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A NOTE ON CHARACTERIZATION OF HIGHER DERIVATIONS AND THEIR PRODUCT

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Abstract. There exists a one to one correspondence between higher derivations $\{d_n\}_{n=0}^{\infty}$ on an algebra $\mathcal{A}$ and the family of sequences of derivations $\{\delta_n\}_{n=1}^{\infty}$ on $\mathcal{A}$. In this paper, we obtain a relation that calculates each derivation $\delta_n (n \in \mathbb{N})$ directly as a linear combination of products of terms of the corresponding higher derivation $\{d_n\}_{n=0}^{\infty}$. Also, we find the general form of the family of inner derivations corresponding to an inner higher derivation.

We show that for every two higher derivations on an algebra $\mathcal{A}$, the product of them is a higher derivation on $\mathcal{A}$. Also, we prove that the product of two inner higher derivations is an inner higher derivation.

Keywords: Derivation, Higher derivation, Inner higher derivation.
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1. Introduction

Let $\mathcal{A}$ be an algebra. A linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ satisfying the Leibniz rule $\delta(xy) = \delta(x)y + x\delta(y)$ for each $x, y \in \mathcal{A}$, is called a *derivation* on $\mathcal{A}$. For a fixed element $a \in \mathcal{A}$, the linear mapping δ_a defined by $\delta_a(x) = ax - xa$ on $\mathcal{A}$ is a derivation on $\mathcal{A}$, which is called the *inner derivation* implemented by a.

A sequence of linear mappings $\mathbf{d} = \{d_n : \mathcal{A} \to \mathcal{A}\}_{n=0}^{\infty}$ with $d_0 = I$ (where I is the identity mapping on $\mathcal{A}$), satisfying the *generalized Leibniz rule* $d_n(xy) = \sum_{i=0}^{n} d_i(x)d_{n-i}(y)$ for each $n = 0, 1, 2, \ldots$ and $x, y \in \mathcal{A}$, is called a *higher derivation* on $\mathcal{A}$.

For example, when $\delta : \mathcal{A} \to \mathcal{A}$ is a derivation, the sequence $\mathbf{d} = \{d_n\}_{n=0}^{\infty}$ of linear mappings on $\mathcal{A}$ defined by $d_0 = I$ and $d_n = \frac{\delta^n}{n!}$, is a higher derivation on $\mathcal{A}$. Such a sequence is called an *ordinary higher derivation*. An ordinary higher derivation is a typical example and is not the only example of a higher derivation.

Mirzavaziri in Theorem 2.5 of [11] showed that there exists a one to one correspondence between higher derivations and the family of sequences of derivations on torsion free algebras. Let $\mathcal{A}$ be a torsion free algebra and $\mathbf{d} = \{d_n\}_{n=0}^{\infty}$ be a higher derivation on $\mathcal{A}$. Then there is a sequence $\boldsymbol{\delta} = \{\delta_n\}_{n=1}^{\infty}$ of derivations

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on $\mathcal{A}$ such that

$$d_n = \sum_{i=1}^{n} \Big(\sum_{\sum_{j=1}^{i} r_j = n} \big(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \big) \ \delta_{r_1} \ldots \delta_{r_i} \Big),$$

where the inner summation is taken over all positive integers r_j with $\sum_{j=1}^{i} r_j = n$. Moreover the sequence $\boldsymbol{\delta} = \{\delta_n\}_{n=1}^{\infty}$ is unique. If we denote this higher derivation by $\mathbf{d}_{\boldsymbol{\delta}}$, then we can say that each higher derivation on a torsion free algebra $\mathcal{A}$ is of the form $\mathbf{d}_{\boldsymbol{\delta}}$ for some sequence $\boldsymbol{\delta}$ of derivations on $\mathcal{A}$.

A notion of an inner higher derivation is given in [12]. The authors characterized all uniformly bounded inner higher derivations on Banach algebras and showed that each uniformly bounded higher derivation on a Banach algebra $\mathcal{A}$ is inner, provided that each derivation on $\mathcal{A}$ is inner.

In section 2, we obtain a relation that calculates each derivation $\delta_n (n \in \mathbb{N})$ directly as a linear combination of products of terms of the corresponding higher derivation $\{d_n\}_{n=0}^{\infty}$. Also, we find the general form of the family of inner derivations corresponding to an inner higher derivation.

Let $\mathcal{A}$ be an associative algebra with identity element $1_{\mathcal{A}}$. A sequence of linear mappings $\mathbf{d} = \{d_n\}_{n=0}^{\infty}$ on $\mathcal{A}$ defined by

$$d_n(x) = \sum_{i=0}^{n} p_i x q_{n-i} \tag{1}$$

for all $x \in \mathcal{A}$ and each non-negative integer n, is called an *inner higher derivation* on $\mathcal{A}$, in which $\mathbf{p} = \{p_n\}_{n=0}^{\infty}$ and $\mathbf{q} = \{q_n\}_{n=0}^{\infty}$ are two sequences in $\mathcal{A}$ satisfying the conditions $p_0 = q_0 = 1_{\mathcal{A}}$,

$$(\mathbf{p} * \mathbf{q})_n = \sum_{i=0}^{n} p_i q_{n-i} = 0, \quad (\mathbf{q} * \mathbf{p})_n = \sum_{i=0}^{n} q_i p_{n-i} = 0$$

for all $n \in \mathbb{N}$ (to see this, refer to [14,15]). Nowicki proved in [13] that if every derivation of $\mathcal{A}$ is inner, then every higher derivation of $\mathcal{A}$ is inner. Xu and Xiao proved in [15] that if every Jordan derivation of $\mathcal{A}$ is inner, then every Jordan higher derivation of $\mathcal{A}$ is also inner.

