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SOME RESULTS ON COMMUTATIVE BI-ALGEBRAS

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Abstract. The notion of a (*branchwise*) *commutative* BI-*algebra* is presented, and some related properties are investigated. We show that the class of commutative BH-algebras is broader than the class of commutative BI-algebras. Moreover, we prove every *singular* BI-algebra is a BH-algebra. Also, we define the *commutative ideals* in BI-algebras and characterize the commutative BI-algebras in terms of commutative ideals.

Keywords: BI-algebra, (Branchwise) Commutative, Distributive, (Commutative) Ideal.
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1. Introduction

Abbott introduced implication algebra, which is a class of abstract algebras for formalizing the logical connective implication in the classical propositional logic [1]. Chen and Oliveira proved that in any implication algebra $(X;*)$, for all $x, y \in X$, the identity $x * x = y * y$ holds and is denoted by the constant 0 [5]. The concept of BCK-*algebras* was introduced by Imai and Iséki [9]. Tanaka introduced commutative BCK-algebras, which is an important class of BCK-algebras and forms a class of lower semilattices [21–23]. Meng proved that implication algebras are dual to *implicative* BCK-algebras [16]. Many interesting extensions of BCI/BCK-algebras were introduced by Iorgulescu, and the basic properties of such algebras are studied [10, 11]. Walendziak investigated the property of commutativity for various generalizations of BCK-algebras [24]. Borumand Saeid et al. introduced BI-*algebras* as a generalization of both an implicative BCK-algebra and a (dual) implication algebra, and they investigated some congruence relations and ideals. They proved that every implicative BCK-algebra is a BI-algebra, but the converse is not true in general [4]. Bandaru introduced the notion of a QI-*algebra*, which is an extension of a BI-algebra and discussed the relation between congruence kernels and ideals when a QI-algebra is distributive [3]. Ahn et al. investigated normal subalgebras in BI-algebras, by using an analytic method, and obtained various conditions for obtaining BI-algebra on the non-negative real numbers [2]. Rezaei and Smarandache [19] introduced the concepts of a Neutro-BI-algebras

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and Anti-BI-algebras. They showed that the class of Neutro-BI-algebras is an alternative of the class of BI-algebras. Niazian defined the notion of hyper BI-algebras as a generalization of BI-algebras and constructed quotient structure related to a (weak) ideal of a hyper BI-algebra [17]. Recently, Rezaei and Soleymani discussed the notions of independent and absorbent subsets of BI-algebras and investigated some of its properties [20].

In this paper, we introduce and study the notion of a (*branchwise*) *commutative BI-algebra* and show that commutative BI-algebras form a class of *lower semilattices.* We show that every commutative BI-algebra is a *commutative BH-algebra.* However, the converse is not valid in general and states that the class of commutative BH-algebras is a broader class than commutative BI-algebras, *dual implication algebras*, and *dual commutative Hilbert algebras.* A set of equivalent conditions is derived for a BI-algebra to become commutative. Some properties of the singular BI-algebras are studied. Also, the concept of the *commutative ideal* in a BI-algebra is introduced and showed that the *extension property* for the commutative ideals in right distributive $BI(A)$-algebras are valid, and also the zero ideal is commutative if and only if all ideals are commutative.

2. **Preliminaries notes**

In this section, we recall the basic definitions and some elementary aspects that we need for this paper.

Definition 2.1. [14] An algebra $(X, *, 0)$ of type $(2, 0)$ (i.e. a non-empty set with a constant 0 and binary operation $*$) is called a BCK-algebra if it satisfies the following axioms, for all $x, y, z \in X$:
(I_1) $((x * y) * (x * z)) * (z * y) = 0$,
(I_2) $x * 0 = x$,
(I_3) $x * y = 0$ and $y * x = 0$ imply $x = y$,
(I_4) $0 * x = 0$.

Hilbert algebras were defined in [6, 7]. Following the terminology of [7], we bring some definitions.

Definition 2.2. [7] A non-empty set X with a binary operation $\cdot$ and a constant 1 is said a *Hilbert algebra* if the following axioms hold, for all $x, y, z \in X$:
(HA_1) $x \cdot (y \cdot x) = 1$,
(HA_2) $(x \cdot (y \cdot z)) \cdot ((x \cdot y) \cdot (x \cdot z)) = 1$,
(HA_3) $x \cdot y = 1$ and $y \cdot x = 1$ imply $x = y$.

By [12], a Hilbert algebra $(X, \cdot, 1)$ is said to be *commutative* if it satisfies the condition (for all $x, y \in X$):

$$(x \cdot y) \cdot y = (y \cdot x) \cdot x.$$

Proposition 2.3. [7] *Every Hilbert algebra satisfies the following properties:*

(H_1) $x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z)$,
(H_2) $x \le y$ *implies* $y \cdot z \le x \cdot z$,
(H_3) $x \le y$ *implies* $z \cdot x \le z \cdot y$.

Definition 2.4. [7] An implication algebra $(X, \cdot, 1)$ is an algebra of type $(2, 0)$ satisfying the following axioms (for all $x, y, z \in X$):
(IA_1) $x \cdot x = 1$,
(IA_2) $(x \cdot y) \cdot x = x$,
(IA_3) $x \cdot (y \cdot z) = y \cdot (x \cdot z)$,
(IA_4) $(x \cdot y) \cdot y = (y \cdot x) \cdot x$.

