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STRUCTURE OF FINITE GROUPS WITH SOME WEAKLY S-SEMIPERMUTABLE SUBGROUPS

H. Jafarian Dehkordy ✉, G.R. Rezaeezadeh, and M. Bibak

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Abstract. Let G be a finite group. If $A \leq G$, recall that A is weakly S-semipermutable in G provided there is $K \trianglelefteq G$ such that KA is S-permutable in G, and $K \cap A$ is S-semipermutable in G. The purpose of this paper is to demonstrate that weakly S-semipermutability of special types of subgroups in a finite group G can help us to determine structural properties of G. For example, given a prime p, a p-soluble finite group G and a Sylow p-subgroup G_p of G, we will show that G is p-supersoluble if the maximal subgroups of G_p are weakly S-semipermutable in G. Moreover, we use the concept of weakly S-semipermutability to prove new criteria for p-nilpotency of finite groups.

Keywords: p-nilpotent, p-supersoluble, Weakly S-semipermutable.
2020 MSC: Primary 20F18, 20D15, 20F16, 20D20, 20D10.

1. Introduction

Throughout this paper, all groups are finite. An active research area in finite group theory is the study of subgroup embedding properties. A problem of particular interest is to study the structure of a group G under the assumption that some given subgroups of G satisfy a given embedding property. The symbol $\pi(n)$ denotes the set of all primes dividing the positive integer n; as usual, $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of G and G_p denotes a Sylow p-subgroup of G when $p \in \pi(G)$. Let G be a group and $A \leq G$. Recall that A is permutable (π-quasi-normal or S-permutable) in G if $AK = KA$ for all subgroups (Sylow subgroups) K of G. These concepts generalize the concept of a normal subgroup and were introduced by Ore [12] in 1939 (Kegel [6] in 1962); furthermore, they were investigated by many other authors. If K is a permutable (π-quasi-normal or S-permutable) subgroup of G, then K is a subnormal subgroup of G, by Ore [12] (Kegel [6]). For all permutable subgroups L of G, $L^G/L_G \subseteq Z_\infty(G/L_G)$ where $Z_\infty(G)$ is the hypercenter of G and L^G is the intersection of all normal subgroups N of G such that $L \leq N$, Maier and Schmid [10]. We consider $\mathfrak{U}$ to be the class of supersoluble groups. Recall that the $\mathfrak{U}$-hypercenter of a group G, denoted by

✉ h.jafarian.d@pnu.ac.ir, ORCID: 0000-0003-0282-9536
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$Z_{\mathfrak{U}}(G)$, is the product of all normal subgroups K of G such that all chief factors of G below K have prime order.

Let G be a group and $A \leq G$. Recall that A is S-semipermutable in G if for any $G_p \in Syl_p(G)$, where $(|A|, p) = 1$, we have $AG_p = G_pA$. It is clear that if $A \leq G$ is S-permutable in G, then A is S-semipermutable in G, but in general, the converse is not true. For example, let A be a Sylow 2-subgroup of S_3, the symmetric group of degree 3. Then A is S-semipermutable in S_3, but it is not S-permutable in S_3.

In [5], a new subgroup embedding property generalizing S-semipermutability was introduced, namely the concept of ν-permutability. We will recall the definition of this notion in Section 2. The purpose of this paper is to show that the ν-permutability of some subgroups of Sylow subgroups of a group G can help us to prove the p-supersolubility or p-nilpotency of G.

We mention that many other generalizations of S-semipermutability appear in the literature. For example, S.E. Mirdamadi and G.R. Rezaeezadeh [11] introduced the concept of SS-semipermutability, which not only generalizes S-semipermutability, but also SS-quasinormality.

2. Preliminaries

G.R. Rezaeezadeh and H. Jafarian Dehkordy [5] define the weakly S-semipermutable (ν-permutable) subgroups.

Definition 2.1. Let G be a group and $A \leq G$. Then A is said to be ***weakly S-semipermutable*** (ν-permutable) in G provided there is $K \trianglelefteq G$ such that KA is S-permutable in G and $K \cap A$ is S-semipermutable in G .

