Shahid Bahonar University of Kerman
Journal of Mahani Mathematical Research
Print ISSN: 2251-7952 Online ISSN: 2645-4505

A NOTE ON 2-PLECTIC VECTOR SPACES

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Article type: Research Article

(Received: 17 January 2023, Received in revised form 08 June 2023)

(Accepted: 15 October 2023, Published Online: 18 October 2023)

Abstract. Among the 2-plectic structures on vector spaces, the canonical ones and the 2-plectic structures induced by the Killing form on semisimple Lie algebras are more interesting. In this note, we show that the group of linear preservers of the canonical 2-plectic structure is noncompact and disconnected and its dimension is computed. Also, we show that the group of automorphisms of a compact semisimple Lie algebra preserving its 2-plectic structure, is compact. Furthermore, it is shown that the 2-plectic structure on a semisimple Lie algebra $\mathfrak{g}$ is canonical if and only if it has an abelian Lie subalgebra whose dimension satisfies in a special condition. As a consequence, we conclude that the 2-plectic structures induced by the Killing form on some important classical Lie algebras are not canonical.

Keywords: 2-plectic structure, Canonical 2-plectic structure, Semisimple Lie algebra.
2020 MSC: Primary 53Dxx, 53D05, 53D20, 11F22.

1. Introduction

A k-plectic manifold (M, ω) consists of a smooth manifold M and a differential $(k+1)$-form ω on M which is closed and for every $x \in M$, ω_x is a k-plectic structure on the tangent space T_xM, in the sense that

$$\iota_v \omega = 0 \Rightarrow v = 0, \qquad v \in T_xM,$$

where $\iota_v \omega$ is the contraction of ω by v.
Simply, a 1-plectic manifold is called a symplectic manifold. Symplectic geometry emerges from the Hamiltonian description of classical mechanics. In classical mechanics, the phase space of a point-particle (or a mechanical system) is a smooth manifold of even dimension. Each point of this manifold is a state of the system which it consists of the position and momentum of the particle in a specific time. The time evolution of the system is represented by a smooth curve in this manifold and it is a solution to an ordinary differential equation called " Hamilton's equation ". Every smooth function on the phase space is an observable of the system. To guarantee that the Hamilton's equation has a solution and to describe the time evolution of observables along the solutions, we need to equip the phase space with a symplectic form. Indeed,

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DOI: 10.22103/jmmr.2023.20889.1389

Publisher: Shahid Bahonar University of Kerman
How to cite: M. Shafiee, *A note on 2-plectic vector spaces*, J. Mahani Math. Res. 2024; 13(1): 443 - 455.

the symplectic form allows us to consider a Poisson bracket on the set of all smooth functions on the phase space (observables).
As the time evolution of a point-particle is a 1-dimensional object in a phase space, the time evolution of a string sweeps out a 2-dimensional object, called its world-sheet, in a spacetime manifold. In general, the time evolution of a $(k-1)$-dimensional physical object (or a $(k-1)$-brane) sweeps out a k-dimensional object, called its world-volume, in a spacetime manifold and physically it is described by k-dimensional field theory.
As we saw in above, symplectic manifolds can be used for phase spaces of 1-dimensional field theories and the symplectic form makes the set of observables into a Poisson algebra. Similarly, it has been shown that k-plectic manifolds can be used as "multiphase" spaces for k-dimensional field theories. This symplectic approach to field theory invented by DeDonder [3] and Weyle [12] in the 1930s. In this case, observables are special $(k-1)$-forms, called Hamiltonian forms, and $(k+1)$-plectic form makes them into a "Lie k- algebra" (for more details refer to [9]).
So, among the k-plectic structures, the "2-plectic" ones may be used to describe a classical string. In [10] and [11] some applications of 2-plectic geometry in string theory and $M2$-brane models has been provided.
While compact semisimple Lie groups can not admit a symplectic structure (because their second cohomology group is trivial), they admit a natural 2-plectic structure induced by their Killing form. So, the class of semisimple Lie groups is an important class of 2-plectic manifolds. In [1] it has been shown that the Lie 2-algebra associated to this 2-plectic Lie group is isomorphic to a **string Lie 2-algebra**.
Another important class of 2-plectic manifolds is the class of bundles of exterior 2-forms on manifolds, i.e, $\wedge^2 T^\star M$, for any arbitrary smooth manifold M.
We know that all symplectic vector spaces of the same dimension are symplectic isomorphic. Therfore, the Lie group $SP(V,\omega)$ of all symplectic isomorphisms on V, where (V,ω) is an arbitrary symplectic vector space of dimension $2n$, is isomorphic to the Lie group $SP(2n,\mathbb{R})$ containing all symplectic isomorphisms on $(\mathbb{R}^{2n},\omega_0)$, where ω_0 is the standard symplectic structure on $\mathbb{R}^{2n}$. The latter group is called the symplectic linear group, and it has been studied extensively. But in the 2-plectic case the situation is very different, i.e, a vector space may admit many differnt 2-plectic structures (see [6]) and hence, instead of one Lie group, we are facing non-isomorphic Lie groups associated to them. So, to the best of my knowledge, there is no study on the Lie groups associated to these 2-plectic structures. However, the linear counterparts of the two mentioned classes of 2-plectic manifolds, semisimple Lie algebras and $V \oplus \wedge^2 V^*$, where V is a real finite dimensional vector space, admit a spesific 2-plectic structure. So, it is valuable to study the Lie group associated to these 2-plectic structures. In this short paper, we obtain some easy results about these groups.
The structure of the paper is as follows. The second section is devoted to some basic definitions and proof of the result about the group of linear preservers of

