} \tag{7}$$

whenever i, j, l are distinct.

3. Examples

In this section we see several examples of C-biconservative spacelike hypersurfaces in $d\mathbb{S}^4$, $\mathbb{L}^4$ and $Ad\mathbb{S}^4$ with constant ordinary mean curvature. First, we have some Riemannian product hypersurfaces (see [3,19,29]). In Examples 3.1 and 3.9 we follow Example 5.3 in [19]. The idea of examples 3.2-3.4 is taken from Example 5.6 in [19]. Examples 3.6 and 3.7 are provided by the author. The idea of examples 3.5, 3.8 and 3.11 are taken from Examples 5.7, 5.2 and 5.5 in [19], respectively.

Example 3.1. Let $r \geq 1$ and $\Gamma_0 = \mathbb{S}^3(r) \subset d\mathbb{S}^4$ defined as

$$\Gamma_0 = \{(y_1, y_2, y_3, y_4, y_5) \in \mathbb{L}^5 | y_2^2 + y_3^2 + y_4^2 + y_5^2 = r^2, y_1 = \pm\sqrt{r^2 - 1}\},$$

with the Gauss map $\mathbf{n}(y) = \frac{-\sqrt{r^2-1}}{r}(0, y_2, y_3, y_4, y_5) + \frac{-r}{\sqrt{r^2-1}}(y_1, 0, 0, 0, 0)$ and only one principal curvature of multiplicity 3 as $\kappa_1 = \kappa_2 = \kappa_3 = \frac{\sqrt{r^2-1}}{r}$. One can see that Γ_0 is C-biconservative and its 1st and 2nd mean curvatures are constants.

Example 3.2. Let $r > 1$ and $\Gamma_1 = \mathbb{H}^1(-\sqrt{r^2-1}) \times \mathbb{S}^2(r) \subset d\mathbb{S}^4$ defined by

$$\Gamma_1 = \{(y_1, y_2, y_3, y_4, y_5) \in \mathbb{L}^5 | -y_1^2 + y_2^2 = 1 - r^2, y_3^2 + y_4^2 + y_5^2 = r^2\},$$

whose Gauss map is $\mathbf{n}(y) = \frac{-r}{\sqrt{r^2-1}}(y_1, y_2, 0, 0, 0) + \frac{-\sqrt{r^2-1}}{r}(0, 0, y_3, y_4, y_5)$. It is C-biconservative because it has constant principal curvatures $\kappa_1 = \frac{r}{\sqrt{r^2-1}}$ and $\kappa_2 = \kappa_3 = \frac{\sqrt{r^2-1}}{r}$. Clearly, Γ_1 is C-biconservative and its 1st and 2nd mean curvatures are constants.

Example 3.3. Let $r > 1$ and $\Gamma_2 = \mathbb{H}^2(-\sqrt{r^2-1}) \times \mathbb{S}^1(r) \subset d\mathbb{S}^4$ defined by

$$\Gamma_2 = \{(y_1, y_2, y_3, y_4, y_5) \in \mathbb{L}^5 | -y_1^2 + y_2^2 + y_3^2 = 1 - r^2, y_4^2 + y_5^2 = r^2\},$$

which has the Gauss vector $\mathbf{n}(y) = \frac{-r}{\sqrt{r^2-1}}(y_1, y_2, y_3, 0, 0) + \frac{-\sqrt{r^2-1}}{r}(0, 0, 0, y_4, y_5)$, constant principal curvatures $\kappa_1 = \kappa_2 = \frac{r}{\sqrt{r^2-1}}$ and $\kappa_3 = \frac{\sqrt{r^2-1}}{r}$, and constant higher order mean curvatures. So, Γ_2 is C-biconservative.

Example 3.4. Let $r > 1$ and $\Gamma_3 = \mathbb{H}^3(-\sqrt{r^2-1}) \subset d\mathbb{S}^4$ defined as

$$\Gamma_3 = \{(y_1, y_2, y_3, y_4, y_5) \in \mathbb{L}^5 | -y_1^2 + y_2^2 + y_3^2 + y_4^2 = 1 - r^2, y_5 = \pm r\},$$

which has the Gauss vector $\mathbf{n}(y) = \frac{-r}{\sqrt{r^2-1}}(y_1, y_2, y_3, y_4, 0) + \frac{-\sqrt{r^2-1}}{r}(0, 0, 0, 0, y_5)$, constant principal curvatures $\kappa_1 = \kappa_2 = \kappa_3 = \frac{r}{\sqrt{r^2-1}}$, and constant higher order mean curvatures. So, Γ_3 is C-biconservative.

