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# ON THE DISTRIBUTIVITY OF THE LATTICE OF RADICAL SUBMODULES

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Abstract. Let $R$ be a commutative ring with identity and $\mathcal{R}(_RM)$ denote the bounded lattice of radical submodules of an $R$-module $M$. We say that $M$ is a radical distributive module, if $\mathcal{R}(_RM)$ is a distributive lattice. It is shown that the class of radical distributive modules contains the classes of multiplication modules and finitely generated distributive modules properly. Also, it is shown that if $M$ is a radical distributive semisimple $R$-module and for any radical submodule $N$ of $M$ with direct sum complement $\tilde{N}$, the complementary operation on $\mathcal{R}(_RM)$ is defined by $N' := \tilde{N} + \mathbf{r}ad\{0\}$, then $\mathcal{R}(_RM)$ with this unary operation forms a Boolean algebra. In particular, if $M$ is a multiplication module over a semisimple ring $R$, then $\mathcal{R}(_RM)$ is a Boolean algebra, which is also a homomorphic image of $\mathcal{R}(_RR)$.

*Keywords*: Radical distributive module, Distributive module, Multiplication module, Semisimple ring, Boolean algebra homomorphism.
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## 1. Introduction

Throughout, all rings are commutative with identity and all modules are unitary.

Let $R$ be a ring and $M$ be an $R$-module. For a submodule $N$ of an $R$-module $M$, $(N : M)$ is the ideal $\{r \in R \mid rM \subseteq N\}$. As usual, $M$ is *faithful* if $(\{0\} : M) = \{0\}$. A proper submodule $P$ of $M$ is called a *prime submodule* if for $r \in R$, $x \in M$, $rx \in P$ implies that either $r \in (P : M)$ or $x \in P$. There is an extensive literature on the topic of prime modules over commutative rings (see, for example, [1, 6, 10]). The *radical* of a submodule $N$ is the intersection of all prime submodules of $M$ containing $N$ and denoted by $\operatorname{rad}_M N$ or simply by $\operatorname{rad} N$ whenever there is no ambiguity in the context. It is noted that if there are no prime submodules of $M$ containing $N$, $\operatorname{rad} N$ is defined to be $M$. For an ideal $A$ of a ring $R$, the radical of $A$ is denoted by $\sqrt{A}$. A submodule $N$ of $M$ is called a *radical submodule* if $\operatorname{rad} N = N$. An $R$-module $M$ is said to be a *radical module* if the zero submodule $\{0\}$ of $M$ is a radical submodule. The set of all radical submodules of $M$, denoted $\mathcal{R}(_RM)$, is a bounded lattice

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with respect to inclusion. Indeed, if $\vee$ and $\wedge$ denote the supremum and the infimum operation symbols respectively, then for every $N, L \in \mathcal{R}({}_RM)$, we have $N \vee L = \text{rad}(N + L)$ and $N \wedge L = N \cap L$. Moreover, it is clear that rad$\{0\}$ and $M$ are respectively the least and the greatest elements of $\mathcal{R}({}_RM)$. Following the literature, an $R$-module $M$ is called a *distributive* $R$-module, if the usual lattice of submodules of $M$ is a distributive lattice. Also, a ring $R$ is called *arithmetical* if it is a distributive $R$-module. We say that an $R$-module $M$ is a *radical distributive* $R$-module, if $\mathcal{R}({}_RM)$ is a distributive lattice. It is shown that every multiplication module (Theorem 2.5), and every finitely generated distributive module (Theorem 2.8) is radical distributive.

Based on the motivation arising from some works by P. F. Smith ( [12–14]) in examining certain mappings between the usual lattice of ideals of a commutative ring $R$ and the usual lattice of submodules of an $R$-module $M$, we have recently examined the properties of the mappings $\rho : \mathcal{R}({}_RR) \to \mathcal{R}({}_RM)$ defined by $\rho(I) = \text{rad}(IM)$ and $\sigma : \mathcal{R}({}_RM) \to \mathcal{R}({}_RR)$ defined by $\sigma(N) = (N : M)$ in [4, 8]. Of course, these mappings are also mentioned in [9, 11], where we aimed to investigate the properties of certain mappings between other types of lattices. In this paper, we continue our investigation, in particular considering when $\rho$ and $\sigma$ are Boolean algebra homomorphisms. If $M$ is a radical distributive semisimple $R$-module and for each submodule $N$ with direct sum complement $\tilde{N}$, the unary operation $'$ on $\mathcal{R}({}_RM)$ is defined by $N' := \tilde{N} + \text{rad}\{0\}$, then $'$ (as the complementary operation) is well-defined and the bounded lattice $\mathcal{R}({}_RM)$ with this operation is a Boolean algebra (Theorem 3.1). It is shown that if $M$ is a radical distributive module over a semisimple ring $R$, then $\rho$ is a Boolean algebra homomorphism (Theorem 3.5). In particular, if $M$ is a multiplication module over a semisimple ring $R$, then $\rho$ is a Boolean algebra epimorphism and $\sigma$ is a Boolean algebra monomorphism (Corollary 3.7).