In this paper, we show that if $\mathbf{d} = \{d_n\}_{n=0}^{\infty}$ is an inner higher derivation on $\mathcal{A}$ of the form (1), then the corresponding sequence of derivations $\boldsymbol{\delta} = \{\delta_n\}_{n=1}^{\infty}$ defined on $\mathcal{A}$ by

$$\delta_n(x) = \sum_{i=1}^{n} \Big(\sum_{\sum_{j=1}^{i} r_j = n} (-1)^{i-1} \ r_1 p_{r_1} p_{r_2} \ldots p_{r_i} \Big) x + x \sum_{i=1}^{n} \Big(\sum_{\sum_{j=1}^{i} r_j = n} (-1)^{i-1} \ r_i q_{r_1} q_{r_2} \ldots q_{r_i} \Big)$$

for all $n \in \mathbb{N}$.

In Section 3, we investigate that for every two higher derivations $\mathbf{a} = \{a_n\}_{n=0}^{\infty}$ and $\mathbf{b} = \{b_n\}_{n=0}^{\infty}$ on algebra $\mathcal{A}$, the sequence $\mathbf{a} * \mathbf{b} = \{(a * b)_n\}_{n=0}^{\infty}$ defined by

$$(a * b)_n(x) = \sum_{i=0}^{n} a_i(b_{n-i}(x))$$

for each $n = 0, 1, 2, \ldots$ and $x \in \mathcal{A}$, is a higher derivation on $\mathcal{A}$. Also, we prove that if $\mathbf{a}$ and $\mathbf{b}$ are two inner higher derivations, then $\mathbf{a} * \mathbf{b}$ is an inner higher derivation.

For a discussion about derivations, Jordan derivations, higher derivations, Jordan higher derivations and their generalizations the reader is referred to [1–8] and [9].

2. On characterization of higher derivations

Throughout the paper, $\mathcal{A}$ denotes an algebra over a field of characteristic zero, and I is the identity mapping on $\mathcal{A}$. Let $\mathbf{d} = \{d_n\}_{n=0}^{\infty}$ be a higher derivation on $\mathcal{A}$ and $\boldsymbol{\delta} = \{\delta_n\}_{n=1}^{\infty}$ be the corresponding sequence of derivations to $\mathbf{d}$. In the next proposition, we obtain a relation that calculates each derivation $\delta_n (n \in \mathbb{N})$ directly as a linear combination of products of terms of the higher derivation $\mathbf{d}$.

Proposition 2.1. *Let $\mathbf{d} = \{d_n\}_{n=0}^{\infty}$ be a higher derivation on $\mathcal{A}$. Then the corresponding sequence of derivations to $\mathbf{d}$ denoted by $\boldsymbol{\delta} = \{\delta_n\}_{n=1}^{\infty}$ defined on $\mathcal{A}$ by*

$$\delta_n = \sum_{i=1}^{n} \Big(\sum_{\sum_{j=1}^{i} r_j = n} (-1)^{i-1} \; r_1 d_{r_1} d_{r_2} \ldots d_{r_i} \Big) \tag{2}$$

for all $n \in \mathbb{N}$.

Proof. By Proposition 2.1 of [11], for higher derivation $\mathbf{d} = \{d_n\}_{n=0}^{\infty}$ there exists a sequence of derivations $\boldsymbol{\delta} = \{\delta_n\}_{n=1}^{\infty}$ such that

$$(n+1)d_{n+1} = \sum_{k=0}^{n} \delta_{k+1} d_{n-k} \tag{3}$$

for each non-negative integer n. So, we get

$$\delta_{n+1} = (n+1)d_{n+1} - \sum_{k=0}^{n-1} \delta_{k+1} d_{n-k} \quad (n \in \mathbb{N}). \tag{4}$$