Example 3.5. Let $0 < r < 1$ and $\Gamma_4 = \mathbb{H}^1(-\sqrt{1-r^2}) \times \mathbb{H}^2(-r) \subset Ad\mathbb{S}^4$ defined by

$$\Gamma_4 = \{(y_1, y_2, y_3, y_4, y_5) \in \mathbb{E}_2^5 | -y_1^2 + y_3^2 = r^2 - 1, -y_2^2 + y_4^2 + y_5^2 = -r^2\},$$

with the Gauss map $\mathbf{n}(y) = \frac{-r}{\sqrt{1-r^2}}(y_1, 0, y_3, 0, 0,) + \frac{\sqrt{1-r^2}}{r}(0, y_2, 0, y_4, y_5)$. It has two distinct constant principal curvatures $\kappa_1 = \frac{r}{\sqrt{1-r^2}}$ and $\kappa_2 = \kappa_3 = \frac{-\sqrt{1-r^2}}{r}$, and constant higher order mean curvatures. So, Γ_4 is C-biconservative.

Example 3.6. Let $0 < r < 1$ and $\Gamma_5 = \mathbb{H}^3(-r) \subset Ad\mathbb{S}^4$ defined by

$$\Gamma_5 = \{(y_1, y_2, y_3, y_4, y_5) \in \mathbb{E}_2^5 | -y_2^2 + y_3^2 + y_4^2 + y_5^2 = -r^2, y_1 = \pm\sqrt{1-r^2}\},$$

with the Gauss map $\mathbf{n}(y) = \frac{-r}{\sqrt{1-r^2}}(y_1, 0, 0, 0, 0) + \frac{\sqrt{1-r^2}}{r}(0, y_2, y_3, y_4, y_5)$, only one constant principal curvature of multiplicity three as $\kappa_1 = \kappa_2 = \kappa_3 = \frac{-\sqrt{1-r^2}}{r}$, and the constant higher order mean curvatures. So, Γ_5 is C-biconservative.

Example 3.7. Let $r > 0$ and $\Gamma_6 = \mathbb{S}^1(\sqrt{1+r^2}) \times \mathbb{S}^2(r) \subset Ad\mathbb{S}^4$ defined by

$$\Gamma_6 = \{(y_1, y_2, y_3, y_4, y_5) \in \mathbb{E}_2^5 | y_1^2 + y_2^2 = 1 + r^2, y_3^2 + y_4^2 + y_5^2 = r^2\},$$

which has the Gauss vector $\mathbf{n}(y) = \frac{-r}{\sqrt{1+r^2}}(y_1, y_2, 0, 0, 0) + \frac{\sqrt{1+r^2}}{r}(0, 0, y_3, y_4, y_5)$, constant principal curvatures $\kappa_1 = \frac{r}{\sqrt{1+r^2}}$ and $\kappa_2 = \kappa_3 = \frac{-\sqrt{1+r^2}}{r}$, and constant higher order mean curvatures. So, Γ_6 is C-biconservative.

Example 3.8. Let $r>0$ and $\Lambda_1 := \mathbb{H}^1(-r)\times\mathbb{E}^2 \subset \mathbb{L}^4$ defined by

$$\Lambda_1 := \{(y_1,y_2,y_3,y_4)\in\mathbb{L}^4 | -y_1^2+y_2^2=-r^2\},$$

which has the Gauss vector $N(y)=\frac{-1}{r}(-y_1,y_2,0,0)$ on M. It has constant principal curvatures $\kappa_1=\frac{1}{r}$ and $\kappa_2=\kappa_3=0$. Λ_1 is C-biconservative.

In similar manner, we may define $\Lambda_2 := \mathbb{H}^2(-r)\times\mathbb{E}^1 \subset \mathbb{L}^4$ and $\Lambda_3 := \mathbb{H}^3(-r)\subset\mathbb{L}^4$ as C-biconservative spacelike hypersurfaces with constant curvatures.