Henkin introduced [8] the concept of a dual Hilbert algebra. In this paper, Hilbert algebras and implication algebras are used in a *dual* form, with the binary operation $*$ and one constant element 0. If put $1 := 0$ and $x \cdot y := y * x$, for all $x, y, z \in X$, then we have:
(DHA_1) $(x * y) * x = 0$,
(DHA_2) $((z * x) * (y * x)) * ((z * y) * x) = 0$,
(DHA_3) $x * y = 0$ and $y * x = 0$ imply $x = y$,
(DIA_1) $x * x = 0$,
(DIA_2) $x * (y * x) = x$,
(DIA_3) $(z * y) * x = (z * x) * y$,
(DIA_4) $x * (x * y) = y * (y * x)$.

A *dual Hilbert algebra* $(X, *, 0)$ is said to be *commutative* if it satisfies (DIA_4).

Corollary 2.5. [7] *Every commutative Hilbert algebra is an implication algebra.*

Definition 2.6. An algebra $(X, *, 0)$ of type $(2, 0)$ is called a

- BH-algebra if it satisfies (I_2), (I_3) and (DIA_1) ([13]).
- BI-*algebra* if satisfies (DIA_1) and (DIA_2) ([4]).

It was introduced a relation $\le$ on a BI-algebra $(X, *, 0)$ by $x \le y$ if and only if $x * y = 0$.

In what follows, let X denote a BI-algebra otherwise stated.

From [4] we have (for all $x, y, z, u \in X$):
(p_1) $x * 0 = x$,
(p_2) $0 * x = 0$,
(p_3) $x * y = (x * y) * y$,
(p_4) if $y * x = x$, then $X = \{0\}$,
(p_5) if $x * (y * z) = y * (x * z)$, then $X = \{0\}$,
(p_6) if $x * y = z$, then $z * y = z$ and $y * z = y$,
(p_7) if $(x * y) * (z * u) = (x * z) * (y * u)$, then $X = \{0\}$.

By routine calculation, we can see that (p_6) is equivalent to (DIA_2). For this, let $x * y = z$. Applying (DIA_2) twice, we get $y = y * (x * y) = y * z$ and $z * y = z * (y * z) = z$. Conversely, let (p_6) hold and $y * x = z$. Then $x * (y * x) = x * z = x$.

Moreover, it was proved that every implication commutative semigroup $(S, \leq, \cdot, *, 1)$ satisfies (p_5) (see [15, Th. 3.15]), and so it is a trivial BI-algebra.

Notice that if the binary operation $*$ is *associative* (i.e. if it satisfies the condition $x*(y*z) = (x*y)*z$, for all $x, y, z \in X$), then X is a trivial BI-algebra. Let $x \in X$. Applying (p_1), (p_2), associativity and (DIA_1) we have $x = x*0 = x*(0*x) = (x*0)*x = x*x = 0$. Hence $X = \{0\}$. Also, if X satisfies the condition $x*(y*z) = z*(y*x)$, then $X = \{0\}$. Since by (p_1) and (p_2), we have $x = x*0 = x*(0*0) = 0*(x*0) = 0$.

A BI-algebra X is called *right distributive* if the following condition holds for it (for all $x, y, z \in X$):

(I_5) $(x*y)*z = (x*z)*(y*z)$.

In any right distributive BI-algebra, we have (for all $x, y \in X$):

(p_8) $x*y \leq x$,

(p_9) $y*(y*x) \leq x$,

(p_{10}) if $x*y = x$, then $y*x = y$.

Let $X := \{0, x\}$. Then there is only one BI-algebra of order two as the following Cayley Table [18].

$*_1$	0	x
0	0	0
x	x	0

Further, let $X := \{0, x, y\}$. Then we have two following BI-algebras [18].

$*_2$	0	x	y
0	0	0	0
x	x	0	x
y	y	y	0

$*_3$	0	x	y
0	0	0	0
x	x	0	0
y	y	0	0

We can see that there are twenty six BI-algebras of order four, which divide into eight classes up to isomorphism with the following Tables [18].

$*_4$	0	x	y	z
0	0	0	0	0
x	x	0	0	0
y	y	0	0	0
z	z	0	0	0

$*_5$	0	x	y	z
0	0	0	0	0
x	x	0	0	0
y	y	0	0	y
z	z	0	z	0

$*_6$	0	x	y	z
0	0	0	0	0
x	x	0	0	x
y	y	z	0	0
z	z	z	0	0

$*_7$	0	x	y	z
0	0	0	0	0
x	x	0	0	x
y	y	z	0	x
z	z	z	0	0

$*_8$	0	x	y	z
0	0	0	0	0
x	x	0	0	x
y	y	0	0	y
z	z	z	z	0

$*_9$	0	x	y	z
0	0	0	0	0
x	x	0	0	x
y	y	z	0	y
z	z	z	z	0

$*_{10}$	0	x	y	z
0	0	0	0	0
x	x	0	x	x
y	y	y	0	x
z	z	z	x	0

$*_{11}$	0	x	y	z
0	0	0	0	0
x	x	0	x	x
y	y	y	0	y
z	z	z	z	0

3. On commutative BI-algebras

In this section, we generalize the concept of a commutative BCK-algebra to the case BI-algebra and study some of the properties. We show that every commutative BI-algebra is a commutative BH-algebra, but the converse is not true in general, and so the class of commutative BI-algebras is a subclass of commutative BH-algebras.

Definition 3.1. A BI-algebra X is called

- *commutative* if for all $a, b \in X$:
$$a * (a * b) = b * (b * a).$$
- *transitive* if for all $a, b, c \in X$:
$$a * b = 0 \text{ and } b * c = 0 \text{ imply } a * c = 0.$$

By [4], every right distributive BI-algebra is transitive.

Example 3.2. (i) Let $X := \{0, x, y, z\}$ with the binary operation "$*_7$". Then $(X, *_7, 0)$ is a commutative BI-algebra.