It is clear that if $K \leq G$ is S-semipermutable in G, then K is ν-permutable in G. However, the converse is not true. For instance, let K denote the subgroup $\langle (12) \rangle$ of S_4, the symmetric group of degree 4. Then K is easily seen to be ν-permutable in S_4, but K is not S-semipermutable in S_4.

Lemma 2.2. *[9, Lemma 2.1 (6)] Let G be a group, $p \in \pi(G)$ be a prime and A be a p-subgroup of G. Then A is S-permutable in G if and only if $O^p(G)$ normalizes A.*

Lemma 2.3. *[9, Lemma 2.2 (3)] Let G be a group and $p \in \pi(G)$ be a prime and $A \leq O_p(G)$. If A is S-semipermutable in G, then A is S-permutable in G.*

Lemma 2.4. *[3, Chapter III, Satz 5.2] Let G be a minimal non-nilpotent group. Then the following hold:*

1. *For some $p \in \pi(G)$, there exists $G_p \in Syl_p(G)$ such that $G_p \trianglelefteq G$ and $G = G_pQ$, where Q is a cyclic non-normal Sylow q-subgroup of G for some prime $q \neq p$.*
2. *If $p > 2$, then G_p has exponent p. If $p = 2$, then G_p has exponent 2 or 4.*
3. *If G_p is abelian, then G_p is elementary abelian.*

(4) $\Phi(G_p) \leq Z(G)$.
(5) $G_p/\Phi(G_p)$ *is a chief factor of* G.

Lemma 2.5. *[13, Lemma 2.2] Let* G *be a group and* $p \in \pi(G)$ *a prime with* $(|G|, p-1) = 1$*. Then the subsequent statements stand:*

(1) *If* $N \trianglelefteq G$ *and* $|N| = p$*, then* $N \leq Z(G)$.
(2) *If* G *has a Sylow* p*-subgroup such that it is cyclic, then* G *is* p*-nilpotent.*
(3) *If* $M \leq G$ *such that* $|G : M| = p$*, then* $M \trianglelefteq G$.

Lemma 2.6. *[5, Lemma 2.7] Let* G *be a group,* A *be a* ν*-permutable subgroup of* G*,* $N \trianglelefteq G$ *and* $L \leq G$*. Then the following hold:*

(1) *If* $A \leq L \leq G$*, then* A *is* ν*-permutable in* L.
(2) *If* $(|A|, |N|) = 1$*, then* AN/N *is* ν*-permutable in* G/N.
(3) *If for some prime* $p \in \pi(G)$*,* A *is a* p*-subgroup of* G*, then* AN/N *is* ν*-permutable in* G/N.

Lemma 2.7. *[5, Lemma 2.8] Let* $N \trianglelefteq G$ *be a minimal normal and elementary abelian subgroup. Then* N *has no nontrivial proper subgroup* K *such that any subgroup of* N *with order* $|K|$ *is* ν*-permutable in* G.

Lemma 2.8. *[2, Theorem 1.8.17] Let* G *be a group with* $\Phi(G) = 1$*. Then* $F(G)$ *is the direct product of all abelian minimal normal subgroups of* G*, where* $\Phi(G)$ *denotes the Frattini subgroup of* G *and* $F(G)$ *denotes the Fitting subgroup of* G.

Lemma 2.9. *[1, Chapter 1, Theorem 7.19] If* $K \trianglelefteq G$*, then* $K \leq Z_{\mathfrak{U}}(G)$ *if and only if* $K/\Phi(K) \leq Z_{\mathfrak{U}}(G/\Phi(K))$

Lemma 2.10. *[8, Lemma 2.4] Let* p *be a prime and* G *a group with* $(|G|, p-1) = 1$*. Suppose that* G_p *is a Sylow* p*-subgroup of* G *such that every maximal subgroup of* G_p *has a* p*-nilpotent supplement in* G*, then* G *is* p*-nilpotent.*