Example 3.9. For a chosen unit vector $\mathbf{a}\in\mathbb{L}^5$ and a real number $r>\sqrt{|\tau|}$ where $\tau=\langle\mathbf{a},\mathbf{a}\rangle$, the subset $\Upsilon^{\mathbf{a}} := \{\mathbf{y}\in d\mathbb{S}^4\subset\mathbb{L}^5 | \langle\mathbf{y},\mathbf{a}\rangle=\sqrt{r^2+\tau}\}$ is a totally umbilic hypersurface in $d\mathbb{S}^4$. Similar to Example 3.2 in [3], the Gauss map is $\mathbf{n}(\mathbf{y})=\frac{1}{r}(\mathbf{a}-\sqrt{r^2+\tau}\mathbf{y})$, so for all i, $\kappa_i=\frac{1}{r}\sqrt{r^2+\tau}$, and for each k, $H_k=(-1)^k[\frac{1}{r}\sqrt{r^2+\tau}]^k$. When $\tau=-1$, we get $\Upsilon^{\mathbf{a}}=\mathbb{S}^n(r)$ and when $\tau=1$, we have $\Upsilon^{\mathbf{a}}=\mathbb{H}^n(-r)$.

Example 3.10. In this example, we follow [29], page 132. Take the function $g: d\mathbb{S}^4\subset\mathbb{L}^5\to\mathbb{R}$ by $g(\mathbf{x})=-x_1+x_2$, and take $\Omega_t := g^{-1}(e^{-t})$, for each $t\in\mathbb{R}$. In fact, $\Omega_t=\{(f(\mathbf{y})+sinh\ t, f(\mathbf{y})+cosh\ t,\mathbf{y})\in d\mathbb{S}^4 | \mathbf{y}\in\mathbb{E}^3\}$, where $f(\mathbf{y})=\frac{-e^t}{2}\sum_{i=1}^3 y_i^2$. With respect to the Gauss map $\mathbf{n}(\mathbf{x})=e^t\omega-\mathbf{x}$ on Ω_t, where $\omega=(-1,1,0,...,0)\in\mathbb{L}^5$. So, we get $\kappa_1=\kappa_2=\kappa_3=1$, so $H_k=(-1)^k$. Hence, Ω_t is C-biconservative.

Example 3.11. Let $\mathbf{b}\in\mathbb{E}_2^5$ be a timelike unit vector $g: Ad\mathbb{S}^4\subset\mathbb{E}_2^5\to\mathbb{R}$ be the function given by rule $g(\mathbf{y})=\langle\mathbf{y},\mathbf{b}\rangle$. For each $0<r\le 1$, $\Pi_r := g^{-1}(-\sqrt{1-r^2})=\mathbb{H}^3(-r)$ is a totally umbilic hypersurface in $Ad\mathbb{S}^4$ with the Gauss map $\mathbf{n}(\mathbf{y})=\frac{1}{r}(\mathbf{b}-\sqrt{1-r^2}\mathbf{y})$ and principal curvatures $\kappa_1=\kappa_2=\kappa_3=\frac{\sqrt{1-r^2}}{r}$. Hence, Π_r is C-biconservative.

4. C-biconservative hypersurfaces with 2 and 3 principal curvatures

In this section, we study C-biconservative Riemannian hypersurfaces in $\mathbb{M}_1^4(c)$ for $c=\pm1$. The similar study has been made for ordinary biconservative hypersurfaces in some papers [11, 25, 27]. Let $\mathbf{x}: M^3\to\mathbb{M}^4(c)$ be a biconservative hypersurface in the Riemannian space form with 2 distinct principal curvatures. By Theorem 4.2 in [7], M^3 is an open part of a rotational hypersurface in $\mathbb{M}^4(c)$ for an appropriately chosen profile curve. In C-biconservative case, we show that a Riemannian hypersurface in $\mathbb{M}_1^4(c)$ with constant ordinary mean curvature has to be of constant scalar curvature. First, we see the next lemma which can be proved by the same manner of similar one in [24].

Lemma 4.1. Let M^3 be a Riemannian hypersurface in $\mathbb{M}_1^4(c)$ with principal curvatures of constant multiplicities. Then the distribution generated by principal directions is completely integrable. Also, each principal curvature of multiplicity greater than one is constant on each integral submanifold of its distribution.

The following theorems may be compared with Theorem 4.27 in [9] which is an alternative to the result of K. Nomizu and B. Smyth about compact CMC hypersurfaces in space forms in [16].