the canonical 2-plectic structures. In section three we consider the linear reduction of 2-plectic structures. In the last section, we study the 2-plectic structure on semisimple Lie algebras induced by the Killing form. In particular it is shown that this structure is not canonical for some Lie algebras. Moreover, we prove that the group $Aut(\mathfrak{g}, \omega)$ of automorphisms of $\mathfrak{g}$, preserving this 2-plectic structure is compact when $\mathfrak{g}$ is compact.

2. 2-*plectic vector space*

Let V be a finite dimensional vector space. An exterior 3-form (or simply a 3-form) ω on V is called a 2-plectic structure, if ω is nondegenerate in the sense that $\iota_v\omega = 0$ if and only if $v = 0$. If ω is a 2-plectic structure on V, then the pair (V, ω) is called a 2-plectic vector space.
Suppose (V_1, ω_1) and (V_2, ω_2) are two 2-plectic vector spaces. A linear isomorphism $\psi : V_2 \to V_1$ is called a 2-plectic isomorphism, whenever $\psi^*\omega_1 = \omega_2$.

Two important examples of 2-plectic vector spaces are as follows:

Example 2.1. *([2]) Let V be any finite dimensional vector space. The 3-form ω on $\mathcal{V} = V \oplus \wedge^2 V^*$ defined by*

$$\omega((u, \alpha), (v, \beta), (w, \gamma)) = \alpha(v, w) - \beta(u, w) + \gamma(u, v)$$

is a 2-plectic structure on $\mathcal{V}$. This 2-plectic structure is called the ***canonical 2-plectic structure****.*

Let $V = \mathbf{R}^n$ and let $\{e_1, ..., e_n\}$ be the standard basis for $\mathbf{R}^n$ and $\{e^1, ..., e^n\}$ be the corresponding dual basis. Then $\{e_1, ..., e_n, e^i \wedge e^j : 1 \le i < j \le n\}$ is a basis for $\mathcal{V}$. Put $E_{ij} = e^i \wedge e^j$ and suppose E^{ij} is the dual of E_{ij}. Then it is easy to see that the 2-plectic structure ω reads

$$\omega = \sum_{i<j} E^{ij} \wedge e^i \wedge e^j.$$

Example 2.2. *([2]) Suppose $\mathfrak{g}$ is a semisimple Lie algebra and denote by K, the Killing form on $\mathfrak{g}$. Since $\mathfrak{g}$ is semisimple then K is nondegenerate. Moreover $K(X, [Y, Z]) = K([X, Y], Z)$, for all X, Y, Z in $\mathfrak{g}$. So the 3-form ω on $\mathfrak{g}$ defined by*

$$\omega(X, Y, Z) = K(X, [Y, Z])$$

is a 2-plectic structure. Let us call this 2-plectic structure the ***standard 2-pectic structure*** *on a semisimple Lie algebra.*

Suppose (V, ω) is a 2-plectic vector space and W is a subspace of V. Put

$$W^{\perp,1} = \{v \in V : \omega(v, w, .) = 0, for \quad all \quad w \in W\},$$

$$W^{\perp,2} = \{v \in V : \omega(v, w_1, w_2) = 0, for \quad all \quad w_1, w_2 \in W\}.$$

Definition 2.3. ([2]) For $l = 1, 2$, the subspace W is called l-isotropic if $W \subseteq W^{\perp,l}$; l-coisotropic if $W^{\perp,l} \subseteq W$ and l-Lagrangian if $W = W^{\perp,l}$.