## 2. **Radical distributive modules**

Let $R$ be a ring and $M$ an $R$-module. Recall $M$ is called radical distributive if the lattice $\mathcal{R}({}_RM)$ of radical submodules of $M$ is a distributive lattice. The following lemma is an immediate consequence of [2, Theorem I.3.2].

**Lemma 2.1.** *Let $R$ be a ring and $M$ be an $R$-module. Then the following statements are equivalent:*

(1) *$M$ is radical distributive;*
(2) $N \cap \text{rad}(L + K) = \text{rad}((N \cap L) + (N \cap K))$ *for all* $N, L, K \in \mathcal{R}({}_RM)$*;*
(3) $\text{rad}(N + (L \cap K)) = \text{rad}(N + L) \cap \text{rad}(N + K)$ *for all* $N, L, K \in \mathcal{R}({}_RM)$.

Given a ring $R$, we shall call an $R$-module $M$ a *chain* module in case the submodules of $M$ are linearly ordered. As noted in [15, Corollary 1 of Proposition 2.3 ], every distributive module over a local ring is a chain module.

**Proposition 2.2.** *Every chain module is radical distributive. In particular, every distributive module over a local ring is radical distributive.*

*Proof.* The result follows immediately from Lemma 2.1 by considering all inclusion cases between any three radical submodules of a chain module. □

As stated in [1, Lemma 2.1], for submodules $N$ and $L$ of an $R$-module $M$ we have $\mathrm{rad}_{M/L}\, N/L = (\mathrm{rad}_M N)/L$. This is used in the proof of the following theorem.

**Theorem 2.3.** *Let $R$ be a ring. Every quotient of any radical distributive $R$-module is radical distributive.*

*Proof.* (1) Let $M$ be a radical distributive $R$-module and $T$ be a submodule of $M$. Let $N$, $L$ and $K$ be submodules of $M$ containing $T$ such that $N/T, L/T, K/T \in \mathcal{R}({}_R M/T)$. Then by [1, Lemma 2.1], $N, L, K \in \mathcal{R}({}_R M)$. Hence by using Lemma 2.1, we have

$$\begin{aligned} N/T \cap \mathrm{rad}_{M/T}(L/T + K/T) &= N/T \cap \mathrm{rad}_{M/T}((L+K)/T) \\ &= N/T \cap (\mathrm{rad}_M(L+K)/T) \\ &= (N \cap \mathrm{rad}_M(L+K))/T \\ &= (\mathrm{rad}_M((N\cap L) + (N \cap K)))/T \\ &= \mathrm{rad}_{M/T}((N\cap L) + (N\cap K)/T) \\ &= \mathrm{rad}_{M/T}((N\cap L)/T + (N\cap K)/T) \\ &= \mathrm{rad}_{M/T}((N/T \cap L/T) + (N/T \cap K/T)). \end{aligned}$$

Thus, by Lemma 2.1 again, we are done.

□

**Lemma 2.4.** *Every ring $R$ is a radical distributive $R$-module.*

*Proof.* By Lemma 2.1, it is enough to show that

$$\sqrt{A + (B \cap C)} = \sqrt{A+B} \cap \sqrt{A+C}$$

for all $A, B, C \in \mathcal{R}({}_R R)$. Clearly $\sqrt{A+(B\cap C)} \subseteq \sqrt{A+B} \cap \sqrt{A+C}$. For the reverse inclusion, let $x \in \sqrt{A+B} \cap \sqrt{A+C}$, so that there are positive integers $m$, $n$ and the elements $a, a' \in A$, $b \in B$ and $c \in C$ such that $x^m = a + b$ and $x^n = a' + c$. Therefore, $x^{m+n} \in A + (B \cap C)$ and hence $x \in \sqrt{A + (B\cap C)}$, as requested. □

An $R$-module $M$ is a *multiplication* module, if for each submodule $N$ of $M$ there exists an ideal $A$ of $R$ such that $N = AM$. In this case, we can take $A = (N : M)$ (see, for example, [3]).