(ii) Consider the BI-algebra $(X, *_{10}, 0)$, where $X := \{0, x, y, z\}$. Then $(X, *_{10}, 0)$ is not commutative, since

$$y *_{10} (y *_{10} z) = y *_{10} x = y \neq z = z *_{10} x = z *_{10} (z *_{10} y).$$

(iii) Let X be a set with a constant 0. Define a binary operation $*_{12}$ on X as follows:

$$x *_{12} y = \begin{cases} 0 & \text{if } x = y; \\ x & \text{if } x \neq y. \end{cases}$$

Then $(X, *_{12}, 0)$ is a commutative BI-algebra.

(iv) Let $X = [0, \infty)$. Define a binary operation $*_{13}$ on X by

$$x *_{13} y = \begin{cases} x & \text{if } y = 0; \\ 0 & \text{if } y \neq 0. \end{cases}$$

Then $(X, *_{13}, 0)$ is not a commutative BI-algebra, since

$$4 *_{13} (4 *_{13} 5) = 4 *_{13} 0 = 4 \neq 5 *_{13} (5 *_{13} 4) = 5 *_{13} 0 = 5.$$

Let Λ be any set and, for each $i \in \Lambda$, let $X_i = (X_i, *_i, 0)$ be a BI-algebra. Suppose $X_i \cap X_j = \{0\}$, for $i \neq j$; $i, j \in \Lambda$. Set $X = \bigcup_{i\in\Lambda} X_i$ and define the binary operation $*_5$ on X by

$$x *_{14} y = \begin{cases} x *_i y & \text{if } x, y \in X_i;\ i \in \Lambda; \\ x & \text{if } x \in X_i, y \in X_j, i \neq j;\ i, j \in \Lambda. \end{cases}$$

Hence $(X, *_{14}, 0)$ is a BI-algebra. The algebra X will be called the *disjoint union* of $(X_i)_{i\in\Lambda}$.

In the next example, we show that the class of commutative and distributive BI-algebras are different. Also, we show that a BI-algebra can be distributive and commutative, simultaneously.

Example 3.3. (i) Let $X := \{0, x, y, z\}$ with the binary operation "$*_{11}$". Then $(X, *_{11}, 0)$ is a right distributive and commutative BI-algebra. Also, $(X, *_4, 0)$ is a distributive BI-algebra, since

$$x *_4 (x *_4 y) = x *_4 0 = x \neq y = y *_4 0 = y *_4 (y *_4 x),$$

Then $(X, *_4, 0)$ is not commutative.

(ii) Consider the transitive BI-algebra $(X, \diamond, 0)$, with $X := \{0, t, u, v\}$ and $\diamond$ is defined in the following table:

$\diamond$	0	t	u	v
0	0	0	0	0
t	t	0	0	t
u	u	v	0	u
v	v	v	v	0

Then $(X, *, 0)$ is not right distributive, since

$$(u \diamond t) \diamond v = v \diamond v = 0 \neq (u \diamond v) \diamond (t \diamond v) = u \diamond t = v.$$

Also, it is not commutative, since

$$t \diamond (t \diamond u) = t \diamond 0 = t \neq u \diamond (u \diamond t) = u \diamond v = u.$$

(iii) Let $X := \{0, x, y, z, t, u, v\}$ be a set with the following table.

$*_{15}$	0	x	y	z	t	u	v
0	0	0	0	0	0	0	0
x	x	0	x	x	x	x	x
y	y	y	0	y	y	y	y
z	z	z	z	0	z	z	z
t	t	t	t	t	0	0	t
u	u	u	u	u	v	0	u
v	v	v	v	v	v	v	0

Then $(X, *_{15}, 0)$ is the disjoint union of X_1 and X_2, and a BI-algebra. Since

$$(u *_{15} t) *_{15} u = v *_{15} u = v \neq 0 = 0 *_{15} 0 = (u *_{15} u) *_{15} (t *_{15} u),$$

$$u *_{15} (u *_{15} t) = u *_{15} v = u \neq t = t *_{15} 0 = t *_{15} (t *_{15} u),$$

it is neither commutative nor right distributive.

(iv) Let $X := \{0, x, y, z, t, u\}$ be a set with the following table.

$*_{16}$	0	x	y	z	t	u
0	0	0	0	0	0	0
x	x	0	0	x	x	0
y	y	t	0	0	x	y
z	z	z	u	0	z	y
t	t	t	0	t	0	0
u	u	t	u	0	x	0

Then $(X, *_{16}, 0)$ is a commutative BI-algebra, but not right distributive, since

$$(z *_{16} y) *_{16} x = u *_{16} x = t \neq (z *_{16} x) *_{16} (y *_{16} x) = z *_{16} t = z.$$

Proposition 3.4. *Let $(L, \vee, \wedge, \neg, 0, 1)$ be a Boolean lattice. Then $(L, *, 0)$ is a commutative BI-algebra, where $*$ is defined by $a * b = \neg b \wedge a$, for all $a, b \in L$.*

Proof. Let $(L, \vee, \wedge, \neg, 0, 1)$ be a Boolean lattice. Then $a * a = \neg a \wedge a = 0$ and so $(L, *)$, satisfies (DIA_1). Also $x*(y*x) = x*(\neg x \wedge y) = \neg(\neg x \wedge y) \wedge x = (x \vee \neg y) \wedge x = x$. Hence $(L, *)$ satisfies (DIA_2) and is a BI-algebra. We have

$$a*(a*b) = \neg(a*b) \wedge a = \neg(\neg b \wedge a) \wedge a = (b \vee \neg a) \wedge a = (a \wedge b) \vee (a \wedge \neg a) = a \wedge b.$$

On the other hand, by changing a with b we have $b * (b * a) = b \wedge a$. Since $a \wedge b = b \wedge a$, we get $a*(a*b) = b*(b*a)$. Therefore, $(L, *, 0)$ is a commutative BI-algebra. □

Proposition 3.5. *Let X be a commutative BI-algebra. Then*

(i) (I_3) *is valid,*
(ii) $x \leq y$ *implies* $x = y * (y * x)$,
(iii) *if* $x * y = y * x$, *then* $x = y$.