Theorem 4.2. Let $\mathbf{x} : M^3 \to \mathbb{M}_1^4(c)$ be a C-biconservative Riemannian hypersurface with constant ordinary mean curvature and at most two distinct principal curvatures. Then, its scalar curvature is constant and M^3 is isoparametric.

Proof. We start the proof with the assumption that the scalar curvature of M^3 is not constant and then H_2 is non-constant. We consider the open subset $\mathrm{U} := \{p \in M^3 : \nabla H_2^2(p) \neq 0\}$ which is non-empty. By assumption, M^3 has two distinct principal curvatures λ and η of multiplicities 2 and 1, respectively. The local orthonormal tangent frame $\{e_1, e_2, e_3\}$ of principal directions on U gives $Se_i = \lambda_i e_i$ for $i = 1, 2, 3$, where (by assumption)

$$\lambda_1 = \lambda_2 = \lambda,\ \lambda_3 = \eta.$$

Also,

$$\mu_{1,2} = \mu_{2,2} = \lambda\eta,\ \mu_{3,2} = \lambda^2,\ 3H = 2\lambda + \eta,\ 3H_2 = \lambda^2 + 2\lambda\eta. \tag{8}$$

The condition (1) gives

$$\mathrm{N}_2(\nabla H_2) = \frac{9}{2} H_2 \nabla H_2,$$

which, using the polar decomposition $\nabla H_2 = \sum_{i=1}^{3} \langle \nabla H_2, e_i\rangle e_i$, gives

$$\langle \nabla H_2, e_i\rangle (\mu_{i,2} - \frac{9}{2} H_2) = 0$$

on U, for $i = 1, 2, 3$. Since $\nabla H_2 \neq 0$ on U, there exists at least one i such that $\langle \nabla H_2, e_i\rangle \neq 0$ on U, so we have

$$\mu_{i,2} = \frac{9}{2} H_2. \tag{9}$$

We consider two possible cases.

Case 1. $\langle \nabla H_2, e_i\rangle \neq 0$, for $i = 1$ or $i = 2$. By equalities (8) and (9), we obtain

$$\lambda\eta = \frac{9}{2}(\frac{2}{3}\lambda\eta + \frac{1}{3}\lambda^2),$$

which gives

$$\lambda(6H - \frac{5}{2}\lambda) = 0. \tag{10}$$

Since H_2 is non-constant on U, formulae (8) gives that λ is not identically zero on U. Hence, by (10), we get $\lambda = \frac{12}{5} H$ and then $\eta = -\frac{9}{5} H$ and $H_2 = -\frac{72}{25} H^2$. The last equality gives that H is non-constant on U which is in contradiction with the assumption of theorem.

Case 2. $\langle \nabla H_2, e_3 \rangle \neq 0$. In a similar way, by equalities (8) and (9), we obtain

$$\lambda^2 = \frac{9}{2}(\frac{2}{3}\lambda\eta + \frac{1}{3}\lambda^2),$$

which gives

$$\lambda(9H - \frac{11}{2}\lambda) = 0. \tag{11}$$

H_2 is non-constant on U, so by (8) λ is not identically zero on U. Hence, by (11), we obtain $\lambda = \frac{18}{11}H$, $\eta = -\frac{3}{11}H_1$ and $H_2 = \frac{216}{121}H^2$. By the last equality, H is non-constant on U which is in contradiction with the assumption of theorem.

Therefore, H_2 and then the scalar curvature of M^3 have to be constant. Finally, we get that M^3 is isoparametric. □

Theorem 4.7 in [13] says that every biconservative hypersurface in $\mathbb{E}^4$ is made up of the following hypersurfaces:
(i) hypersurfaces with constant mean curvature,
(ii) some rotational hypersurfaces with non-constant mean curvature,
(iii) some generalized cylinders over surfaces of revolution lying in $\mathbb{E}^3$ with non-constant mean curvature,
(iv) some $O(2) \times O(2)$-invariant hypersurfaces with non-constant mean curvature.

Also, one can find a similar result for biconservative hypersurface in $\mathbb{E}^m$ (where $m \geq 5$) by Theorem 1 in [25]. Now, we pay attention to C-biconservative Riemannian hypersurfaces with 3 distinct principal curvatures. We show that such a hypersurface with constant mean curvature has constant scalar curvature.

Theorem 4.3. Let $\mathbf{x} : M^3 \to \mathbb{M}_1^4(c)$ be C-biconservative Riemannian hypersurface with constant ordinary mean curvature and three distinct principal curvatures. Then, the scalar curvature of M^3 is constant.