Theorem 2.11. *[5, Theorem 3.2] Let* G *be a group,* $p \in \pi(G)$ *with* $(|G|, p-1) = 1$ *and* G_p *is a Sylow* p*-subgroup of* G*. If every maximal subgroup of* G_p *is* ν*-permutable in* G*, then* G *is* p*-nilpotent.*

Theorem 2.12. *[5, Theorem 3.3] Let* G *be a group,* $p \in \pi(G)$ *with* $(|G|, p-1) = 1$ *and* G_p *be a Sylow* p*-subgroup of* G*. If every cyclic subgroup of* G_p *with order* p *or* 4 *(if* G_p *is a nonabelian 2-group) has a* p*-nilpotent supplement in* G *or is* ν*-permutable in* G*, then* G *is* p*-nilpotent.*

3. Main Results

Theorem 3.1. *Let* G *be a* p*-soluble group and* $G_p \in Syl_p(G)$ *where* $p \in \pi(G)$*. If each of the maximal subgroups of* G_p *is* ν*-permutable in* G*, then* G *is* p*-supersoluble.*

Proof. Assume that the theorem is false and consider a counterexample G with minimal order.

Step 1. $O_{p'}(G) = 1$.

Assume that $O_{p'}(G) \neq 1$, then $G/O_{p'}(G)$ is p-supersoluble by Lemma 2.6 (2) and the choice of G, so G is p-supersoluble. That is a contradiction.

Step 2. $N := O_p(G)$ is the unique minimal normal subgroup of G, and we have $\Phi(G) = 1$.

Let N be a minimal normal subgroup of G, then N is an abelian p-subgroup of G by Step 1. Hence $N \leq O_p(G) \leq G_p$. If $N = G_p$, then G/N is obviously p-supersoluble. If $N < G_p$, then G/N is p-supersoluble by Lemma 2.6 (3) and the choice of G. Since the class of all p-supersoluble groups is a saturated formation, it follows that N is the unique minimal normal subgroup of G and that $\Phi(G) = 1$. Lemma 2.8 shows that $O_p(G) = N$, and so $O_p(G)$ is the unique minimal normal subgroup of G.

Step 3. Final contradiction.

There is a maximal subgroup D of G such that $G = ND$, $N \cap D = 1$ by Step 2. Set $D_p := G_p \cap D$. Then $G_p = ND_p$. We have $D_p < G_p$ because otherwise $N \leq G_p = D_p \leq D$, which is contrary to the choice of D. Assume that G_p^* is a maximal subgroup of G_p with $D_p \leq G_p^*$. By hypothesis, G_p^* is ν-permutable in G. Hence, there is a normal subgroup K of G such that KG_p^* is S-permutable in G and such that $K \cap G_p^*$ is S-semipermutable in G. If $K = 1$, then G_p^* is S-permutable in G, and Lemma 2.2 implies that $G_p^* = N$, whence $G_p = ND_p = G_p^*$, a contradiction. So $K \neq 1$, and Step 2 implies that $N \leq K$. Then $N \cap G_p^* = N \cap (K \cap G_p^*)$ is easily seen to be S-semipermutable in G since N and $K \cap G_p^*$ are S-semipermutable in G. Lemma 2.3 implies that $N \cap G_p^*$ is S-permutable in G. Since $G_p \leq N_G(N \cap G_p^*)$, Lemma 2.2 now implies that $N \cap G_p^* \trianglelefteq G$. Then $N \cap G_p^* = 1$ or $N \cap G_p^* = N$. If $N \cap G_p^* = N$, then $N \leq G_p^*$ and $G_p = ND_p = G_p^*$. That is a contradiction. If $N \cap G_p^* = 1$, then $G_p = ND_p = NG_p^*$. Since $|G_p| = |N| \left|G_p^*\right| / \left|N \cap G_p^*\right| = |N| \left|G_p^*\right|$, we have $|N| = |G_p| / \left|G_p^*\right| = p$, and since G/N is p-supersoluble by the argumentation in Step 2, it follows that G is p-supersoluble. This contradiction completes the proof. $\square$

Theorem 3.2. *Let G be a group and $p \in \pi(G)$ with $(|G|, p-1) = 1$. Let G_p be a Sylow p-subgroup of G. Suppose that any maximal subgroup of G_p, that does not have a p-nilpotent supplement in G, is ν-permutable in G. Then G is p-nilpotent.*

Proof. Assume that the theorem is false, and consider a counterexample with minimal order.