Proof. (i) Assume $x \leq y$ and $y \leq x$. Using (p_1) and commutative law, we have

$$x = x * 0 = x * (x * y) = y * (y * x) = y * 0 = y.$$

(ii) Assume $x, y \in X$ and $x \leq y$. Using (p_1) and commutative law, we obtain

$$x = x * 0 = x * (x * y) = y * (y * x).$$

(iii) Let $x * y = y * x$. Applying (DIA_2) and commutative law, we get

$$x = x * (y * x) = x * (x * y) = y * (y * x) = y * (x * y) = y.$$

□

Corollary 3.6. *Every commutative BI-algebra is a commutative BH-algebra.*

The following example shows that the converse of Corollary 3.6, is not valid.

Example 3.7. Let $X := \{0, x, y, z\}$ be a set with the following table.

$*_{17}$	0	x	y	z
0	0	y	z	x
x	x	0	z	z
y	y	y	0	y
z	z	x	x	0

Then $(X, *_{17}, 0)$ is a commutative BH-algebra, but not a BI-algebra, since

$$x *_{17} (y *_{17} x) = x *_{17} y = z \neq x.$$

Proposition 3.8. *Let X be a BI-algebra and $x, y \in X$. If X satisfies one of the following conditions:*

(i) $x * y = x$,
(ii) $x = (x * y) * (0 * y)$,
(iii) $x \leq x * y$,
(iv) $(x * z) * (x * y) = (y * z) * (y * x)$, *for all $x, y, z \in X$,*

then X is a commutative BI-algebra.

Proof. (i) Assume $x, y \in X$ and $x * y = x$. By (DIA_1), we have

$$x * (x * y) = x * x = 0 = y * y = y * (y * x).$$

(ii) The proof is obvious by (i) and (p_2).
(iii) The proof is obvious (since $x * (x * y) = 0 = y * (y * x)$).
(iv) Put $z = 0$ and applying (p_1), we have

$$x * (x * y) = (x * 0) * (x * y) = (y * 0) * (y * x) = y * (y * x). \qquad \square$$

Lemma 3.9. *Let X be a commutative BI-algebra, $0, x, y, z$ be distinct elements of X and $x * y = y * z = 0$. Then*

(i) $z * y \neq y * x$,
(ii) $z * y \notin \{0, x, y, z\}$ *and* $y * x \notin \{0, x, y, z\}$.

Proof. (i) Let $z * y = y * x$. Using commutative law, (p_1) and (BI), we get $y * (y * x) = x * (x * y) = x * 0 = x$ and $y * (y * x) = y * (z * y) = y$, which is a contradiction.

(ii). Assume $y * z = 0$. Then $z * (z * y) = y * (y * z) = y * 0 = y$, and so we have the following cases:
Case 1. If $z * y = 0$, then $y = y * 0 = y * (y * z) = z * (z * y) = z * 0 = z$,
Case 2. If $z * y = x$, then by (p_6), $x * y = x \neq 0$,
Case 3. If $z * y = y$, then by (DIA_1), $y * (z * y) = 0 \neq y$,
Case 4. If $z * y = z$, then $y = z * (z * y) = 0$,
which are contradiction. Therefore, $z * y \notin \{0, x, y, z\}$.
Similarly, $y * x \notin \{0, x, y, z\}$. $\square$

Corollary 3.10. *Let $X = \{0, x, y, z\}$, $(X, *, 0)$ be a commutative BI-algebra and $x \leq y$. Then $y \not\leq z$.*

Proof. Let $X = \{0, x, y, z\}$ be a commutative BI-algebra and $x \leq y$. On the contrary, if $y \leq z$, then $x * y = y * z = 0$. By Lemma 3.9 (ii), $z * y \notin X$, which is a contradiction. Therefore, $y \not\leq z$. □

The following theorem shows that if X is a commutative BI-algebra of order ≤ 5, then $(X, \leq)$ is a poset. The converse is not true in general. Consider the BI-algebra given in Example 3.2(ii). Then $(X, \leq)$ is a poset, but it is not commutative.

Theorem 3.11. *If X is a commutative BI-algebra and $|X| \leq 5$, then $\leq$ is transitive.*

Proof. Assume $X := \{0, x, y, z\}$, $x \leq y$ and $y \leq z$. Using Corollary 3.10, if $x \leq y$, then $y \not\leq z$. This shows that $\leq$ is transitive. Now, suppose $X = \{0, x, y, z, t\}$ is a commutative BI-algebra, $x \leq y$ and $y \leq z$. Using Lemma 3.9(ii), we have $z * y \notin \{0, x, y, z\}$ and $y * x \notin \{0, x, y, z\}$. It follows that $z * y = y * x = t$, which is a contradiction with the fact that $z * y \neq y * x$ (by Lemma 3.9(i)). Hence if $x \leq y$, then $y \not\leq z$, and so $\leq$ is transitive. □

In the following example, we show that every commutative BI-algebra is not transitive in general.

Example 3.12. Consider Example 3.3(iv), $(X, *_{16}, 0)$ is a commutative BI-algebra, but not transitive, since

$$x \leq y \text{ and } y \leq z, \text{ while } x \not\leq z.$$

Example 3.13. (i) Consider the commutative BI-algebra $(X, *_7, 0)$, it is easily seen that $\leq$ is a transitive relation on X.

(ii) Let X be a set with a constant 0. Consider Example 3.2(iii), $(X, *_{12}, 0)$ is a right distributive and commutative BI-algebra.