Now we use induction on n. For $n = 1$ we have $\delta_1 = d_1$. Suppose that δ_k is defined for $k \leq n$ as equation (2). For $n + 1$ the right side of equation (2) is equal to

$$\sum_{k=1}^{n+1}\Big(\sum_{\sum_{j=1}^{k} r_j=n+1}(-1)^{k-1}\ r_1 d_{r_1} d_{r_2}\dots d_{r_k}\Big)$$

$$= (n+1)d_{n+1} + \sum_{k=2}^{n+1}\Big(\sum_{\sum_{j=1}^{k} r_j=n+1}(-1)^{k-1}\ r_1 d_{r_1} d_{r_2}\dots d_{r_k}\Big)$$

$$= (n+1)d_{n+1} + \sum_{k=1}^{n}\Big(\sum_{\sum_{j=1}^{k+1} r_j=n+1}(-1)^{k}\ r_1 d_{r_1} d_{r_2}\dots d_{r_k} d_{r_{k+1}}\Big)$$

$$= (n+1)d_{n+1} - \sum_{k=1}^{n}\sum_{r_{k+1}=1}^{n-(k-1)}\Big(\sum_{\sum_{j=1}^{k} r_j=n+1-r_{k+1}}(-1)^{k-1}\ r_1 d_{r_1} d_{r_2}\dots d_{r_k}\Big)d_{r_{k+1}}$$

$$= (n+1)d_{n+1} - \sum_{k=1}^{n}\sum_{i=1}^{n-(k-1)}\Big(\sum_{\sum_{j=1}^{k} r_j=n+1-i}(-1)^{k-1}\ r_1 d_{r_1} d_{r_2}\dots d_{r_k}\Big)d_{i}$$

$$= (n+1)d_{n+1} - \sum_{i=1}^{n}\sum_{k=1}^{n-(i-1)}\Big(\sum_{\sum_{j=1}^{k} r_j=n-(i-1)}(-1)^{k-1}\ r_1 d_{r_1} d_{r_2}\dots d_{r_k}\Big)d_{i}$$

$$= (n+1)d_{n+1} - \sum_{i=1}^{n}\delta_{n-(i-1)}d_i$$

$$= (n+1)d_{n+1} - \sum_{i=0}^{n-1}\delta_{n-i}d_{i+1}$$

$$= (n+1)d_{n+1} - \sum_{k=0}^{n-1}\delta_{k+1}d_{n-k} = \delta_{n+1}.$$

This completes the proof. □

Example 2.2. *Using Proposition 2.1, the five terms of $\{\delta_n\}$ are*

$$\begin{aligned}
&\delta_1 = d_1,\\
&\delta_2 = 2d_2 - d_1^2,\\
&\delta_3 = 3d_3 - 2d_2d_1 - d_1d_2 + d_1^3,\\
&\delta_4 = 4d_4 - 3d_3d_1 - 2d_2^2 - d_1d_3 + 2d_2d_1^2 + d_1d_2d_1 + d_1^2d_2 - d_1^4,\\
&\delta_5 = 5d_5 - 4d_4d_1 - 3d_3d_2 - 2d_2d_3 - d_1d_4 + 3d_3d_1^2 + d_1d_3d_1 + d_1^2d_3 + 2d_2^2d_1\\
&\quad + 2d_2d_1d_2 + d_1d_2^2 - 2d_2d_1^3 - d_1d_2d_1^2 - d_1^2d_2d_1 - d_1^3d_2 + d_1^5.
\end{aligned}$$

Example 2.3. *Let $M_n(\mathbb{C})$ be the algebra of all $n \times n$ complex matrices and let $A \in M_n(\mathbb{C})$ be an arbitrary matrix. Define the sequences $\boldsymbol{P} = \{P_n\}_{n=0}^{\infty}$ and $\boldsymbol{Q} = \{Q_n\}_{n=0}^{\infty}$ in $M_n(\mathbb{C})$ by $P_n = \frac{(-1)^n}{n!}A^n$ and $Q_n = \frac{1}{n!}A^n$ for each non-negative integer n. Then $P_0 = Q_0 = I_n$ (the $n \times n$ identity matrix) and*

$$\begin{aligned}(\boldsymbol{P} * \boldsymbol{Q})_n &= \sum_{i=0}^{n} P_i Q_{n-i} = \sum_{i=0}^{n} \frac{(-1)^i}{i!(n-i)!} A^i A^{n-i} \\ &= \sum_{i=0}^{n} \frac{(-1)^i}{n!} \binom{n}{i} A^n = \frac{A^n}{n!} \sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0.\end{aligned}$$

Similarly, $(\boldsymbol{Q} * \boldsymbol{P})_n = \sum_{i=0}^{n} Q_i P_{n-i} = 0$.

Then the sequence of mappings $\mathbf{d} = \{d_n\}_{n=0}^{\infty}$ defined on $M_n(\mathbb{C})$ by $d_0 = I$ and

$$d_n(X) = \sum_{i=0}^{n} P_i X Q_{n-i} = \sum_{i=0}^{n} \frac{(-1)^i}{n!} \binom{n}{i} A^i X A^{n-i} \quad (n \in \mathbb{N}),$$

for all $X \in M_n(\mathbb{C})$, is an inner higher derivation on $M_n(\mathbb{C})$.

Now by Proposition 2.1, the sequence of derivations $\{\delta_n\}_{n=1}^{\infty}$ corresponding to $\mathbf{d}$, defined by $\delta_1(X) = XA - AX$ and $\delta_k = 0$ for all $k \geq 2$.

In the next proposition, we find the general form of the family of derivations corresponding to an inner higher derivation.