Step 1. G has a unique minimal normal subgroup N, G/N is p-nilpotent, and we have $\Phi(G) = 1$.

Let N be a minimal normal subgroup of G. We show that G/N is p-nilpotent. Of course, this is the case when $|G/N|$ is not divisible by p. Therefore, we assume now that $|G/N|$ is divisible by p. Then G/N is a group with $p \in \pi(G/N)$

and $(|G/N|, p-1) = 1$. Also, G_pN/N is a Sylow p-subgroup of G/N. Considering the canonical group isomorphism $G_p/(G_p \cap N) \longrightarrow G_pN/N$, we see that any maximal subgroup of G_pN/N is the image of some maximal subgroup of G_p. So, by hypothesis and Lemma 2.6 (3), any maximal subgroup of G_pN/N is ν-permutable in G/N or has a p-nilpotent supplement in G/N. Consequently, the group G/N satisfies the hypotheses of the theorem and the minimality of G implies that G/N is p-nilpotent.
Since the class of p-nilpotent groups is a saturated formation, it follows that N is the only minimal normal subgroup of G and that $\Phi(G) = 1$.

Step 2. $O_{p'}(G) = 1$.
Assume that $O_{p'}(G) \neq 1$. Then it follows from Step 1 that $G/O_{p'}(G)$ is p-nilpotent. Thus G is p-nilpotent. That is a contradiction, and so we have $O_{p'}(G) = 1$.

Step 3. $O_p(G) = 1$.
Assume that $O_p(G) \neq 1$. Then $N \leq O_p(G)$. Since $\Phi(G) = 1$ by Step 1, there is a maximal subgroup M of G with $N \nleq M$. Then $G = NM$. We have $N \cap M \trianglelefteq M$. Since $N \leq O_p(G)$ is minimal normal in G, we have that N is abelian, and so we have $N \cap M \trianglelefteq N$. It follows that $N \cap M \trianglelefteq G$. Since N is minimal normal in G and $N \nleq M$, it follows that $N \cap M = 1$. Hence, $M \cong M/(N \cap M) \cong MN/N = G/N$, and so we see from Step 1 that M is p-nilpotent.
Now, let G_p^* be a maximal subgroup of G_p which is ν-permutable in G. We claim that G_p^* has a p-nilpotent supplement in G. We have $G_p^* \neq 1$ because otherwise G_p would have order p, and Burnside's p-nilpotency criterion would imply that G is p-nilpotent. Since G_p^* is ν-permutable in G, there is a normal subgroup K of G such that KG_p^* is S-permutable in G and such that $K \cap G_p^*$ is S-semipermutable in G. If $K = 1$, then G_p^* is S-permutable in G, and Lemma 2.2 shows that $G_p^* \neq 1$ is in fact normal in G. Then $N \leq G_p^*$ and hence $G = NM = G_p^*M$, so that M is a p-nilpotent supplement of G_p^* in G. If $K \neq 1$, then $N \leq K$ by Step 1, and we have $N \cap G_p^* = N \cap (K \cap G_p^*)$. Since N and $K \cap G_p^*$ are S-semipermutable in G, it is easy to see that $N \cap G_p^* = N \cap (K \cap G_p^*)$ is S-semipermutable in G. Then it follows from Lemma 2.3 that $N \cap G_p^*$ is S-permutable in G, and since $N \cap G_p^*$ is normal in G_p, it follows from Lemma 2.2 that $N \cap G_p^*$ is normal in G. The minimal normality of N in G implies that $N \cap G_p^* = 1$ or $N \cap G_p^* = N$. If $N \cap G_p^* = 1$, then $|N| = p$, and since G/N is p-nilpotent by Step 1, it follows that G is p-nilpotent, a contradiction. Thus $N \cap G_p^* = N$ and hence $N \leq G_p^*$. So we have $G = NM = G_p^*M$, so that M is a p-nilpotent supplement of G_p^* in G.
By the preceding paragraph and by hypothesis, any maximal subgroup of G_p has a p-nilpotent supplement in G. Lemma 2.10 implies that G is p-nilpotent. This contradiction shows that $O_p(G) = 1$.