Borumand Saeid et al. [4], proved that any right distributive BI-algebra is transitive. In the following example, we show that the converse of [4, Pro. 3.14] and Theorem 3.9 are not valid in general.

Example 3.14. Let $X := \{0, x, y, z\}$ with the binary operation "$*_9$".Then $(X, *_9, 0)$ is a transitive BI-algebra, but not right distributive, since

$$(y *_9 x) *_9 z = z *_9 z = 0 \neq (y *_9 z) *_9 (x *_9 z) = y *_9 x = z.$$

Also, it is not commutative, since

$$x *_9 (x *_9 y) = x *_9 0 = x \neq y *_9 (y *_9 x) = y *_9 z = y.$$

Theorem 3.15. *Let X be a commutative BI-algebra. Then $(X, \leq)$ is a chain if and only if $|X| = 2$.*

Proof. Assume $X = \{0, a\}$. Using (p_2), we have $0 * a = 0$, and so $0 \leq a$.
Conversely, let $|X| > 2$. Then there are $x, y \in X \setminus \{0\}$ such that $x \neq y$. Since $(X; \leq)$ is a chain, we get $x < y$ or $y < x$. Without the loss of generality, let

$x < y$. Applying Proposition 3.5(ii), we obtain $x = y * (y * x)$. Now, consider the following cases:
Case 1. if $y * x = y$, then $x = y * (y * x) = y * y = 0$,
Case 2. if $y * x = x$, then $x = x * (y * x) = x * x = 0$,
Case 3. if $y * x = 0$, then $x = y$ by Proposition 3.5(i),
Case 4. if $y * x \notin \{0, x, y\}$, then there is $z \in X \setminus \{0, x, y\}$, such that $y * x = z$. By (p_6), $x * z = x$ and $z * x = z$. But by assumption, $(X; \leq)$ is a chain, we get $x < z$ or $z < x$, and so $x * z = 0$ or $z * x = 0$, which is a contradiction. Therefore, $|X| = 2$. □

The following, we first introduce the notion of *atoms* in BI-algebras and next study some of their properties. A non-zero element $a \in X$ is said to be an *atom* of X if for any $x \in X$, $x \leq a$ implies $a = x$ or $x = 0$. Let $\mathrm{A}(X)$ denote the set of all atoms of X. Further, X is said to be *singular* if every non-zero element of X is an atom of X (i.e. $\mathrm{A}(X) = X$). Obviously, every singular BI-algebra is transitive.

Example 3.16. (i) Consider the BI-algebra $(X, *_{13}, 0)$ given in Example 3.2(iv). It is not singular nor commutative, since

$$3 *_{13} (3 *_{13} 4) = 3 *_{13} 0 = 3 \neq 4 = 4 *_{13} 0 = (4 *_{13} 3),$$

also $3 *_{13} 4 = 0$, which means that $3 \leq 4$, but $3 \neq 4$ and $3 \neq 0$.

(ii) Consider the BI-algebra $(X, *_{12}, 0)$ given in Example 3.2(iii). It is singular, right distributive and commutative.

(iii) Let $X := \{0, x, y, z\}$ with the binary operation "$*_{10}$". Then $(X, *_{10}, 0)$ is a singular and transitive BI-algebra, but not right distributive, since

$$(y *_{10} x) *_{10} z = y *_{10} z = x \neq 0 = x *_{10} x = (y *_{10} z) *_{10} (x *_{10} z).$$

Also, it is not commutative, since

$$y *_{10} (y *_{10} z) = y *_{10} x = y \neq z = z *_{10} x = z *_{10} (z *_{10} y).$$

(iv) Let $X := \{0, x, y, z\}$ with the binary operation "$*_7$". Then $(X, *_7, 0)$ is a commutative and right distributive BI-algebra, but not singular, since $x \leq y$, while $x \neq 0$ and $x \neq y$.

(v) Let $X := \{0, x, y, z\}$ with the binary operation "$*_4$". Then $(X, *_4, 0)$ is a right distributive BI-algebra, but not singular, since $x \leq y$, but $x \neq 0$ and $x \neq y$. Also, it is not commutative, since

$$x *_4 (x *_4 z) = x *_4 0 = x \neq z *_4 (z *_4 x) = z *_4 0 = z.$$

(vi) Let $X := \{0, x, y, z\}$ with the binary operation "$*_9$". Then $(X, *_9, 0)$ is a BH-algebra, but not singular, since $x \leq y$, but $x \neq 0$ and $x \neq y$. Also, it is not a commutative BI-algebra, since

$$x *_9 (x *_9 y) = x *_9 0 = x \neq y *_9 (y *_9 x) = y *_9 z = y.$$

Further, X is not a right distributive, since

$$(y *_9 x) *_9 z = z *_9 z = 0 \neq (y *_9 z) *_9 (x *_9 z) = y *_9 x = z.$$

(vii) Let $X := \{0, x, y, z\}$ with the binary operation "$*_5$". Then $(X, *_5, 0)$ is a BI-algebra, but not singular, since $z \leq x$, but $z \neq 0$ and $z \neq x$. Also, it is not commutative, since

$$x *_5 (x *_5 y) = x *_5 0 = x \neq y *_5 (y *_5 x) = y *_5 0 = y.$$

Proposition 3.17. *Let X be a singular BI-algebra. Then*

(i) *$(X, \leq)$ is a poset,*
(ii) *X is a BH-algebra.*

Proof. Let X be a singular BI-algebra,
(i) Assume $x \leq y$ and $y \leq z$, for some $x, y, z \in X$. By defintion of atom, $x = y$ or $x = 0$. If $x = y$, then $x = y \leq z$. If $x = 0$, then $0 = x \leq z$. Therefore, $\leq$ is transitive. By (DIA_1), $\leq$ is reflexive .