Proposition 2.4. *Let $\mathcal{A}$ be an associative algebra with identity element $1_{\mathcal{A}}$ and $\mathbf{d} = \{d_n\}_{n=0}^{\infty}$ be an inner higher derivation on $\mathcal{A}$ defined by*

$$d_n(x) = \sum_{i=0}^{n} p_i x q_{n-i}$$

in which $\boldsymbol{p} = \{p_n\}_{n=0}^{\infty}$ and $\boldsymbol{q} = \{q_n\}_{n=0}^{\infty}$ are sequences in $\mathcal{A}$ such that $p_0 = q_0 = 1_{\mathcal{A}}$ and $(\boldsymbol{p} * \boldsymbol{q})_n = (\boldsymbol{q} * \boldsymbol{p})_n = 0$ for all $n \in \mathbb{N}$. Then the corresponding sequence of derivations to $\mathbf{d}$ denoted by $\boldsymbol{\delta} = \{\delta_n\}_{n=1}^{\infty}$ defined on $\mathcal{A}$ by

$$\begin{aligned}\delta_n(x) = &\sum_{i=1}^{n} \Big(\sum_{\sum_{j=1}^{i} r_j = n} (-1)^{i-1}\ r_1 p_{r_1} p_{r_2} \ldots p_{r_i} \Big) x \\ &+ x \sum_{i=1}^{n} \Big(\sum_{\sum_{j=1}^{i} r_j = n} (-1)^{i-1}\ r_i q_{r_1} q_{r_2} \ldots q_{r_i} \Big)\end{aligned} \tag{5}$$

for all $n \in \mathbb{N}$.

Proof. Put

$$A_n = \sum_{i=1}^{n} \Big(\sum_{\sum_{j=1}^{i} r_j = n} (-1)^{i-1} \; r_1 p_{r_1} p_{r_2} \dots p_{r_i} \Big),$$

$$B_n = \sum_{i=1}^{n} \Big(\sum_{\sum_{j=1}^{i} r_j = n} (-1)^{i-1} \; r_i q_{r_1} q_{r_2} \dots q_{r_i} \Big).$$

Then $\delta_n(x) = A_n x + x B_n$. First we prove the following equations, using induction on n.

$$\text{(i)} \quad (n+1)p_{n+1} = \sum_{k=0}^{n} A_{k+1} p_{n-k},$$

$$\text{(ii)} \quad (n+1)q_{n+1} = \sum_{k=0}^{n} q_{n-k} B_{k+1}.$$

Proof of (i): For $n = 1$ we have $A_1 = p_1$. Suppose that A_k is defined for all $k = 1, 2, \dots, n$ as equation (i). Then for $n + 1$ we have

$$\begin{aligned}
A_{n+1} &= \sum_{k=1}^{n+1} \Big(\sum_{\sum_{j=1}^{k} r_j = n+1} (-1)^{k-1} \; r_1 p_{r_1} p_{r_2} \dots p_{r_k} \Big) \\
&= (n+1)p_{n+1} + \sum_{k=2}^{n+1} \Big(\sum_{\sum_{j=1}^{k} r_j = n+1} (-1)^{k-1} \; r_1 p_{r_1} p_{r_2} \dots p_{r_k} \Big) \\
&= (n+1)p_{n+1} + \sum_{k=1}^{n} \Big(\sum_{\sum_{j=1}^{k+1} r_j = n+1} (-1)^{k} \; r_1 p_{r_1} p_{r_2} \dots p_{r_k} p_{r_{k+1}} \Big) \\
&= (n+1)p_{n+1} - \sum_{k=1}^{n} \sum_{r_{k+1}=1}^{n-(k-1)} \Big(\sum_{\sum_{j=1}^{k} r_j = n+1-r_{k+1}} (-1)^{k-1} \; r_1 p_{r_1} p_{r_2} \dots p_{r_k} \Big) p_{r_{k+1}} \\
&= (n+1)p_{n+1} - \sum_{k=1}^{n} \sum_{i=1}^{n-(k-1)} \Big(\sum_{\sum_{j=1}^{k} r_j = n+1-i} (-1)^{k-1} \; r_1 p_{r_1} p_{r_2} \dots p_{r_k} \Big) p_i \\
&= (n+1)p_{n+1} - \sum_{i=1}^{n} \sum_{k=1}^{n-(i-1)} \Big(\sum_{\sum_{j=1}^{k} r_j = n-(i-1)} (-1)^{k-1} \; r_1 p_{r_1} p_{r_2} \dots p_{r_k} \Big) p_i \\
&= (n+1)p_{n+1} - \sum_{i=1}^{n} A_{n-(i-1)} p_i = (n+1)p_{n+1} - \sum_{i=0}^{n-1} A_{n-i} p_{i+1} \\
&= (n+1)p_{n+1} - \sum_{k=0}^{n-1} A_{k+1} p_{n-k}.
\end{aligned}$$

This proves the validity of equation (i).