In [8, p. 37], it has been shown that the mapping $\rho : \mathcal{R}({}_R R) \to \mathcal{R}({}_R M)$ defined by $\rho(A) = \mathrm{rad}(AM)$ is always a lattice homomorphism. In particular if $M$ is a multiplication $R$-module, then for each ideal $A$ of $R$ we see that

$\rho(\sqrt{A}) = \text{rad}(\sqrt{A}M) = \text{rad}(AM)$ which shows $\rho$ is a lattice epimorphism. Regarding this fact and Lemma 2.4, we have the following result:

**Theorem 2.5.** *Every multiplication module is radical distributive. In particular, every cyclic R-module is radical distributive.*

*Proof.* Let $M$ be a multiplication $R$-module, and $N, L, K \in \mathcal{R}(_RM)$. As remarked previously, $\rho$ is a lattice epimorphism and hence there are $A, B, C \in \mathcal{R}(_RR)$ such that $\rho(A) = N$, $\rho(B) = L$ and $\rho(C) = K$. Now by using Lemma 2.4, we have

$$\begin{aligned}\text{rad}(N + (L \cap K)) &= N \vee (L \wedge K) = \rho(A) \vee (\rho(B) \wedge \rho(C)) = \rho(A \vee (B \wedge C)) \\ &= \rho((A \vee B) \wedge (A \vee C)) = (\rho(A) \vee \rho(B)) \wedge (\rho(A) \vee \rho(C)) \\ &= (N \vee L) \wedge (N \vee K) = \text{rad}(N + L) \cap \text{rad}(N + K).\end{aligned}$$

Therefore, by Lemma 2.1, $M$ is a radical distributive $R$-module. The "in particular part follows" from [3, Corollary 2.4]. □

It is easily seen that every proper subspace of a vector space is prime as a submodule. We use this fact in the following theorem.

**Theorem 2.6.** *Let $V$ be a vector space over a field $F$. Then $V$ is a radical distributive module if and only if* $\dim_F V \leq 1$.

*Proof.* $\Rightarrow$) We may assume that $\dim_F V = 2$ (The proof for higher dimensions is done similarly). Clearly $V \cong F \oplus F$. Now by considering $N = \{(x, x) \mid x \in F\}$, $L = F \oplus \{0\}$, $K = \{0\} \oplus F$ and, we have

$$N \cap \text{rad}(L + K) = N \cap \text{rad}(V) = N \cap V = N, \text{ and}$$

$$\text{rad}((N \cap K) + (L \cap K)) = \text{rad}(\{0\} + \{0\}) = \text{rad}\{0\} = \{0\}.$$

Thus by Lemma 2.1, $V$ is not a radical distributive module.
$\Leftarrow$) The converse holds by Theorem 2.5. □

Now consider the following lemma.

**Lemma 2.7.** *If $M$ is a distributive $R$-module such that any finite intersection of its radical submodules is a radical submodule, then $M$ is a radical distributive module. In particular, if $(N : M)$ and $(L : M)$ are comaximal ideals for all non-zero distinct proper submodules $N$ and $L$ of $M$, then $M$ is a radical distributive $R$-module.*

*Proof.* Let $N, L, K \in \mathcal{R}(_RM)$. By the hypothesis,

$$\text{rad}(N + (L \cap K)) = \text{rad}((N + L) \cap (N + K)) = \text{rad}(N + L) \cap \text{rad}(N + K).$$

Now, the result follows from Lemma 2.1. For the "In particular" part first note that, if $N$ or $L$ is $M$ or the zero-submodule or equal, then it is eaily seen that $\mathbf{r}ad(N \cap L) = \mathbf{r}adN \cap \mathbf{r}adL$. This equality also holds for non-zero distinct proper submodules $N$ and $L$ by [10, Theorem 2.6]. Now the result follows from the previous part. □

**Theorem 2.8.** *Every finitely generated distributive module is radical distributive.*

*Proof.* Let $M$ be a finitely generated distributive $R$-module. Since $M$ is distributive, by [15, Theorem 1.6], we have $(Rx : Ry) + (Ry : Rx) = R$ for all $x, y \in M$. This result together with $M$ being a finitely generated module follows that $M$ is a multiplication $R$-module by [12, Corollary 3.9]. Now apply Theorem 2.5. □

It is evident that every non-arithmetical ring $R$ may be considered as a finitely generated radical distributive $R$-module which is not distributive. In the following, we give an example of a non-finitely generated and non-multiplication distributive module which is radical distributive. Therefore the class of radical distributive module contains both classes of finitely generated distributive modules and multiplication modules properly.