Example 2.4. (1) *In Example* 2.1, $V = V \oplus 0$ *is a* 2*-Lagrangian subspace, while,* $\wedge^2 V^* = 0 \oplus \wedge^2 V^*$ *is a* 1*-Lagrangian subspace.*

(2) *If* (V, ω) *is a* 2*-plectic vector space, then every subspace of dimension* 1 *is* 1*-isotropic and every subspace of codimension* 1 *is* 2*-coisotropic. Morover any* 2*-dimensional subspace is* 2*-isotropic.*

(3) *If* (V, ω) *is a* 2*-plectic vector space, then a linear isomorphism* $\psi : V \to V$ *is a* 2*-plectic isomorphism if and only if its graph*

$$\Gamma_\psi = \{(v, \psi(v)) : v \in V\}$$

is a 2*-Lagrangian subspace of* $(V \times V, (-\omega) \times \omega)$.

We recall that the real symplectic group of degree $2n$, denoted by $SP(2n, \mathbf{R})$, is the group of $2n \times 2n$ matrices X satisfying $X^t J_0 X = J_0$, where

$$J_0 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

and I denotes the $n \times n$ identity matrix. The Lie algebra $sp(2n, \mathbf{R})$ of the Lie group $SP(2n, \mathbf{R})$ is given by the set of $2n \times 2n$ matrices X satisfying

$$X^t J_0 + J_0 X = 0.$$

The dimension of this Lie algebra is $n(2n + 1)$.

Now consider the Lie algebra $sp(4, \mathbf{R})$. The set $\mathcal{E} = \{e_1, ..., e_{10}\}$ is a basis for this Lie algebra where

$$e_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$e_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, e_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, e_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$e_7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, e_8 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, e_9 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$e_{10} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

After computing $[e_i, e_j]$, for $i, j = 1, ..., 10$ and using the fact that $K(X, Y) = (2n+2)tr(XY)$, for $X, Y \in sp(2n, \mathbf{R})$ one can see that the standard 2-plectic structure ω on $sp(4, \mathbf{R})$ reads

$$12\omega = e^1 \wedge e^2 \wedge e^5 + e^1 \wedge e^3 \wedge e^8 + e^1 \wedge e^4 \wedge e^9 + e^2 \wedge e^4 \wedge e^8$$

$$-e^2 \wedge e^5 \wedge e^6 + e^2 \wedge e^7 \wedge e^9 - e^3 \wedge e^5 \wedge e^9 - e^4 \wedge e^5 \wedge e^{10}$$

$$-e^4 \wedge e^8 \wedge e^9 - e^4 \wedge e^6 \wedge e^9 + e^6 \wedge e^7 \wedge e^{10},$$

where $\{e^1, ..., e^{10}\}$ is the corresponding dual basis.

Example 2.5. *The subspace $W = span\{e_1, ..., e_7\}$ is a subalgebra in $sp(4, \mathbf{R})$. It is easy to see that $W^{\perp,2} = span\{e_1 - e_6, e_3, e_4, e_7\}$. So W is a 2-coisotropic subspace.*

It is well known that if (V, ω) is a symplectic vector space and L is an arbitrary Lagrangian subspace of V, then (V, ω) is symplectic isomorphic to symplectic vector space $(L \oplus L^*, \Omega)$, where Ω is defined by

$$\Omega((u, \alpha), (v, \beta)) = \beta(u) - \alpha(v).$$

This means that all symplectic vector spaces of the same dimension look the same. But this is not true in the general multisymplectic case ([2], [6]). However since the 2-plectic structure appeared in Example 2.1 is a generalization of Ω, it is called the canonical 2-plectic vector space. In the following a 2-plectic structure will be called canonical if it is 2-plectic isomorphic to a canonical 2-plectic structure.

Suppose (V, ω) is a 2-plectic vector space and L is a subspace of V. Denote by $L^\perp \subset V^*$ the annihilator of L. The 3-form ω induces a linear map $\widehat{\omega} : V \to \wedge^2 V^*$ defined by $\widehat{\omega}(v) = \iota_v \omega$. If L is 1-Lagrangian, then $\widehat{\omega}(L) \subseteq \wedge^2 L^\perp$. The following lemma is a trivial useful result.