Proof of (ii): For $n = 1$ we have $B_1 = q_1$. Suppose that B_k is defined for all $k = 1, 2, \ldots, n$ as equation (ii). Then for $n + 1$ we have

$$
\begin{aligned}
&B_{n+1}\\
&= \sum_{k=1}^{n+1} \Big(\sum_{\sum_{j=1}^{k} r_j = n+1} (-1)^{k-1}\ r_k q_{r_1} q_{r_2} \ldots q_{r_k} \Big)\\
&= (n+1)q_{n+1} + \sum_{k=2}^{n+1} \Big(\sum_{\sum_{j=1}^{k} r_j = n+1} (-1)^{k-1}\ r_k q_{r_1} q_{r_2} \ldots q_{r_k} \Big)\\
&= (n+1)q_{n+1} + \sum_{k=1}^{n} \Big(\sum_{\sum_{j=1}^{k+1} r_j = n+1} (-1)^{k}\ r_{k+1} q_{r_1} q_{r_2} \ldots q_{r_k} q_{r_{k+1}} \Big)\\
&= (n+1)q_{n+1} - \sum_{k=1}^{n} \Big(\sum_{r_1=1}^{n-(k-1)} q_{r_1} \sum_{\sum_{j=2}^{k+1} r_j = n+1-r_1} (-1)^{k-1}\ r_{k+1} q_{r_2} \ldots q_{r_{k+1}} \Big)\\
&= (n+1)q_{n+1} - \sum_{k=1}^{n} \Big(\sum_{i=1}^{n-(k-1)} q_i \sum_{\sum_{j=2}^{k+1} r_j = n+1-i} (-1)^{k-1}\ r_{k+1} q_{r_2} \ldots q_{r_{k+1}} \Big)\\
&= (n+1)q_{n+1} - \sum_{i=1}^{n} \Big(q_i \sum_{k=1}^{n-(i-1)} \Big(\sum_{\sum_{j=1}^{k} r_j = n-(i-1)} (-1)^{k-1}\ r_k q_{r_1} \ldots q_{r_k} \Big) \Big)\\
&= (n+1)q_{n+1} - \sum_{i=1}^{n} q_i B_{n-(i-1)}\\
&= (n+1)q_{n+1} - \sum_{i=0}^{n-1} q_{i+1} B_{n-i}\\
&= (n+1)q_{n+1} - \sum_{k=0}^{n-1} q_{n-k} B_{k+1}.
\end{aligned}
$$

This proves the validity of equation (ii). Now we have

$$
\begin{aligned}
\sum_{k=0}^{n} \delta_{k+1} d_{n-k}(x) &= \sum_{k=0}^{n} \delta_{k+1} \Big(\sum_{i=0}^{n-k} p_i x q_{n-k-i} \Big) = \sum_{k=0}^{n} \sum_{i=0}^{n-k} \delta_{k+1}(p_i x q_{n-k-i})\\
&= \sum_{k=0}^{n} \sum_{i=0}^{n-k} \Big(A_{k+1} p_i x q_{n-k-i} + p_i x q_{n-k-i} B_{k+1} \Big)\\
&= \sum_{k=0}^{n} \sum_{i=0}^{n-k} \Big(A_{k+1} p_i x q_{n-k-i} + p_{n-k-i} x q_i B_{k+1} \Big).
\end{aligned}
$$

In the summation $\sum_{k=0}^{n}\sum_{i=0}^{n-k}$, we have $0 \le k+i \le n$. Thus if we put $k+i = r$, then we can write it as the form $\sum_{r=0}^{n}\sum_{k+i=r}$. Putting $i = r - k$, we indeed have

$$
\begin{aligned}
&\sum_{k=0}^{n}\delta_{k+1}d_{n-k}(x)\\
&=\sum_{r=0}^{n}\sum_{k=0}^{r}\Big(A_{k+1}p_{r-k}xq_{n-r}+p_{n-r}xq_{r-k}B_{k+1}\Big)\\
&=\sum_{r=0}^{n}\Big((r+1)p_{r+1}xq_{n-r}+(r+1)p_{n-r}xq_{r+1}\Big)\\
&=\sum_{r=0}^{n}(r+1)p_{r+1}xq_{n-r}+\sum_{r=0}^{n}(r+1)p_{n-r}xq_{r+1}\\
&=(n+1)p_{n+1}x+\sum_{r=0}^{n-1}(r+1)p_{r+1}xq_{n-r}+(n+1)xq_{n+1}+\sum_{r=0}^{n-1}(r+1)p_{n-r}xq_{r+1}\\
&=(n+1)p_{n+1}x+\sum_{r=0}^{n-1}(r+1)p_{r+1}xq_{n-r}+(n+1)xq_{n+1}+\sum_{r=0}^{n-1}(n-r)p_{r+1}xq_{n-r}\\
&=(n+1)p_{n+1}x+(n+1)\sum_{r=0}^{n-1}p_{r+1}xq_{n-r}+(n+1)xq_{n+1}\\
&=(n+1)p_{n+1}x+(n+1)\sum_{r=1}^{n}p_{r}xq_{n+1-r}+(n+1)xq_{n+1}\\
&=(n+1)\sum_{r=0}^{n+1}p_{r}xq_{n+1-r}\\
&=(n+1)d_{n+1}(x).
\end{aligned}
$$