Let $x \leq y$ and $y \leq x$. Thus $x*y = y*x = 0$. By definition of atom, $x = y$ or $x = 0$ or $y = 0$. If $y = 0$, then by (p_1), $0 = x * y = x * 0 = x$ and so $x = y = 0$. In a similar way, $x = 0$ implies $x = y = 0$. Hence $\leq$ is antisymmetric and is a poset.

(ii) By (p_2) and (i), the proof is clear. □

Proposition 3.18. *Let X be a right distributive BI-algebra and $a \in X$. Then*

(i) *$a \in \mathrm{A}(X)$ implies $x * (x * a) = a$ or $x * (x * a) = 0$, for any $x \in X$,*
(ii) *$a = x * (x * a)$, for all $x \in X$ implies $a \in \mathrm{A}(X)$.*

Proof. (i) Assume X is a BI-algebra and $a \in \mathrm{A}(X)$. Since for any $x \in X$, applying (p_9), we have $x * (x * a) \leq a$. Then $x * (x * a) = a$ or $x * (x * a) = 0$.

(ii) Suppose $a \in X$ satisfies for any $x \in X$, $a = x * (x * a)$. If $x \leq a$, then $x * a = 0$, and so $a = x * (x * a) = x * 0 = x$. Hence $a \in \mathrm{A}(X)$. □

Now, we recorded some definitions of respect to BI-algebra and show that commutative BI-algebras generalize properties of *commutative Hilbert algebras* and *implication algebras.*

Theorem 3.19. (i) *Every dual implication algebra is a commutative BI-algebra,*
(ii) *every dual commutative Hilbert algebra is a dual implication algebra,*
(iii) *every dual commutative Hilbert algebra is a commutative BI-algebra,*
(iv) *the class of right distributive commutative BI-algebras and dual commutative Hilbert algebras coincide.*

Proof. (i) By definition of dual implication algebra and commutative BI-algebra the proof is clear.

(ii) By Corollary 2.5 the proof is clear.

(iii) By definition of dual commutative Hilbert algebra and commutative BI-algebra, the proof is clear.

(iv) Assume X is a right distributive commutative BI-algebra. Then from (p_8), (DHA_1) is valid. By (DIA_1) and (I_5),

$$((z * x) * (y * x)) * ((z * y) * x) = ((z * y) * x) * ((z * y) * x).$$

Hence (DHA_2) holds. Using Proposition 3.5(i), (DHA_3) holds.

Conversely, let X be a dual commutative Hilbert algebra. Then by (ii), X is a dual implication algebra and by (i), is a commutative BI-algebra. By Proposition 2.3(H_1), X is distributive. □

The following example shows that the converse of Theorem 3.19, is not valid in general.

Example 3.20. (i) Consider the commutative BI-algebra $(X, *_{16}, 0)$ given in Example 3.12. It is not a dual implication algebra ((DIA_3) is not valid), since

$$(y *_{16} t) *_{16} z = x *_{16} z = x \neq (y *_{16} z) *_{16} t = 0 *_{16} t = 0.$$

(ii) Consider the commutative BI-algebra $(X, *_{16}, 0)$ is given in Example 3.12. It is not a commutative dual Hilbert algebra ((DHA_2) is not valid), since

$$((z*_{16}x)*_{16}(y*_{16}x))*_{16}((z*_{16}y)*_{16}x) = (z*_{16}t)*_{16}(u*_{16}x) = z*_{16}t = z \neq 0.$$

(iii) Let $X := \{0, x, y, z, t\}$ be a set with the following table.

$*_{18}$	0	x	y	z	t
0	0	0	0	0	0
x	x	0	t	x	x
y	y	z	0	x	y
z	z	z	t	0	z
t	t	t	t	t	0

Then $(X, *_{18}, 0)$ is a dual implication algebra, but not a Hilbert algebra, since

$$(z *_{18} y) *_{18} z = t *_{18} z = t \neq 0.$$

We are applying [7, Prop. 1.4], in every dual Hilbert algebra right distributivity holds. Now, from Theorem 3.19, we get *right distributive commutative BI-algebras* are equivalent to *dual commutative Hilbert algebras.* It is known that every implication algebras are a special case of Hilbert algebras, R. Halaš showed that (dual) commutative Hilbert algebras are just the (dual) implication algebras [7], so right distributive commutative BI-algebras are equivalent to (dual) implication algebras.

4. On branchwise commutative BI-algebras

In this section, we discuss branchwise commutative BI-algebras and investigate some of their properties. We also show that $BI(A)$-algebra X is branchwise commutative if and only if each branch of X is a semilattice w.r.t. to $\wedge$ defined by $x \wedge y = x * (x * y)$. We can see that $x \wedge y \leq x, y$. Also, we have $x \wedge x = x$ and $x \wedge 0 = 0 \wedge x = 0$.

For any $a \in X$, put $\mathrm{B}(a) = \{x \in X : a \leq x\}$.
Notice that, since $a \leq a$ we have $a \in \mathrm{B}(a)$, and so $\mathrm{B}(a) \neq \emptyset$. Applying ($p_2$), $B(0) = X$. If a is an atom of X, then the set $\mathrm{B}(a)$ is called a *branch* of X determined by element a.

Example 4.1. Let $X := \{0, x, y, z, t\}$ with the following table.

$*_{19}$	0	x	y	z	t
0	0	0	0	0	0
x	x	0	0	0	x
y	y	t	0	y	x
z	z	t	z	0	x
t	t	t	0	0	0

Then $(X, *_{19}, 0)$ is a BI-algebra. Also, $\mathrm{B}(x) = \{x, y, z\}$ and $\mathrm{B}(t) = \{y, z, t\}$ are branches of X.