**Example 2.9.** *As usual $\mathbb{Z}$ denotes the ring of integers, and $\mathbb{Q}$ is the set of rational numbers which is known as a non-finiltely generated and a non-multiplication $\mathbb{Z}$-module. Note that by [15, Proposition 3.3(ii)], $\mathbb{Q}$ is a distributive $\mathbb{Z}$-module. Now since $\{0\}$ is the only prime submodule of $\mathbb{Q}$, $\mathcal{R}({}_{\mathbb{Z}}\mathbb{Q}) = \{\{0\}, \mathbb{Q}\}$ which follows that $\mathbb{Q}$ is a radical distributive $\mathbb{Z}$-module.*

Our next example shows that the usual distributivity alone dose not imply the radical distributivity.

**Example 2.10.** *Let $R = \mathbb{Z}$ be the ring of integers, and consider non-finitely generated $R$-module $M = \mathbb{Z}(p^\infty) \oplus \mathbb{Z}_p$ for some prime number $p$. Note that every submodule of $M$ has the form $N \oplus L$ where $N$ is the submodule of the chain module $\mathbb{Z}(p^\infty)$ and $L$ is the submodule of the simple $R$-module $\mathbb{Z}_p$. It is easily seen that $M$ is a distributive $R$-module. Furthermore, by [6,* Example 3. 7*], $M$ is not a multiplication module and $pM$ is the only prime submodule of $M$. Now, if we consider $N = \{0\} \oplus \mathbb{Z}_p$ and $L = K = pM = \mathbb{Z}(p^\infty) \oplus \{\bar{0}\}$, then*

$$N \cap \operatorname{rad}(L + K) = N \cap \operatorname{rad} L = N \cap L = \{0\} \quad \text{and}$$

$$\operatorname{rad}((N \cap L) + (N \cap K)) = \operatorname{rad}\{0\} = pM.$$

*Hence, by Lemma 2.1, $M$ is not a radical distributive $R$-module. Moreover, contrary to Lemma 2.7, $(N : M) + (L : M) = \{0\}$.*

## 3. **A Boolean algebra of radical submodules of semisimple modules**

Recall that a Boolean algebra $B$ is a distributive bounded lattice with the least element $\mathbf{0}$, greatest element $\mathbf{1}$, binary operations $\vee$ and $\wedge$ and unary operation $' : B \longrightarrow B$ such that $a \wedge a' = \mathbf{0}$ and $a \vee a' = \mathbf{1}$ for all $a \in B$. In this section, we introduce a binary operation on $\mathcal{R}({}_RM)$ for any radical distributive semisimple $R$-module, which makes $\mathcal{R}({}_RM)$ into a Boolean algebra.

Let $M$ be a semisimple $R$-module and $N$ a radical submodule of $M$. Then there is a submodule $\tilde{N}$ of $M$ such that $M = N \oplus \tilde{N}$. Note that if $M$ is a radical module, then by [7, Lemma 4.1], $N, \tilde{N} \in \mathcal{R}(_RM)$. We define $'$ on $\mathcal{R}(_RM)$ by $N' := \tilde{N} + \mathrm{rad}\{0\}$, and show in next theorem that $'$ is a unary operation on $\mathcal{R}(_RM)$.