This completes the proof. □

Example 2.5. *Using Proposition 2.4, the four terms of sequence of derivations $\{\delta_n\}$ are defined on $\mathcal{A}$ as follows:*

$$
\begin{aligned}
\delta_1(x) &= p_1x + xq_1,\\
\delta_2(x) &= (2p_2 - p_1^2)x + x(2q_2 - q_1^2),\\
\delta_3(x) &= (3p_3 - 2p_2p_1 - p_1p_2 + p_1^3)x + x(3q_3 - q_2q_1 - 2q_1q_2 + q_1^3),\\
\delta_4(x) &= (4p_4 - 3p_3p_1 - 2p_2^2 - p_1p_3 + 2p_2p_1^2 + p_1p_2p_1 + p_1^2p_2 - p_1^4)x\\
&\quad + x(4q_4 - q_3q_1 - 2q_2^2 - 3q_1q_3 + q_2q_1^2 + q_1q_2q_1 + 2q_1^2q_2 - q_1^4).
\end{aligned}
$$

The next corollaries follows from Proposition 2.4.

Corollary 2.6. *Let $\mathcal{A}$ be an associative algebra with identity element $1_{\mathcal{A}}$. For every inner higher derivation $\mathbf{d} = \{d_n\}_{n=0}^{\infty}$ on $\mathcal{A}$, there exists a unique sequence of inner derivations $\boldsymbol{\delta} = \{\delta_n\}_{n=1}^{\infty}$ on $\mathcal{A}$ satisfying the equation (2).*

Corollary 2.7. *Let $\mathcal{A}$ be an associative algebra with identity element $1_{\mathcal{A}}$ and $\mathbf{d} = \{d_n\}_{n=0}^{\infty}$ be an inner higher derivation on $\mathcal{A}$ defined by*

$$d_n(x) = \sum_{i=0}^{n} p_i x q_{n-i}$$

in which $\boldsymbol{p} = \{p_n\}_{n=0}^{\infty}$ and $\boldsymbol{q} = \{q_n\}_{n=0}^{\infty}$ are sequences in $\mathcal{A}$ such that $p_0 = q_0 = 1_{\mathcal{A}}$ and $(\boldsymbol{p} * \boldsymbol{q})_n = (\boldsymbol{q} * \boldsymbol{p})_n = 0$ for all $n \in \mathbb{N}$. Then

$$\sum_{i=1}^{n} \Big(\sum_{\sum_{j=1}^{i} r_j = n} (-1)^{i-1} \left(r_1 p_{r_1} p_{r_2} \dots p_{r_i} + r_i q_{r_1} q_{r_2} \dots q_{r_i} \right) \Big) = 0$$

for all $n \in \mathbb{N}$.

Proof. It is an immediate consequence of the fact that $\delta_n(1_{\mathcal{A}}) = 0$ for all $n \in \mathbb{N}$. □

3. The product of higher derivations

In this section, we show that the product of two higher derivations is a higher derivation. Also, we show that the product of two inner higher derivations is an inner higher derivation. To prove the main results, we first need a lemma.

Lemma 3.1. *For any sequence $\{x_{i,k}\}_{i,k=0}^{n}$ in an algebra $\mathcal{A}$, we have*

$$\text{(i)} \qquad \sum_{i=0}^{n} \sum_{k=0}^{i} x_{i,k} = \sum_{i=0}^{n} \sum_{k=i}^{n} x_{k,i},$$

$$\text{(ii)} \qquad \sum_{i=0}^{n} \sum_{k=0}^{n-i} x_{i,k} = \sum_{k=0}^{n} \sum_{i=0}^{n-k} x_{i,k}.$$

Proof. (i) The right side of the equation is equal to

$$\sum_{i=0}^{n} \sum_{k=i}^{n} x_{k,i} = \sum_{i=0}^{n} \sum_{k=0}^{n-i} x_{k+i,i}.$$

In the summation $\sum_{i=0}^{n} \sum_{k=0}^{n-i}$, we have $0 \leq i+k \leq n$. Thus if we put $i+k = r$, then we can write it as the form $\sum_{r=0}^{n} \sum_{i+k=r}$. Putting $k = r - i$, we indeed have

$$\sum_{i=0}^{n} \sum_{k=0}^{n-i} x_{k+i,i} = \sum_{r=0}^{n} \sum_{i=0}^{r} x_{r,i}.$$

Now, renaming the indix r and i by i and k, respectively in the right side summation, we get the required result.