Proof. Assuming H_2 to be non-constant, we take $\mathrm{U} = \{p \in M^3 : \nabla H_2^2(p) \neq 0\}$. According to a suitable (local) orthonormal tangent frame $\{e_1, e_2, e_3\}$ on M^3, the shape operator S has a diagonal matrix form, such that $Se_i = \lambda_i e_i$ and then, $\mathrm{N}_2 e_i = \mu_{i,2} e_i$ for $i = 1, 2, 3$. By equality (1) and decomposition $\nabla H_2 = \sum_{i=1}^{3} e_i(H_2)e_i$, for $i = 1, 2, 3$ we obtain

$$e_i(H_2)(\mu_{i,2} - \frac{9}{2}H_2) = 0. \tag{12}$$

Around every point $\mathbf{p} \in \mathrm{U}$ there exists a neighborhood such that $e_i(H_2) \neq 0$ on which for at least one i. So, we can assume that $e_1(H_2) \neq 0$ and then we have $\mu_{1,2} = \frac{9}{2}H_2$, (locally) on U, which gives $\lambda_2\lambda_3 = \frac{9}{2}H_2$. We affirm three claims.

Claim 1: $e_2(H_2) = e_3(H_2) = 0$.
If $e_2(H_2) \neq 0$ or $e_3(H_2) \neq 0$, then by (12) we get $\mu_{1,2} = \mu_{2,2} = \frac{9}{2}H_2$ or

$\mu_{1,2} = \mu_{3,2} = \frac{9}{2}H_2$, which give $\lambda_2(\lambda_1 - \lambda_3) = 0$ or $\lambda_3(\lambda_2 - \lambda_1) = 0$. By assumption, λ_i's are mutually distinct, so we get $\lambda_3 = 0$ or $\lambda_2 = 0$, then $H_2 = 0$ on U. This contradicts with the definition of U.

Claim 2: $e_2(\lambda_1) = e_3(\lambda_1) = 0$.
By assumption H is constant on M. So, $e_2(\lambda_1) = e_2(3H - \lambda_1 - \lambda_2) = -e_2(\lambda_1) - e_2(\lambda_2)$. Also, by recent results, $e_2(H_2) = 0$ and $\lambda_2\lambda_3 = \frac{9}{2}H_2$, we get

$$e_2(\lambda_1\lambda_3) + e_2(\lambda_1\lambda_2) = e_2(3H_2 - \frac{9}{2}H_2) = 0,$$

which gives $\lambda_1 e_2(\lambda_2 + \lambda_3) + (\lambda_2 + \lambda_3)e_2\lambda_1 = 0$, and then we have

$$\lambda_1 e_2(3H - \lambda_1) + (\lambda_2 + \lambda_3)e_2\lambda_1 = \lambda_1 e_2(-\lambda_1) + (\lambda_2 + \lambda_3)e_2\lambda_1 = (-\lambda_1 + \lambda_2 + \lambda_3)e_2\lambda_1 = 0.$$

Therefore, assuming $e_2(\lambda_1) \neq 0$, we get $\lambda_1 = \lambda_2 + \lambda_3$ which gives contradiction

$$e_2(\lambda_1) = e_2(\lambda_2 + \lambda_3) = e_2(3H - \lambda_1) = -e_2(\lambda_1).$$

Consequently, $e_2(\lambda_1) = 0$.

Similarly, one can show $e_3(\lambda_1) = 0$. So, Claim 2 is affirmed.

Claim 3: $e_2(\lambda_3) = e_3(\lambda_2) = 0$.
Using the notations

$$\nabla_{e_i} e_j = \sum_{k=1}^{3} \omega_{ij}^k e_k, \ (i, j = 1, 2, 3), \tag{13}$$

and the compatibility condition $\nabla_{e_k} < e_i, e_j >= 0$, we have

$$\omega_{ki}^i = 0, \ \omega_{ki}^j + \omega_{kj}^i = 0, \ (i, j, k = 1, 2, 3) \tag{14}$$

and applying the Codazzi equation (see [17], page 115, Corollary 34(2))

$$(\nabla_V S)W = (\nabla_W S)V, \ (\forall V, W \in \chi(M)) \tag{15}$$

on the basis $\{e_1, e_2, e_3\}$, we get for distinct $i, j, k = 1, 2, 3$

$$(a)\ e_i(\lambda_j) = (\lambda_i - \lambda_j)\omega_{ji}^j, \ (b)\ (\lambda_i - \lambda_j)\omega_{ki}^j = (\lambda_k - \lambda_j)\omega_{ik}^j. \tag{16}$$