We say that a BI-algebra X is *branchwise commutative* if the axiom $x \wedge y = y \wedge x$ holds, for x and y belonging to the same branch. Clearly, any commutative BI-algebra is branchwise commutative. Note that the BI-algebra $(X, *_7, 0)$, where $X = \{0, x, y, z\}$ is branchwise commutative, but not commutative.

Proposition 4.2. *Let X be a right distributive commutative BI-algebra. Then $x * y = x$ if and only if $x \wedge y = 0$.*

Proof. Assume that $x * y = x$. Applying (DAI_1) and commutative law, we get

$$x \wedge y = x * (x * y) = x * x = 0.$$

Conversely, let $x \wedge y = 0$. Using (p_8), we get

$$x * y = x * (x * (x * y)).$$

Thus, $x * y = x * (x \wedge y) = x * 0 = x$. □

Proposition 4.3. *Let X be a right distributive branchwise commutative BI-algebra and $a \in X$. Then*

(i) *$(X, \leq)$ is transitive and reflexive,*
(ii) *$(B(a), \leq)$ is a poset.*

Proof. (i). Obviously, $\leq$ is reflexive. Now, we prove $\leq$ is transitive. Let $x, y, z \in X$ and assume $x \leq y$ and $y \leq z$. Then $x * y = y * z = 0$. We have

$$x * z = (x * z) * 0 = (x * z) * (y * z) = (x * y) * z = 0 * z = 0.$$

Hence $x \leq z$. It follows that $\leq$ is transitive.

(ii). By (i), $(\mathrm{B}(a), \leq)$ is transitive and reflexive. Observe that $\leq$ is also anti-symmetric, for all $x, y \in B(a)$. Indeed, let $x, y \in \mathrm{B}(a)$. Suppose $x \leq y$ and $y \leq x$. Then $x * y = y * x = 0$. Using (p_1) and branchwise commutative law,

$$x = x * 0 = x * (x * y) = y * (y * x) = y * 0 = y.$$

Consequently, $(X, \leq)$ is a poset. □

Definition 4.4. A BI-algebra with the condition (A) or a $BI(A)$-algebra for short, is a BI-algebra X such that the operation $*$ is anti-tonic in the first variable, that condition (A) is satisfied:
(A) if $x, y \in X$ such that $x \leq y$, then $z * y \leq z * x$, for all $z \in X$.

Example 4.5. (i) Let $X := \{0, x, y, z\}$ with the binary operation "$*_{11}$". Then $(X, *_{11}, 0)$ is singular, commutative and right distributive $BI(A)$-algebra.

(ii) Let $X := \{0, x, y, z\}$ with the binary operation "$*_8$". Then $(X, *_8, 0)$ is a right distributive $BI(A)$-algebra, but not commutative. Since

$$x *_8 (x *_8 y) = x *_8 0 = x \neq y *_8 (y *_8 x) = y *_8 0 = y.$$

(iii) Let $X := \{0, x, y, z, t\}$ with the following table.

$*_{20}$	0	x	y	z	t
0	0	0	0	0	0
x	x	0	0	x	0
y	y	z	0	x	y
z	z	z	0	0	0
t	t	z	t	x	0

Then $(X, *_{20}, 0)$ is a transitive and commutative BI-algebra, but not satisfies (A), since $z \leq y$, but $t *_{20} y \not\leq t *_{20} z$. Also, it is not right distributive, since

$$(y *_{20} z) *_{20} t = x *_{20} t = 0 \neq (y *_{20} t) *_{20} (z *_{20} t) = y *_{20} 0 = y.$$

(iv) Let $X := \{0, x, y, z\}$ with the binary operation "$*_{10}$". Then $(X, *_{10}, 0)$ is a transitive and singular BI-algebra, but not satisfies (A), since $0 \leq y$, but $z *_{10} y \not\leq z *_{10} 0$. Also, it is not commutative nor right distributive, since

$$y *_{10} (y *_{10} z) = y *_{10} x = y \neq z = z *_{10} x = z *_{10} (z *_{10} y).$$

and

$$(x *_{10} y) *_{10} z = x *_{10} z = x \neq (x *_{10} z) *_{10} (y *_{10} z) = x *_{10} x = 0.$$

(v) Let $X := \{0, x, y, z, t\}$ with the following table.

$*_{21}$	0	x	y	z	t
0	0	0	0	0	0
x	x	0	t	t	z
y	y	0	0	y	y
z	z	0	z	0	z
t	t	0	t	t	0

Then $(X, *_{21}, 0)$ is a $BI(A)$-algebra, but not right distributive, since

$$(x *_{21} t) *_{21} y = z *_{21} y = z \neq (x *_{21} y) *_{21} (t *_{21} y) = t *_{21} t = 0.$$

Proposition 4.6. *Every commutative and right distributive BI-algebra is a $BI(A)$-algebra.*

Proof. Let X be a commutative and right distributive BI-algebra. Then by Theorem 3.19(iv), X is commutative Hilbert algebra. By duality and proposition 2.3(H_3), X is a $BI(A)$-algebra. $\square$

Proposition 4.7. *Let X be a $BI(A)$-algebra. Then*

(i) $y * x \leq y$, *for all* $x, y \in X$,

(ii) *X is transitive.*

Proof. (i) By (P_2), we have $0 * x = 0$, Thus $0 \leq x$, for all $x \in X$. Applying (A) and (P_1), we get $y * x \leq y * 0 = y$.