(ii) In the left side summation, we have $0 \leq i+k \leq n$. Thus if we put $i+k=r$, then we have

$$\sum_{i=0}^{n}\sum_{k=0}^{n-i} x_{i,k} = \sum_{r=0}^{n}\sum_{i=0}^{r} x_{i,r-i}.$$

Using Lemma 3.1 (i), we get

$$\sum_{r=0}^{n}\sum_{i=0}^{r} x_{i,r-i} = \sum_{i=0}^{n}\sum_{r=i}^{n} x_{r-i,i} = \sum_{i=0}^{n}\sum_{r=0}^{n-i} x_{r,i}.$$

Now, renaming the indix i and r by k and i, respectively on the right side summation, we get the required result. $\square$

Theorem 3.2. *Suppose that* $\boldsymbol{a} = \{a_n\}_{n=0}^{\infty}$ *and* $\boldsymbol{b} = \{b_n\}_{n=0}^{\infty}$ *are two higher derivations on* $\mathcal{A}$*.Then the sequence* $\boldsymbol{a} * \boldsymbol{b} = \{(a*b)_n\}_{n=0}^{\infty}$ *defined by*

$$(a*b)_n(x) = \sum_{i=0}^{n} a_i(b_{n-i}(x)) \tag{6}$$

for each $n = 0, 1, 2, \ldots$ *and* $x \in \mathcal{A}$*, is a higher derivation on* $\mathcal{A}$*.*

Proof. Trivially each $(a*b)_n$ is linear. Also for all $x, y \in \mathcal{A}$ we have

$$\begin{aligned}
(a*b)_n(xy) &= \sum_{i=0}^{n} a_i(b_{n-i}(xy)) \\
&= \sum_{i=0}^{n} a_i\Big(\sum_{j=0}^{n-i} b_j(x)b_{n-i-j}(y)\Big) = \sum_{i=0}^{n}\sum_{j=0}^{n-i} a_i(b_j(x)b_{n-i-j}(y)) \\
&= \sum_{i=0}^{n}\sum_{j=0}^{n-i}\sum_{k=0}^{i} a_k(b_j(x))a_{i-k}(b_{n-i-j}(y)) \\
&= \sum_{i=0}^{n}\sum_{k=0}^{i}\sum_{j=0}^{n-i} a_k(b_j(x))a_{i-k}(b_{n-i-j}(y)).
\end{aligned}$$

Using Lemma 3.1 (i), for sequence $\{x_{i,k}\} = \{\sum_{j=0}^{n-i} a_k(b_j(x))a_{i-k}(b_{n-i-j}(y))\}$, we conclude that

$$\begin{aligned}
(a*b)_n(xy) &= \sum_{i=0}^{n}\sum_{k=i}^{n}\sum_{j=0}^{n-k} a_i(b_j(x))a_{k-i}(b_{n-k-j}(y)) \\
&= \sum_{i=0}^{n}\sum_{k=0}^{n-i}\sum_{j=0}^{n-i-k} a_i(b_j(x))a_k(b_{n-k-i-j}(y)).
\end{aligned}$$

Using Lemma 3.1 (ii), we can write $\sum_{k=0}^{n-i}\sum_{j=0}^{n-i-k} x_{k,j} = \sum_{j=0}^{n-i}\sum_{k=0}^{n-i-j} x_{k,j}$, in which $x_{k,j} = a_i(b_j(x))a_k(b_{n-k-i-j}(y))$. Thus

$$(a*b)_n(xy) = \sum_{i=0}^{n}\sum_{j=0}^{n-i}\sum_{k=0}^{n-i-j} a_i(b_j(x))a_k(b_{n-k-i-j}(y)).$$

In the summation $\sum_{i=0}^{n}\sum_{j=0}^{n-i}$, we have $0 \le i+j \le n$. Thus if we put $i+j=r$, then we can write it as the form $\sum_{r=0}^{n}\sum_{i+j=r}$. Putting $j = r-i$, we indeed have

$$\begin{aligned}(a*b)_n(xy) &= \sum_{r=0}^{n}\sum_{i=0}^{r}\sum_{k=0}^{n-r} a_i(b_{r-i}(x))a_k(b_{n-r-k}(y))\\ &= \sum_{r=0}^{n}\Big(\sum_{i=0}^{r} a_i(b_{r-i}(x))\Big)\Big(\sum_{k=0}^{n-r} a_k(b_{n-r-k}(y))\Big)\\ &= \sum_{r=0}^{n}(a*b)_r(x)(a*b)_{n-r}(y).\end{aligned}$$

This shows that $\mathbf{a}*\mathbf{b} = \{(a*b)_n\}_{n=0}^{\infty}$ is a higher derivation and completes the proof. □

Corollary 3.3. *Let $\alpha, \beta : \mathcal{A} \to \mathcal{A}$ be two derivations. The sequence $\{d_n\}_{n=0}^{\infty}$ which is defined by*

$$d_0 = I, \quad d_n = \sum_{i=0}^{n} \frac{\alpha^i \beta^{n-i}}{i!(n-i)!} \quad (n \ge 1),$$

is a higher derivation on $\mathcal{A}$.

Proof. Suppose that $\mathbf{a} = \{a_n\}_{n=0}^{\infty}$ is the higher derivation corresponding to the sequence of derivations $\{\alpha_n\}_{n=1}^{\infty} = \{\alpha, 0, 0, \dots\}$ and $\mathbf{b} = \{b_n\}_{n=0}^{\infty}$ is the higher derivation corresponding to the sequence of derivations $\{\beta_n\}_{n=1}^{\infty} = \{\beta, 0, 0, \dots\}$. Then by Theorem 3.2, the sequence $\{(a*b)_n\}_{n=0}^{\infty}$ which is defined as above, is a higher derivation on $\mathcal{A}$. □

In the next theorem, we show that if $\mathbf{a}$ and $\mathbf{b}$ are two inner higher derivations, then $\mathbf{a}*\mathbf{b}$ is an inner higher derivation.