Lemma 2.6. *$\widehat{\omega}(L) = \wedge^2 L^\perp$ if and only if $dimL = \frac{N(N-1)}{2}$, where N is the codimension of L in V.*

Theorem 2.7. *Suppose L is a 1-Lagrangian subspace of (V, ω) with codimension N and $dimL = \frac{N(N-1)}{2}$. If W is a 1-isotropic subspace of V with*

$$dimW > \frac{(N-1)(N-2)}{2} + 1,$$

then W is contained in L. In particular, L is unique.

Proof. At first consider two linearly independent vectors v_1 and v_2 in V such that $span\{v_1, v_2\} \cap L = 0$. Suppose v_1^* and v_2^* are dual of v_1 and v_2, respectively. Thus $v_1^* \wedge v_2^* \in \wedge^2 L^\perp$, and hence by previous Lemma there is a vector $u \in L$ such that $\iota_u \omega = v_1^* \wedge v_2^*$. So $\omega(u, v_1, v_2) = v_1^* \wedge v_2^*(v_1, v_2) = 1 \neq 0$. This means that $span\{v_1, v_2\}$ can not be 1-isotropic. Now, we can conclude that if W is 1-isotropic, then the codimension of $W \cap L$ in W is at most 1, i.e, $dimW - dim(W \cap L) \leq 1$. So $dim(W + L) - dimL \leq 1$. If the equality holds, then $W + L$ has codimension $N - 1$ in V, $dim(\wedge^2(W + L)^\perp) = \frac{(N-1)(N-2)}{2}$, whereas

$$dim(W \cap L) \geq dimW - 1 > \frac{(N-1)(N-2)}{2},$$

which is a contradiction. Since $\widehat{\omega}_{|W\cap L} : W \cap L \to \wedge^2(W+L)^\perp$ is injective. □

We recall that if $\psi : V \to V$ is a linear transformation and α is a p-form on V, then $\psi^*\alpha$ is a p-form on V defined by

$$\psi^*\alpha(v_1, ..., v_p) = \alpha(\psi(v_1), ..., \psi(v_p)),$$

for all $v_1, ..., v_p$ in V. For simplicity the p-form $(\psi^{-1})^*\alpha$ is denote by $\psi_*\alpha$. Let $SP_2(n, \omega)$ denote the group of all 2-plectic automorphisms of $(\mathcal{V}, \omega)$, where $V = \mathbf{R}^n$.

Theorem 2.8. *If* $\psi = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *is an element of* $SP_2(n, \omega)$*, where* A*,* B*,* C*, and* D *are* $n \times n$*,* $n \times (\frac{n(n-1)}{2})$*,* $(\frac{n(n-1)}{2}) \times n$*, and* $(\frac{n(n-1)}{2}) \times (\frac{n(n-1)}{2})$ *real matrices, respectively, then*

(1) $B = 0$*,*

(2) $GL(n, \mathbf{R}) \subset SP_2(n, \omega)$ *and* $D = A_*$*. In particular* $SP_2(n, \omega)$ *is not compact.*

(3) *For* $n > 2$*,* $SP_2(n, \omega)$ *is disconnected and its dimension is given by*

$$\dim(SP_2(n, \omega)) = \frac{1}{3} n(n^2 + 3n - 1).$$

Proof. (1) By Example 2.4, $L = \wedge^2 V^*$ is a 1-Lagrangian subspace and since ψ is a 2-plectic isomorphism, then $\psi(L)$ is also a 1-Lagrangian subspace of the same dimension. But Theorem 2.7 says that such a 1-Lagrangian subspace is unique. So, $\psi(L) = L$. This shows that $B = 0$.