Step 4. N is not solvable.
By Steps 2 and 3, we have $O_{p'}(N) \leq O_{p'}(G) = 1$ and $O_p(N) \leq O_p(G) = 1$.

Therefore, a minimal normal subgroup of N can neither be a p'-group nor a p-group. Thus, N cannot be p-solvable. In particular, N cannot be solvable.

Step 5. Final contradiction.

Because of Lemma 2.10, there is a maximal subgroup of G_p which does not have a p-nilpotent supplement in G, say G_p^*. By hypothesis, G_p^* is ν-permutable in G. Hence, there is a normal subgroup K of G such that KG_p^* is S-permutable in G and such that $K \cap G_p^*$ is S-semipermutable in G.

Assume that $K \cap G_p^* \neq 1$. Then $(K \cap G_p^*)^G$, the normal closure of $K \cap G_p^*$ in G, is nontrivial, whence $N \leq (K \cap G_p^*)^G$. Since $(K \cap G_p^*)^G$ is solvable by [4, Theorem A], it follows that N is solvable. This contradicts Step 4. Therefore, we have $K \cap G_p^* = 1$.

Since G_p^* is maximal in G_p, it follows that $K \cap G_p$ has order at most p. Since $K \cap G_p$ is a Sylow p-subgroup of K, it follows from Burnside's p-nilpotency criterion that K is p-nilpotent. As $O_{p'}(K) \leq O_{p'}(G) = 1$ by Step 2, it then follows that K is a p-group. Thus $K \leq O_p(G) = 1$ by Step 3.

Consequently, $G_p^* = KG_p^*$ is S-permutable in G, and Lemma 2.2 shows that G_p^* is in fact normal in G. It follows that $G_p^* = 1$ because otherwise $N \leq G_p^*$, so that N would be solvable, which is not true by Step 4. As $G_p^* = 1$, we have that G_p has order p. Now, Burnside's p-nilpotency contradiction implies that G is p-nilpotent. This contradiction completes the proof. □

Theorem 3.3. *Let G be a group, $L \trianglelefteq G$, and $L_p \in Syl_p(L)$, where $p \in \pi(L)$ with $(|L|, p-1) = 1$. Assume that each of the maximal subgroups of L_p, that has no p-supersoluble supplement in G, is ν-permutable in G. Then all chief factors of G between L and $O_{p'}(L)$ are cyclic.*

Proof. Assume that the theorem is false and consider a counterexample (G, L) for which $|G|\,|L|$ is minimal.

Step 1. L is p-nilpotent.

By hypothesis, any maximal subgroup of L_p is ν-permutable in G or has a p-supersoluble supplement in G. Using Lemma 2.6 (1), we conclude that any maximal subgroup of L_p is ν-permutable in L or has a p-supersoluble supplement in L. Since $(|L|, p-1) = 1$, any p-supersoluble supplement of a maximal subgroup of L_p is p-nilpotent. So Theorem 3.2 implies that L is p-nilpotent.

Step 2. $L_p = L$.

$O_{p'}(L) \in Hall_{p'}(L)$ by Step 1. Let $O_{p'}(L) \neq 1$, then all chief factors of $G/O_{p'}(L)$ between $L/O_{p'}(L)$ and $O_{p'}(L)/O_{p'}(L)$ are cyclic by Lemma 2.6 (2), and the choice of (G, L). So all chief factors of G between L and $O_{p'}(L)$ are cyclic. That is a contradiction.

Step 3. $\Phi(L_p) = 1$.

Let $\Phi(L_p) \neq 1$, then all chief factors of $G/\Phi(L_p)$ below $L/\Phi(L_p)$ are cyclic by Lemma 2.6 (3) and the choice of (G, L). So all chief factors of G below L are cyclic by Lemma 2.9. That is a contradiction.