Theorem 3.4. *Let $\mathcal{A}$ be an associative algebra with identity element $1_{\mathcal{A}}$ and let $\boldsymbol{a} = \{a_n\}_{n=0}^{\infty}$ and $\boldsymbol{b} = \{b_n\}_{n=0}^{\infty}$ be two inner higher derivations on $\mathcal{A}$. Then $\boldsymbol{a}*\boldsymbol{b} = \{(a*b)_n\}_{n=0}^{\infty}$ is an inner higher derivation on $\mathcal{A}$.*

Proof. By hypothesis, there exist sequences $\mathbf{p} = \{p_n\}_{n=0}^{\infty}$, $\mathbf{q} = \{q_n\}_{n=0}^{\infty}$, $\mathbf{r} = \{r_n\}_{n=0}^{\infty}$ and $\mathbf{s} = \{s_n\}_{n=0}^{\infty}$ in $\mathcal{A}$ such that

$$a_n(x) = \sum_{i=0}^{n} p_i x q_{n-i}, \quad p_0 = q_0 = 1_{\mathcal{A}},\ (\mathbf{p} * \mathbf{q})_n = (\mathbf{q} * \mathbf{p})_n = 0,$$

$$b_n(x) = \sum_{i=0}^{n} r_i x s_{n-i}, \quad r_0 = s_0 = 1_{\mathcal{A}},\ (\mathbf{r} * \mathbf{s})_n = (\mathbf{s} * \mathbf{r})_n = 0,$$

for all $x \in \mathcal{A}$ and each non-negative integer n.

By Theorem 3.2, $\mathbf{a} * \mathbf{b}$ is a higher derivation on $\mathcal{A}$. We show that $\mathbf{a} * \mathbf{b}$ is an inner higher derivation. Using Lemma 3.1, we have

$$\begin{aligned}
(a * b)_n(x) &= \sum_{i=0}^{n} a_i(b_{n-i}(x)) \\
&= \sum_{i=0}^{n} \sum_{j=0}^{n-i} a_i(r_j x s_{n-i-j}) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} \sum_{k=0}^{i} p_k r_j x s_{n-i-j} q_{i-k} \\
&= \sum_{i=0}^{n} \sum_{k=0}^{i} \sum_{j=0}^{n-i} p_k r_j x s_{n-i-j} q_{i-k} = \sum_{i=0}^{n} \sum_{k=i}^{n} \sum_{j=0}^{n-k} p_i r_j x s_{n-k-j} q_{k-i} \\
&= \sum_{i=0}^{n} \sum_{k=0}^{n-i} \sum_{j=0}^{n-k-i} p_i r_j x s_{n-k-i-j} q_k = \sum_{i=0}^{n} \sum_{j=0}^{n-i} \sum_{k=0}^{n-i-j} p_i r_j x s_{n-k-i-j} q_k
\end{aligned}$$

for all $x \in \mathcal{A}$ and each non-negative integer n. If we put $i + j = m$, then we conclude that

$$\begin{aligned}
(a * b)_n(x) &= \sum_{m=0}^{n} \sum_{i=0}^{m} \sum_{k=0}^{n-m} p_i r_{m-i} x s_{n-m-k} q_k \\
&= \sum_{m=0}^{n} \big(\sum_{i=0}^{m} p_i r_{m-i}\big) x \big(\sum_{k=0}^{n-m} s_{n-m-k} q_k\big) \\
&= \sum_{m=0}^{n} (\mathbf{p} * \mathbf{r})_m x (\mathbf{s} * \mathbf{q})_{n-m}
\end{aligned}$$

for all $x \in \mathcal{A}$ and each non-negative integer n. Since $\mathbf{a}*\mathbf{b}$ is a higher derivation, we have

$$\big((\mathbf{p} * \mathbf{r}) * (\mathbf{s} * \mathbf{q})\big)_n = \big((\mathbf{s} * \mathbf{q}) * (\mathbf{p} * \mathbf{r})\big)_n = 0,$$

for all $n \in \mathbb{N}$ and also $(\mathbf{p} * \mathbf{r})_0 = (\mathbf{s} * \mathbf{q})_0 = 1_{\mathcal{A}}$. This completes the proof. □

4. Conclusion

In this paper, we obtained a relation that calculates each derivation $\delta_n (n \in \mathbb{N})$ directly as a linear combination of products of terms of the corresponding higher derivation $\{d_n\}_{n=0}^{\infty}$. Also, we found the general form of the family of inner derivations corresponding to an inner higher derivation. We showed that for every two higher derivations on an algebra $\mathcal{A}$, the product of them is a higher derivation on $\mathcal{A}$. Also, we proved that the product of two inner higher derivations, is an inner higher derivation.

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