(2) Let $A \in GL(n, \mathbf{R})$. Then it is easy to see that $\psi = \begin{pmatrix} A & 0 \\ 0 & A_* \end{pmatrix}$ is an element of $SP_2(n, \omega)$. It proves the first statement and in particular it shows that $SP_2(n, \omega)$ is not compact. To prove that $D = A_*$, at first we prove that if $\varphi = \begin{pmatrix} I & 0 \\ B & C \end{pmatrix}$ is an element of $SP_2(n, \omega)$, then $C = I$, where I denotes the

identity. Rename E_{ij}s by $f_1, f_2, f_3, ...$, it is easy to see that

$$\varphi^* e^j = e^j,$$

$$\varphi^*(E^{lk}) = \varphi^* f^i = \sum b_{ij} e^j + \sum c_{ij} f^j.$$

So,

$$\varphi^*(E^{lk} \wedge e^l \wedge e^k) = (\sum b_{ij} e^j + \sum c_{ij} f^j) \wedge e^l \wedge e^k.$$

Hence the equation $\varphi^*\omega = \omega$ shows that $c_{ii} = 1$ and $c_{ij} = 0$ for $i \neq j$. Thus $C = I$. Now, since $\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$ and $\begin{pmatrix} A^{-1} & o \\ 0 & A^* \end{pmatrix}$ are elements of $SP_2(n, \omega)$, then their product $\begin{pmatrix} I & 0 \\ CA^{-1} & DA^* \end{pmatrix}$ is also an element of $SP_2(n, \omega)$. Therefore $D = A_*$.

(3) An element $\begin{pmatrix} A & 0 \\ B & A_* \end{pmatrix}$ in $SP_2(n, \omega)$ decomposes uniquely as

$$\begin{pmatrix} A & 0 \\ B & A_* \end{pmatrix} = \begin{pmatrix} I & 0 \\ BA^{-1} & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & A_* \end{pmatrix}.$$

So, $SP_2(n, \omega)$ is isomorphic to $GL(n, \mathbf{R}) \oplus H(n, \omega)$, where $H(n, \omega) \subset SP_2(n, \omega)$ consists all matrices of the form $\begin{pmatrix} I & 0 \\ B & I \end{pmatrix}$. Since $GL(n, \mathbf{R})$ is disconnected, then $SP_2(n, \omega)$ is disconnected. Furthermore, this shows that to prove the last assertion, it is enough to calculate the dimension of $H(n, \omega)$. So, let $\varphi = \begin{pmatrix} I & 0 \\ B & I \end{pmatrix}$ be an element of $H(n, \omega)$. Then $\varphi^* e^i = e^i$, $i = 1, ..., n$, and

$$\varphi^*(E^{lk}) = \varphi^*(f^j) = f^j + \sum_{i=1}^{n} b_{ij} e^i.$$

Thus

$$\begin{aligned} \varphi^*(E^{lk} \wedge e^l \wedge e^k) &= (E^{lk} + \textstyle\sum_{i=1}^n b_{ij} e^i) \wedge e^l \wedge e^k \\ &= E^{lk} \wedge e^l \wedge e^k + \textstyle\sum_{i=1}^n b_{ij} e^i \wedge e^l \wedge e^k, \end{aligned}$$

and hence

$$\varphi^*\omega = \omega + \sum_{l<k} b_{ij} e^i \wedge e^l \wedge e^k.$$

The last term has to be zero. So, for l fixed and $k > l$ one can obtain $n - k$ different equations. Thus we have $\sum_{k=1}^{n-2} k(n-k)$ different equations and correspondence to each of them we have to remove a parameter. Hence all free parameters are equal to

$$n^2 + \frac{1}{2} n^2 (n-1) - \sum_{k=1}^{n-2} k(n-k-1) = \frac{1}{3} n(n^2 + 3n - 1).$$

□

Note that for $n = 2$ the dimension of $\mathcal{V}$ is 3. Thus ω is a volume form and hence the group of its linear preservers is $SL(2,\omega)$ which is connected and its dimension is 8.

The following important theorem has been proved in [2].

Theorem 2.9. *The 2-plectic vector space (V,ω) is canonical if and only if it has a 1-Lagrangian subspace L with codimension N such that* $\dim L = \frac{N(N-1)}{2}$.

3. Linear reduction

In symplectic geometry, there is a basic theory called the **symplectic reduction**. This theory formalizes the well-known classical fact that if a symmetry group of dimension d acts on a system, then the number of degrees of freedom of the system may be reduced by d. In classical mechanics, the phase space of the system is a symplectic manifold. Thus in geometric language it means that if a symmetry Lie group of dimension d acts on a symplectic manifold of dimension $2n$ in the Hamiltonian way, then one can construct a new symplectic manifold of dimension $2n - 2d$ (see [7]). In this procedure, the coisotropic submanifolds have crucial role. In multisymplectic case there is no such a theory. In the following we obtain some results concerning linear reduction of 2-plectic structures.