Also, by a straightforward computation of components of the identity $(\nabla_{e_i} S)e_j - (\nabla_{e_j} S)e_i \equiv 0$ for distinct $i, j = 1, 2, 3$, we get $[e_2,\ e_3](H_2) = 0$, $\omega_{12}^1 = \omega_{13}^1 = \omega_{13}^2 = \omega_{21}^3 = \omega_{32}^1 = 0$ and

$$\begin{aligned} \omega_{21}^2 &= \frac{e_1(\lambda_2)}{\lambda_1 - \lambda_2}, \ \omega_{31}^3 = \frac{e_1(\lambda_3)}{\lambda_1 - \lambda_3}, \\ \omega_{23}^2 &= \frac{e_3(\lambda_2)}{\lambda_3 - \lambda_2}, \ \omega_{32}^3 = \frac{e_2(\lambda_3)}{\lambda_2 - \lambda_3}. \end{aligned} \tag{17}$$

Therefore, the covariant derivatives $\nabla_{e_i} e_j$ simplify to $\nabla_{e_1} e_k = 0$ for $k = 1, 2, 3$, and

(18)
$$\nabla_{e_2} e_1 = \frac{e_1(\lambda_2)}{\lambda_1 - \lambda_2} e_2, \ \nabla_{e_3} e_1 = \frac{e_1(\lambda_3)}{\lambda_1 - \lambda_3} e_3, \nabla_{e_2} e_2 = \frac{e_1(\lambda_2)}{\lambda_2 - \lambda_1} e_1,$$
$$\nabla_{e_3} e_2 = \frac{e_2(\lambda_3)}{\lambda_2 - \lambda_3} e_3, \nabla_{e_2} e_3 = \frac{e_3(\lambda_2)}{\lambda_3 - \lambda_2} e_2, \ \nabla_{e_3} e_3 = \frac{e_1(\lambda_3)}{\lambda_3 - \lambda_1} e_1 + \frac{e_2(\lambda_3)}{\lambda_3 - \lambda_2} e_2.$$

Now, the Gauss equation for $< R(e_2, e_3)e_1, \ e_2 >$ and $< R(e_2, e_3)e_1, \ e_3 >$ show that

$$e_3 \left(\frac{e_1(\lambda_2)}{\lambda_1 - \lambda_2} \right) = \frac{e_3(\lambda_2)}{\lambda_3 - \lambda_2} \left(\frac{e_1(\lambda_3)}{\lambda_1 - \lambda_3} - \frac{e_1(\lambda_2)}{\lambda_1 - \lambda_2} \right), \tag{19}$$

$$e_2 \left(\frac{e_1(\lambda_3)}{\lambda_1 - \lambda_3} \right) = \frac{e_2(\lambda_3)}{\lambda_2 - \lambda_3} \left(\frac{e_1(\lambda_3)}{\lambda_1 - \lambda_3} - \frac{e_1(\lambda_2)}{\lambda_1 - \lambda_2} \right). \tag{20}$$

We also have the Gauss equation for $< R(e_1, e_2)e_1, \ e_2 >$ and $< R(e_3, e_1)e_1, \ e_3 >$, which give the following relations

(21)
$$e_1 \left(\frac{e_1(\lambda_2)}{\lambda_1 - \lambda_2} \right) + \left(\frac{e_1(\lambda_2)}{\lambda_1 - \lambda_2} \right)^2 = \lambda_1 \lambda_2, \ e_1 \left(\frac{e_1(\lambda_3)}{\lambda_1 - \lambda_3} \right) + \left(\frac{e_1(\lambda_3)}{\lambda_3 - \lambda_1} \right)^2 = \lambda_1 \lambda_3.$$

Finally, we obtain from the Gauss equation for $< R(e_3, e_1)e_2, e_3 >$ that

$$e_1 \left(\frac{e_2(\lambda_3)}{\lambda_2 - \lambda_3} \right) = \frac{e_1(\lambda_3) e_2(\lambda_3)}{(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_3)}. \tag{22}$$