Step 4. All of the maximal subgroups of L_p are ν-permutable in G.

Let M be a maximal subgroup of L_p and B be p-supersoluble supplement of M in G, then $G = MB = L_pB$ and $L_p \cap B \neq 1$. Since $L_p \cap B \lhd B$, there exists a minimal normal subgroup N of B with $N \leq L_p \cap B$. It is clear that $|N| = p$. Since L_p is an elementary abelian subgroup and $G = L_pB$, we have that $N \lhd G$. So all chief factors of G/N below L_p/N are cyclic by Lemma 2.6 (3) and by the choice of (G, L), then all chief factors of G below L_p are cyclic. That is a contradiction.

Step 5. L_p is not a minimal normal subgroup of G.

Suppose that L_p is a minimal normal subgroup of G. Since all maximal subgroups of L_p are ν-permutable in G by Step 4, we obtain a contradiction to Lemma 2.7.

Step 6. Let N be a minimal normal subgroup of G with $N \leq L_p$. Then $L_p/N \leq Z_{\mathfrak{U}}(G/N)$. Moreover, N is the only minimal normal subgroup of G contained in L_p, and we have $|N| > p$.

The hypothesis is still true for $(G/N, L_p/N)$ by Lemma 2.6 (3), so all chief factors of G/N below L_p/N are cyclic by the choice of (G, L). Hence $L_p/N \leq Z_{\mathfrak{U}}(G/N)$. If $|N| = p$, then all chief factors of G below L_p are cyclic. That is a contradiction. If N_1 is another minimal normal subgroups of G contained in L_p, then $N_1N/N_1 \leq L_p/N_1$, then $|N| = p$ by G-isomorphism $N_1N/N_1 \cong N$. That is a contradiction.

Step 7. Final contradiction.

Let N be a minimal normal subgroup of G with $N \leq L_p$. Since L_p is elementary abelian by Step 3, N has a complement in L_p, say S. Now, let R be a maximal subgroup of N which is normal in some Sylow p-subgroup of G. Set $A := RS$. Then A is maximal in L_p. Hence, A is ν-permutable in G, and so there is a normal subgroup K of G such that KA is S-permutable in G and such that $K \cap A$ is S-semipermutable in G. Assume that $K \cap L_p = 1$. Then $A = KA \cap L_p$ is S-permutable in G, which implies that $R = N \cap A$ is S-permutable in G. Since R is normal in a Sylow p-subgroup of G, it follows from Lemma 2.2 that R is normal in G. So we have $|N| = p$, which is a contradiction. Therefore, $1 \neq K \cap L_p \trianglelefteq G$. The previous step shows that $N \leq K \cap L_p$. Then $R = N \cap A = N \cap (K \cap A)$ is S-permutable in G, and since R is normal in a Sylow p-subgroup of G, it follows that R is normal in G. Thus $|N| = p$, which is a contradiction in completing the proof. □

Theorem 3.4. *Let G be a group and $L \trianglelefteq G$. Assume that, for each $p \in \pi(L)$ and each non-cyclic Sylow p-subgroup L_p of L, any maximal subgroup of L_p is ν-permutable in G or has a p-supersoluble supplement in G. Then all chief factors of G below L are cyclic.*

Proof. Assume that the theorem is false and consider a counterexample (G, L) for which $|G|\,|L|$ is minimal. Let q be the smallest prime divisor of $|L|$, then L is q-nilpotent by Lemma 2.6 (1), and Theorem 3.2. Assume that $L_{q'} \in Hall_{q'}(L)$. If $L_{q'} = 1$, then all chief factors below L are cyclic by Theorem 3.3. That is a contradiction. So we can assume that $L_{q'} \neq 1$, and we have $L_{q'} \lhd G$. So all chief factors of $G/L_{q'}$ below $L/L_{q'}$ are cyclic. On the other hand, all chief factors of G below $L_{q'}$ are cyclic by the choice of (G, L). Therefore it follows that all chief factors of G below L are cyclic. That is a contradiction. □