**Theorem 3.1.** *Let $M$ be a radical distributive semisimple $R$-module and $'$ be as above. Then $\mathcal{R}(_RM)$ is a Boolean algebra with the least element* $\mathrm{rad}\{0\}$ *and the greatest element $M$.*

*Proof.* Let $N \in \mathcal{R}(_RM)$, and $N'$ be as remarked above. First, we show that $N' \in \mathcal{R}(_RM)$. Note that

$$\begin{aligned} M/\mathrm{rad}\{0\} &= (N \oplus \tilde{N})/\mathrm{rad}\{0\} = (N/\mathrm{rad}\{0\}) \oplus (\tilde{N} + \mathrm{rad}\{0\})/\mathrm{rad}\{0\} \\ &= (N/\mathrm{rad}\{0\}) \oplus (N'/\mathrm{rad}\{0\}) \end{aligned}$$

Now since $M/\mathrm{rad}\{0\}$ is a radical module, we conclude that by [7, Lemma 4.1], $N'/\mathrm{rad}\{0\}$ is a radical submodule of $M/\mathrm{rad}\{0\}$. Thus we have

$$\mathrm{rad}(N')/\mathrm{rad}\{0\} = \mathrm{rad}_{M/\mathrm{rad}_M\{0\}}(N'/\mathrm{rad}\{0\}) = N'/\mathrm{rad}\{0\},$$

which implies that $\mathrm{rad}(N') = N'$, i.e., $N' \in \mathcal{R}(_RM)$. Moreover, note that this $N'$ is unique. Because if $N_1$ and $N_2$ are direct sum complements of the radical submodule $N$ of $M$, so that $M = N \oplus \tilde{N}_1 = N \oplus \tilde{N}_2$, then

$$M/\mathrm{rad}\{0\} = N/\mathrm{rad}\{0\} \oplus N_1'/\mathrm{rad}\{0\} = N/\mathrm{rad}\{0\} \oplus N_2'/\mathrm{rad}\{0\}.$$

Let $y_1 \in N_1'$. Then there exist $x \in N$, $y_2 \in N_2'$ such that $y_1 + \mathrm{rad}\{0\} = x + y_2 + \mathrm{rad}\{0\}$. It follows that $x + \mathrm{rad}\{0\} = y_1 - y_2 + \mathrm{rad}\{0\} \in N/rad\{0\} \cap (N_1'/\mathrm{rad}\{0\} + N_2'/\mathrm{rad}\{0\})$. But since $M/\mathrm{rad}\{0\}$ is radical distributive, we have $x + \mathrm{rad}\{0\} \in rad((N/\mathrm{rad}\{0\} \cap N_1'/\mathrm{rad}\{0\}) + (N/\mathrm{rad}\{0\} \cap N_2'/\mathrm{rad}\{0\})) = \mathrm{rad}(\mathrm{rad}\{0\}) = \mathrm{rad}\{0\}$. Therefore $y_1 + \mathrm{rad}\{0\} = y_2 + \mathrm{rad}\{0\}$ and so $N_1'/\mathrm{rad}\{0\} \subseteq N_2'/\mathrm{rad}\{0\}$. Similarly $N_2'/\mathrm{rad}\{0\} \subseteq N_1'/\mathrm{rad}\{0\}$ and so $N_1'/\mathrm{rad}\{0\} = N_2'/\mathrm{rad}\{0\}$. It follows that $N_1' = N_2'$, as desired. Now, we check that $N \wedge N' = \mathrm{rad}\{0\}$. For this, let $x \in N \wedge N' = N \cap (\tilde{N} + \mathrm{rad}\{0\})$. Then $x = y + z$ for some $y \in \tilde{N}, z \in \mathrm{rad}\{0\}$. Hence $x - z = y \in N \cap \tilde{N}$. It follows that $y = 0$ and so $x \in \mathrm{rad}\{0\}$. Therefore $N \wedge N' = N \cap N' = N \cap (\tilde{N} + \mathrm{rad}\{0\}) = \mathrm{rad}\{0\}$. Moreover, $N \vee N' = \mathrm{rad}(N + N') = \mathrm{rad}(N + \tilde{N} + \mathrm{rad}\{0\}) = \mathrm{rad}(M) = M$, which completes the proof. □

From now on, $\mathcal{R}(_RM)$ is assumed to be an algebra with the above assumptions and operations.