Theorem 3.1. *If W is a subspace of (V,ω) and $\dim(\frac{W}{W\cap W^{\perp,2}})$ equals 3 or greater than 4, then $\frac{W}{W\cap W^{\perp,2}}$ is a 2-plectic vector space. In particular if W is 2-coisotropic and $\dim(\frac{W}{W^{\perp,2}})$ equals 3 or greater than 4, then $\frac{W}{W^{\perp,2}}$ is a 2-plectic vector space.*

Proof. Denote $w + W\cap W^{\perp,2} \in \frac{W}{W\cap W^{\perp,2}}$ by $\bar{w}$. Now, define $\bar{\omega}$ on $\frac{W}{W\cap W^{\perp,2}}$ by

$$\bar{\omega}(\bar{w}_1,\bar{w}_2,\bar{w}_3) = \omega(w_1,w_2,w_3).$$

It is easy to see that $\bar{\omega}$ is well-defined. Moreover, if $\bar{\omega}(\bar{w}_1,\bar{w}_2,\bar{w}_3) = 0$, for all $\bar{w}_2,\bar{w}_3 \in \frac{W}{W\cap W^{\perp,2}}$, then $\omega(w_1,w_2,w_3) = 0$, for all $w_1,w_2 \in W$. So, $w_1 \in W\cap W^{\perp,2}$ and hence $\bar{w}_1 = 0$. This shows that $\bar{\omega}$ is a 2-plectic form. □

Proposition 3.2. (1) *If W is a hyperplane of a 2-plectic vector space (V,ω), then W is 2-coisotropic.*

(2) *If V is a vector space of dimension $n \geq 3$ and U is a k-dimensional subspace of V, then $\mathcal{U} = U\times\wedge^2V^*$ is a 2-coisotropic subspace of dimension* $k + \frac{n(n-1)}{2} - \frac{(n-k)(n+k-1)}{2}$.

Proof. The first statement has been proved in [2]. To prove the second statement choose a basis $\{e_1,...,e_n\}$ for V such that the first kth of them are a basis for U. Then it is easy to see that

$$\mathcal{U}^{\perp,2} = span(\{E_{ij} : k+1 \leq i < j \leq n\}) \cup \{E_{ij} : 1 \leq i \leq k, k+1 \leq j \leq n\}.$$

□

Example 3.3. *In the Example* 2.5*, we see that the subspace* $W = span\{e_1, ..., e_7\}$ *is a* 2*-coisotropic subspace of* $sp(4, \mathbf{R})$*. So,* $\frac{W}{W^{\perp,2}}$ *is a* 2*-plectic vector space. Its* 2*-plectic structure is canonical since it is* 3*-dimensional.*

4. 2-plectic structure on semisimple Lie algebras

As we saw in Example 2.5, the Killing form K on a semisimple Lie algebra $\mathfrak{g}$ induces the standard 2-plectic structure defined by

$$\omega(X, Y, Z) = K(X, [Y, Z]).$$

Suppose $\{e_1, ..., e_n\}$ is a basis for $\mathfrak{g}$ and $\{e^1, ..., e^n\}$ is the corresponding dual basis. If C^k_{ij} and K_{ij} denote the structure constants of $\mathfrak{g}$ and coefficients of K, respectively, with respect to this basis, ω reads

$$\omega = \sum K_{il} C^l_{jk} e^i \wedge e^j \wedge e^k.$$

In this section, at first we study some properties of the automorphisms of $\mathfrak{g}$ preserving this 2-plectic structure. Then we show that for some familiar semisimple Lie algebras, this structure is not canonical.

4.1. **The group** $Aut(\mathfrak{g}, \omega)$**.** We recall that a linear isomorphism on $\mathfrak{g}$ which preserves the Lie bracket is called an automorphism. Suppose G is a connected Lie group with Lie algebra $\mathfrak{g}$. For every $g \in G$, let $ad_g(x) = gxg^{-1}$ be the inner automorphism of G induced by g. Then

$$Ad_g = T_e ad_g,$$

the tangent map of ad_g at identity, is an automorphism of $\mathfrak{g}$ which is called an inner automorphism. The set of all such automorphisms of $\mathfrak{g}$ is denoted by $Int\mathfrak{g}$. It is well known that $Int\mathfrak{g}$ is a normal subgroup of automorphisms $Aut\mathfrak{g}$ of $\mathfrak{g}$ and elements of $Int\mathfrak{g}$ preserve the Killing form (see [8]). Let $Aut(\mathfrak{g}, \omega) \subset SP_2(\mathfrak{g}, \omega)$ be the automorphisms of $\mathfrak{g}$ preserving the standard 2-plectic structure ω.