Theorem 3.5. *Let G be a group, $L \trianglelefteq G$ such that G/L is p-supersoluble, where $p \in \pi(L)$ with $(|L|, p-1) = 1$, and $L_p \in Syl_p(L)$. If each of the maximal subgroups of L_p is ν-permutable in G, then G is p-supersoluble.*

Proof. By Theorem 3.3, every chief factor between L and $O_{p'}(L)$ is cyclic. In particular, every chief factor between L and $O_{p'}(L)$ has order p or p'-order. Of course, any chief factor of G below $O_{p'}(L)$ has p'-order. Since G/L is p-supersoluble by hypothesis, we also have that any chief factor between G and L has order p or p'-order. Consequently, any chief factor of G has order p or p'-order. So G is p-supersoluble. □

Theorem 3.6. *Let G be a group, p be the smallest prime dividing the order of G, and $G_p \in Syl_p(G)$. If there exists a subgroup D of G_p with $1 < |D| < |G_p|$ such that all subgroups K of G_p with $|K| = |D|$ or $|K| = 2\,|D|$ (If G_p is a non-abelian 2-group) is ν-permutable in G, then G is p-nilpotent.*

Proof. We closely follow the proof of Theorem 3.2 in [7]. Assume that the theorem is false and consider a counterexample G with minimal order.

Step 1. $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, then $G/O_{p'}(G)$ is p-nilpotent by Lemma 2.6 (2) and the choice of G. Then G is p-nilpotent. That is a contradiction.

Step 2. $|D| > p$

Assume that $|D| = p$. Since G is not p-nilpotent, there is a minimal non-p-nilpotent subgroup G_1 of G. By Satz 5.4 in Chapter IV of [3], G_1 is minimal non-nilpotent. Then $G_1 = P_1 \rtimes Q$, where $P_1 \in Syl_p(G_1)$ and $Q \in Syl_q(G_1)$ for $q \neq p$ by Lemma 2.4. Let $x \in P_1 \setminus \Phi(P_1)$ and $E = \langle x \rangle$. Then $|E| = p$ or $|E| = 4$ by Lemma 2.4, where $|E| = 4$ is only possible when P_1 is a non-abelian 2-group. Hence E is ν-permutable in G, thus in G_1 by Lemma 2.6 (1). Since $x \in P_1 \setminus \Phi(P_1)$ was arbitrarily chosen, we have that $\langle x \rangle$ is ν-permutable in G_1 for every $x \in P_1 \setminus \Phi(P_1)$. Since $\Phi(P_1) \leq Z(G_1)$, we also have that $\langle x \rangle$ is ν-permutable in G_1 for any $x \in \Phi(P_1)$. Then G_1 is p-nilpotent by Theorem 2.12. This contradiction shows that $|D| > p$.

Step 3. $|G_p : D| > p$.

According to the previous content and Theorem 2.11, it is easy to see.

Step 4. If N is a minimal normal subgroup of G with $N \leq G_p$, then $|N| \leq |D|$.

Assume that $|N| > |D|$. Since N is minimal normal in G and a p-group, we have N is an elementary abelian. By hypothesis, every subgroup of N with order $|D|$ is ν-permutable in G. This is a contradiction to Lemma 2.7.

Step 5. Suppose that N is a minimal normal subgroup of G with $N \leq G_p$, then G/N is p-nilpotent.

If $|N| < |D|$, then G/N is p-nilpotent by Lemma 2.6 (3) and the choice of G. So $|N| = |D|$ by Step 4. Let $N \leq K \leq G_p$ and $|K/N| = p$. Since N is not cyclic by Step 2, every subgroup containing N is not cyclic. Hence there is a maximal subgroup $L \neq N$ of K such that $K = NL$. Of course $|N| = |D| = |L|$, thus L is ν-permutable in G. Then $K/N = LN/N$ is ν-permutable in G/N by Lemma 2.6 (3). If $p = 2$ and G_p/N is non-abelian, assume that X/N is a cyclic subgroup of G_p/N with $|X/N| = 4$. Since X is not cyclic and X/N is cyclic, there exists a maximal subgroup L of X such that N is not contained in L. Thus $X = LN$ and $|L| = 2\,|D|$, hence L is ν-permutable in G and $X/N = LN/N$ is ν-permutable in G/N. Then G/N is p-nilpotent by Theorem 2.12.