**Corollary 3.2.** *Let $M$ be a finitely generated distributive semisimple $R$-module. Then $\mathcal{R}(_RM)$ is a Boolean algebra.*

*Proof.* Follows from Theorem 2.8(2) and Theorem 3.1. □

**Corollary 3.3.** *Let $M$ be a semisimple multiplication $R$-module. Then $\mathcal{R}(_RM)$ is a Boolean algebra.*

*Proof.* Follows from Theorem 2.5 and Theorem 3.1. □

**Corollary 3.4.** *Let $M$ be a radical distributive semisimple $R$-module. Then $\mathcal{R}({}_RM)$ is a Boolean ring with the operations*

$N + L = \text{rad}((N \cap (L_1 + \text{rad}\{0\})) + ((N_1 + \text{rad}\{0\}) \cap L))$ *and* $N \cdot L = N \cap L$

*in which $N_1$ and $L_1$ are direct-sum complement of $N$ and $L$ respectively.*

*Proof.* Follows from [2, Theorem IV.2.3]. □

Let $B$ and $C$ be Boolean algebras. A function $f : B \to C$ is called a *Boolean algebra homomorphism*, if $f$ is a lattice homomorphism such that $f(\mathbf{0}) = \mathbf{0}$, $f(\mathbf{1}) = \mathbf{1}$ and $f(b') = f(b)'$ for all $b \in B$. It is noted that a lattice homomorphism $f$ between two Boolean algebras preserves $'$ if and only if it preserves both $\mathbf{0}$ and $\mathbf{1}$.

As stated before, $\rho : \mathcal{R}({}_RR) \to \mathcal{R}({}_RM)$ defined by $\rho(A) = \text{rad}(AM)$ is a lattice homomorphism. Now, we give conditions under which $\rho$ is a Boolean algebra homomorphism.

**Theorem 3.5.** *Let $R$ be a semisimple ring. If $M$ is a radical distributive $R$-module, then $\rho : \mathcal{R}({}_RR) \to \mathcal{R}({}_RM)$ defined by $\rho(A) = \text{rad}(AM)$ is a Boolean algebra homomorphism.*

*Proof.* By Theorem 3.1, $\mathcal{R}({}_RR)$ and $\mathcal{R}({}_RM)$ are Boolean algebras. Now, since

$$\rho(\sqrt{0}) = \text{rad}(\sqrt{0}M) = \text{rad}(\{0\}M) = \text{rad}\{0\} \text{ and}$$

$$\rho(R) = \text{rad}(RM) = \text{rad}(M) = M,$$

$\rho$ is a Boolean algebra homomorphism. □

Note that by [8, Corollary 2.14], for any finitely generated $R$-module $M$, the mapping $\sigma : \mathcal{R}({}_RM) \to \mathcal{R}({}_RR)$ defined by $\sigma(N) = (N : M)$ is a lattice homomorphism if and only if $(Rx : Ry) + (Ry : Rx) = R$ for all elements $x, y \in M$. Moreover, by [15, Theorem 1.6], the later statement holds if and only if $M$ is a distributive $R$-module. Combining these facts, we conclude the following:

**Theorem 3.6.** *Let $R$ be a semisimple ring. If $M$ is a finitely generated distributive module, then $\sigma : \mathcal{R}({}_RM) \to \mathcal{R}({}_RR)$ defined by $\sigma(N) = (N : M)$ is a Boolean algebra homomorphism.*

*Proof.* By the remark above, $\sigma$ is a lattice homomorphism. Also, by Theorem 2.8(2), $M$ is a radical distributive module and so by Theorem 3.1, $\mathcal{R}({}_RR)$ and $\mathcal{R}({}_RM)$ are Boolean algebras. Moreover, since $M$ is finitely generated, we have

$$\sigma(\text{rad}\{0\}) = (\text{rad} : M) = \sqrt{(\{0\} : M)} = \sqrt{\{0\}} \quad \text{and} \quad \sigma(M) = (M : M) = R.$$

Thus $\sigma$ is a Boolean algebra homomorphism. □

It is well-known that every semisimple ring is Artinian ( [5, Corollary 2.6]). Thus by [3, Corollary 2.9], every multiplication module over a semisimple ring is cyclic. In this case, we have the following:

**Corollary 3.7.** *Let $R$ be a semisimple ring and $M$ be a multiplication $R$-module. Then*

(1) *$\rho$ is a Boolean algebra epimorphism*
(2) *$\sigma$ is a Boolean algebra monomorphism.*

*Proof.* First note that $\mathcal{R}({}_RR)$ and $\mathcal{R}({}_RM)$ are Boolean algebras by Corollary 3.3.
(1) By Theorem 3.5, $\rho$ is a Boolean algebra homomorphism. Moreover by the remark mentioned before Theorem 2.5, $\rho$ is surjective.
(2) By Theorem 3.6, $\sigma$ is a Boolean algebra homomorphism. The injectivity of $\sigma$ is clear. □

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