Theorem 4.1. $Int\mathfrak{g}$ *is a normal subgroup of* $Aut(\mathfrak{g}, \omega)$*.*

Proof. It is enough to show that $Int\mathfrak{g} \subseteq Aut(\mathfrak{g}, \omega)$. Suppose $\psi \in Int\mathfrak{g}$. Since ψ preserves the Killing form K, then

$$\begin{aligned} \psi^*\omega(X, Y, Z) = \omega(\psi X, \psi Y, \psi Z) &= K(\psi X, [\psi Y, \psi Z]) \\ &= K(\psi X, \psi[Y, Z]) \\ &= \psi^* K(X, [Y, Z]) \\ &= K(X, [Y, Z]) \\ &= \omega(X, Y, Z), \end{aligned}$$

for $X, Y, Z \in \mathfrak{g}$. □

Let $Iso(K)$ denote the set of all linear isomorphisms of $\mathfrak{g}$ preserving the Killing form. When $\mathfrak{g}$ is semisimple, K is negative definite. So $Iso(K)$ is a compact Lie group.

Theorem 4.2. (1) *If $\psi \in Aut\mathfrak{g}$ then ψ preserves ω iff it preserves K.*

(2) *If $\psi : \mathfrak{g} \longrightarrow \mathfrak{g}$ is a linear isomorphism and ψ preserves ω and K, then it belongs to $Aut\mathfrak{g}$.*

Proof. (1) Since $\mathfrak{g}$ is semisimple, then $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. So for $Y \in \mathfrak{g}$, there are $Z, W \in \mathfrak{g}$ such that $Y = [Z, W]$. Now let ψ preserves ω, then

$$\begin{aligned} \psi^* K(X,Y) = K(\psi X, \psi Y,) &= K(\psi X, \psi[Z,W]) \\ &= K(\psi X, [\psi Z, \psi W]) \\ &= \psi^* \omega(X,Z,W) \\ &= \omega(X,Z,W) \\ &= K(X,[Y,Z]) \\ &= K(X,Y). \end{aligned}$$

The converse is trivial.
(2) Since $\psi^*\omega = \omega$ and $\psi^* K = K$, then

$$\begin{aligned} K(\psi X, [\psi Z, \psi W]) &= \psi^* \omega(X,Y,Z) \\ &= \omega(X,Y,Z) \\ &= K(X,[Y,Z]) \\ &= \omega(X,Z,W) \\ &= K(\psi X, \psi[Y,Z]), \end{aligned}$$

for all $X, Y, Z \in \mathfrak{g}$. So $\psi([Y,Z]) = [\psi Y, \psi Z]$, for all $Y, Z \in \mathfrak{g}$. □

If $Aut(\mathfrak{g}, K)$ denotes the set of all automorphisms of $\mathfrak{g}$ preserving K, then Theorem 4.2 shows that $Aut(\mathfrak{g}, \omega) = Aut(\mathfrak{g}, K)$. Indeed it shows that:

$$SP_2(\mathfrak{g}, \omega) \bigcap Iso(K) = Aut(\mathfrak{g}, \omega) = Aut(\mathfrak{g}, K)$$

Corollary 4.3. *If $\mathfrak{g}$ is compact, then $Aut(\mathfrak{g}, \omega)$ is compact.*

Proof. By the Theorems 4.1 and 4.2, $Aut(\mathfrak{g}, \omega) = Aut\mathfrak{g} \cap Iso(K)$. When $\mathfrak{g}$ is compact $Aut\mathfrak{g}$ is compact and hence $Aut(\mathfrak{g}, \omega)$ is compact. □

4.2. **Relation with canonical 2-plectic structures.** Theorem 2.9 gives a criterion for a 2-plectic structure to be canonical. Using this criterion, we propose a new criterion based on which one can determine whether the standard 2-plectic structure on a semisimple Lie algebra is canonical or not. As a conclusion, we show that on some familiar Lie algebras their standard 2-plectic structure is not canonical. Again let $\mathfrak{g}$ be a semisimple Lie algebra and ω be the standard 2-plectic structure on $\mathfrak{g}$.