Step 6. $O_p(G) = 1$.

Assume that $O_p(G) \neq 1$. Let N be a minimal normal subgroup of G with $N \leq O_p(G)$, thus G/N is p-nilpotent by Step 5. Since the class of p-nilpotent groups is a formation, we see from the previous step that N is the only minimal normal subgroup of G contained in $O_p(G)$. Since $O_{p'}(G) = 1$, we have $\Phi(G) \leq O_p(G)$. Now, if $\Phi(G) \neq 1$, then it follows that $N \leq \Phi(G)$, whence $G/\Phi(G)$ and hence G is p-nilpotent. This contradiction shows that $\Phi(G) = 1$. Hence G has a maximal subgroup M such that $M \cap N = 1$ and $G = MN$. In particular, $M \cong G/N$ is p-nilpotent. Then $M = M_pM_{p'} = (M \cap G_p)M_{p'}$, where $M_{p'}$ is the normal p-complement of M. Let S be a maximal subgroup of $M_p = G_p \cap M$. Thus $NSM_{p'}$ is p-nilpotent by Step 3 and the choice of G, so G is p-nilpotent. That is a contradiction.

Step 7. Every minimal normal subgroup of G is not p-nilpotent.

Let L be a minimal normal subgroup of G such that L is p-nilpotent. Then $L_{p'} \leq O_{p'}(G) = 1$, thus L is a p-subgroup and therefore $L \leq O_p(G) = 1$ by Step 6. That is a contradiction.

Step 8. G is non-abelian simple group.

Assume that G is not a simple group, thus, there is non-trivial normal subgroup L of G. If $|L_p| > |D|$, then L is p-nilpotent by the choice of G. That is a contradiction to Step 7. If $|L_p| \leq |D|$, then there is $P^* \leq G_p$ such that $G_p \cap L \leq P^*$ and $|P^*| = p\,|D|$, hence $P^* \in Syl_p(P^*L)$. All maximal subgroups of P^* are ν-permutable in P^*L by Lemma 2.6 (1), then P^*L is p-nilpotent by Theorem 2.11, and therefore L is p-nilpotent. That is a contradiction to Step 7.

Step 9. Final contradiction.

Let H be a subgroup of G_p with order $|D|$. By hypothesis, H is ν-permutable in G. We show that H is S-semipermutable in G. Since H is ν-permutable in G, there is a normal subgroup T of G such that TH is S-permutable in G and such that $T \cap H$ is S-semipermutable in G. Since G is simple, we have $T = 1$ or $T = G$. If $T = 1$, then $H = TH$ is S-permutable and hence S-semipermutable in G, as wanted. Also, if $T = G$, then $H = H \cap T$ is S-semipermutable in G. Now, let Q be a Sylow q-subgroup of G for some $q \in \pi(G)$ with $q \neq p$. Then HQ is a subgroup of G since H is S-semipermutable in G. We have $G \neq HQ$ since G is non-abelian simple. Also, since H is S-semipermutable in G, we have $HQ^g = Q^gH$ for all $g \in G$. Applying Hilfssatz 4.10 of Chapter VI of [3], we conclude that H or Q is contained in a proper normal subgroup of G. This is a contradiction since G is simple, completing the proof. □

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Hassan Jafarian Dehkordy
Orcid number: 0000-0003-0282-9536
Department of Mathematics
Payme Noor University, P.O.Box 19395-4697
Tehran, Iran
Email address: h.jafarian.d@pnu.ac.ir

Gholamreza Rezaeezadeh
Orcid number: 0000-0002-0442-7269
Faculty of Mathematical Sciences
Shahrekord University
Shahrekord, Iran
Email address: rezaeezadeh@sci.sku.ac.ir

Masoumeh Bibak
Orcid number: 0000-0002-9479-7596
Department of Mathematics
Payme Noor University, P.O.Box 19395-4697
Tehran, Iran
Email address: m.bibak62@gmail.com