On the other hand, by considering the condition (1), from Claim 1 we get

(23)
$$-\mu_{1,1} e_1 e_1(H_2) + \left(\mu_{2,1} \frac{e_1(\lambda_2)}{\lambda_2 - \lambda_1} + \mu_{3,1} \frac{e_1(\lambda_3)}{\lambda_3 - \lambda_1} \right) e_1(H_2) - 9H_2^2 (H_1 - \frac{3}{2}\lambda_1) = 0.$$

By differentiating (23) along e_2 and e_3, and using (19) and (20), respectively, we obtain

$$e_2 \left(\frac{e_1(\lambda_2)}{\lambda_2 - \lambda_1} \right) = \frac{e_2(\lambda_3)}{\lambda_2 - \lambda_3} \left(\frac{e_1(\lambda_3)}{\lambda_1 - \lambda_3} - \frac{e_1(\lambda_2)}{\lambda_1 - \lambda_2} \right), \tag{24}$$

$$e_3 \left(\frac{e_1(\lambda_3)}{\lambda_3 - \lambda_1} \right) = \frac{e_3(\lambda_2)}{\lambda_3 - \lambda_2} \left(\frac{e_1(\lambda_2)}{\lambda_1 - \lambda_2} - \frac{e_1(\lambda_3)}{\lambda_1 - \lambda_3} \right). \tag{25}$$

Using (18), we find that

$$[e_1, \ e_2] = \frac{e_1(\lambda_2)}{\lambda_2 - \lambda_1} e_2. \tag{26}$$

Applying both sides of the equality (26) on $\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1}$, using (24), (21), and (22), we deduce that

$$\frac{e_2(\lambda_3)}{\lambda_2-\lambda_3}\left(\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}+\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}\right)=0. \tag{27}$$

(27) shows that $e_2(\lambda_3)=0$ or

$$\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}=\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1}. \tag{28}$$

From equation (28), by differentiating on its both sides along e_1 and applying (21), we get $\lambda_2=\lambda_3$, which is a contradiction, so we have to confirm the result $e_2(\lambda_3)=0$.

Analogously, using (18), we find that $[e_1,\ e_3]=\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}e_3$. By a similar manner, we deduce that

$$\frac{e_3(\lambda_2)}{\lambda_3-\lambda_2}\left(\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1}+\frac{e_1(\lambda_3)}{\lambda_1-\lambda_3}\right)=0, \tag{29}$$

and one can show that $e_3(\lambda_2)$ necessarily has to be vanished.

Hence, we have obtained $e_2(\lambda_3)=e_3(\lambda_2)=0$ which, by applying the Gauss equation for $< R(e_2,e_3)e_1,e_3 >$, gives the following equality

$$\frac{e_1(\lambda_3)e_1(\lambda_2)}{(\lambda_3-\lambda_1)(\lambda_2-\lambda_1)}=\lambda_2\lambda_3. \tag{30}$$

Finally, using (21), differentiating (30) along e_1 gives

$$\lambda_2\lambda_3\left(\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}+\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}\right)=0, \tag{31}$$

which implies $\lambda_2\lambda_3=0$ (since we have seen above that $\left(\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}+\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}\right)\neq 0$). Therefore, we obtain $H_2=0$ on U, which is a contradiction. Hence H_2 is constant on M^3. □

Remark 4.4. From condition 1, it is clear that each Riemannian hypersurface $\mathbf{x}: M^3\to\mathbb{M}^4_1(c)$, whose H_1 and H_2 are constant, is C-biconservative. But, if H_1 is not assumed constant, it is difficult to prove that M^3 is C-biconservative. So, we have the following open problem.

Open problem 4.5. Dose every Riemannian hypersurface with constant scalar curvature in $\mathbb{M}^4_1(c)$ have to be C-biconservative?

5. **Conclusion**

Biconservative hypersurfaces having conservative stress-energy tensor with respect to the bi-energy contain all minimal and constant mean curvature hypersurfaces. It is proven that any biconservative hypersurface with constant scalar curvature is ether an open part of a certain rotational hypersurface or a constant mean curvature hypersurface. The purpose of this paper was to study an extension of biconservativity condition (namely, C-biconservativity) of Riemannian hypersurfaces with constant mean curvature in the Lorentz 4-space form $\mathbb{M}_1^4(c)$. We tried to show that such a hypersurface has constant scalar curvature. This aim leads us to an open problem claiming that every C-biconservative hypersurface with constant scalar curvature has constant mean curvature.

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