Proposition 4.4. *Let L be a subspace of $\mathfrak{g}$ then:*

(1) *the space* $L^{\perp,1}$ *is the centralizer of* L*;*
(2) *if* L *is* 1*-Lagrangian, then* L *is an abelian subalgebra;*
(3) *every maximal abelian subalgebra of* $\mathfrak{g}$ *is a* 1*-Lagrangian subspace.*

Proof. It is enough to prove the first statement. By definition $X \in L^{\perp,1}$ if and only if $\omega(X,Y,Z) = 0$, for all $Y \in L$ and all $Z \in \mathfrak{g}$. Thus

$$K([X,Y],Z) = K(X,[Y,Z]) = 0,$$

for all $Y \in L$ and all $Z \in \mathfrak{g}$. Since K is nondegenerate, then $[X,Y] = 0$ for all $Y \in L$. □

The following theorem is a consequence of Theorems 2.9 and 4.4.

Theorem 4.5. *The standard* 2*-plectic structure on a semisimple lie algebra* $\mathfrak{g}$ *of dimension* n *is canonical if and only if it has an abelian Lie subalgebra of dimension* p *satisfying* $p = \frac{(n-p)(n-p-1)}{2}$.

Proof. If the 2-plectic structure on $\mathfrak{g}$ is canonical, then it is 2-plectic isomorphic to 2-plectic vector space $(\mathcal{V},\omega)$ mentioned in Example 2.1. So, V is a 1-Lagrangian subspace of $\mathfrak{g}$ of dimension p satisfying the condition $p = \frac{(n-p)(n-p-1)}{2}$. Proposition 4.4 says that this subspace is an abelian subalgebra. Conversely, if $\mathfrak{g}$ has a such subalgebra, Theorem 2.9 says that the 2-plectic structure on $\mathfrak{g}$ is canonical. □

Theorem 4.6. *The standard* 2*-plectic structure on* $su(n)$ *and* $sl(n,\mathbf{R})$ *for* $n \geq 3$ *is not canonical.*

Proof. Suppose L is a 1-Lagrangian subspace of $su(n)$. Then L is abelian. On the other hand it is well known that the dimension of the maximal abelian subalgebra of $su(n)$ is equal to $n-1$ (the same statement is true for $sl(n,\mathbf{R})$ (see [4]). Then dimension of L is at most $n-1$. Now an easy calculation shows that $dimL$ does not satisfy the equation in the previous Theorem. So the 2-plectic structure is not canonical. □

Theorem 4.7. *The standard* 2*-plectic structure on* $so(n)$*, for* $n \geq 4$*,* $sl(n,\mathbb{C})$ *and* $sp(2n,\mathbf{R})$ *is not canonical.*

Proof. It is well known that the dimension of the maximal abelian subalgebra of $so(n)$ and $sp(2n,\mathbf{R})$ are at most $\frac{n}{2}$ and $\frac{n(n+1)}{2}$, respectively (see [4], [5]). Also for $n = 2k$, the maximal abelian subalgebra of $sl(n,\mathbb{C})$ has dimension k^2 and it consists of all matrices of the form $\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$, in which A is an arbitrary k by k matrix. While for n odd, the Cartan subalgebra of $sl(n,\mathbb{C})$ consisting of all diagonal matrices, is a maximal abelian subalgebra of dimension $n-1$. Now an argument similar to the previous Theorem proves that the 2-plectic structure of these Lie algebras is not canonical. □

5. Conclusion

In spite of the symplectic case a vector space may accept non-equivalent 2-plectic structures and hence there are non-isomprphic Lie groups associated to these different 2-plectic structures. However, semisimple Lie algebras and vector spaces of the form $V \oplus \wedge^2 V^\star$, where V is an arbitrary vector space, admit a special 2-plectic structure which are called standard and canonical 2-plectic structure, respectively. So, the Lie groups of linear preservings of these two structures are special. Although these groups are not as important as the symplectic linear group, their study is interesting mathematically. In this paper we proved some results about these groups, but there are still intersting problems about these groups to be studied. For example, we should identify the fundemental groups assciated with these Lie groups, and whether these groups are simisimple or not.
At last we have to point out that so far we have not found a semisimple Lie algebra of dimension greater than 3 whose 2-plectic structure induced by the Killing form is canonical. So we conjecture that the similar results, mentioned in the last two Theorems, are true for all semisimple Lie algebras of dimension greater than 3.

6. Aknowledgement

I would like to thank Vahid Vahedi for helpful conversations.

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