(ii) Let X be a $BI(A)$, $x \leq y$ and $y \leq z$, for some $x, y, z \in X$. Then $x * z \leq x * y = 0$ and so $x * z = 0$. Therefore, $x \leq z$ and X is transitive. □

Lemma 4.8. *If X is commutative and $a \in X$, then $y * (y * a) = a$, for all $y \in B(a)$.*

Proof. Let $y \in B(a)$. By definition of $B(a)$, we get $a*y = 0$. By commutativity, $y * (y * a) = a * (a * y) = a * 0 = a$. □

Theorem 4.9. *Let X be a right distributive BI-algebra. The following statements are equivalent:*

(i) *X is branchwise commutative.*
(ii) *Each branch of X is a $\wedge$-semilattice w.r.t. $\wedge$ defined by $x \wedge y = x*(x*y)$.*

Proof. (i) $\Longrightarrow$ (ii): Suppose X is branchwise commutative. Let $a \in \mathrm{A}(X)$. By Proposition 4.3, $(\mathrm{B}(a), \leq)$ is a poset. Let $x, y \in \mathrm{B}(a)$. Applying (p_9), we get

$$x * (x * y) = y * (y * x) \leq x,\ y.$$

Observe that $x * (x * y) \in \mathrm{B}(a)$. Since $a \leq x$, by using Proposition 4.6 and (A), we see that $y * x \leq y * a$ and hence $y * (y * a) \leq y * (y * x)$. By Lemma 4.8, $a \leq y*(y*x)$, and so $a \leq x*(x*y)$. Then $x*(x*y)$ belongs to $\mathrm{B}(a)$ and it is a lower bound of x and y. Now, we show that $x*(x*y)$ is the greatest lower bound of x and y. Let $z \in \mathrm{B}(a)$ be another lower bound of x and y. Therefore, $z \leq x$ and $z \leq y$. By branchwise commutativity, $z = z*0 = z*(z*x) = x*(x*z)$ and similarly, $y * (y * z) = z$. Since $x \leq z$, applying Proposition 4.6 and (A) twice, we obtain $y * (y * x) \leq y * (y * z) = z$, and so $x * (x * y) \leq x * (x * z) = z$, that is, $x * (x * y) = y * (y * x) \leq z$ and $x * (x * y) \leq z$. Thus, $y * (y * x) = x * (x * y)$ is the greatest lower bound of x and y. Then $x \wedge y = x * (x * y) = y * (y * x)$ and $\mathrm{B}(a)$ is a semilattice w.r.t. $\wedge$.

(ii) $\Longrightarrow$ (i): Let x and y belong to the same branch. By assumption, $x \wedge y = x * (x * y)$. Since $x \wedge y = y \wedge x$, we get $x * (x * y) = y * (y * x)$. Thus, X is branchwise commutative. □

Corollary 4.10. *Any right distributive commutative*

(i) *BI-algebra is a $\wedge$-semilattice w.r.t. $\leq$,*
(ii) *BH-algebra is a $\wedge$-semilattice w.r.t. $\leq$.*

Proof. (i) Let X be a commutative BI-algebra. Then X is a commutative BH-algebra and therefore has only one branch. Consequently, from Theorem 4.9, we obtain is a $\wedge$-semilattice.

(ii) Since every commutative BI-algebra is a BH-algebra, the proof is clear. □

5. On commutative ideals in BI-algebras

In this section, we generalize some results proved by Huang (see [25]) for commutative ideals of BCI-algebras to the case of commutative ideals of BI-algebras.

Recall from [4] that a subset I of X is called an *ideal* of X if it satisfies the following conditions:

(I) $0 \in I$,

(II) $y \in I$ and $x * y \in I$ imply $x \in I$, for any $x, y \in X$.

We can see that, X and $\{0\}$ are ideals of X. We will call X and $\{0\}$ a *trivial ideal* and a *zero ideal*, respectively. An ideal I is called to be *proper* if $I \neq X$. Denote the set of all ideals of X by $I(X)$.

Proposition 5.1. [4] Let I be an ideal of X. If $y \in I$ and $x \leq y$, then $x \in I$.

Definition 5.2. An ideal I of X is said to be *commutative* if it satisfies the following condition (for all $x, y \in X$):
(CI) $x * y \in I$ implies $x * (y * (y * x)) \in I$.

Denote the set of all commutative ideals of X by $CI(X)$.

Example 5.3. *Let $X := \{0, x, y, z, t\}$ be a set with the following table.*

$*_{22}$	0	x	y	z	t
0	0	0	0	0	0
x	x	0	t	t	z
y	y	0	0	t	y
z	z	0	t	0	z
t	t	0	t	t	0

Then $(X, *_{22}, 0)$ is a BI-algebra, $I_1 = \{0, t\} \in CI(X)$ and $I_2 = \{0, z\} \in I(X)$, but $I_2 \notin CI(X)$, since

$y *_{22} x = 0 \in I_2,$ but $y *_{22} (x *_{22} (x *_{22} y)) = y *_{22} (x *_{22} t) = y *_{22} z = t \notin I_2.$

Theorem 5.4. *Let I be a subset of X. Then $I \in CI(X)$ if and only if satisfies in the following conditions:*

(i) *$0 \in I$,*

(ii) *$(x * y) * z \in I$ and $z \in I$ imply $x * (y * (y * x)) \in I$, for any $x, y, z \in X$.*

Proof. Let $I \in CI(X)$. Then it is obvious that I satisfies in (i) and (ii). Conversely, let I satisfies in (i) and (ii). At first we prove that $I \in I(X)$. Let $x * y \in I$ and $y \in I$. Then by $(x * 0) * y = x * y \in I$, $y \in I$ and (ii), we get that $x = x*(0*(0*x) \in I$. Thus, $I \in I(X)$. Let $x*y \in I$. Since $(x*y)*0 = x*y \in I$ and using (ii), we get $x * (y * (y * x)) \in I$. Therefore, $I \in CI(X)$. □

In fact, it is easy to show that the following remark.

Remark 5.5. (i) *Let $\{I_i\}_{i \in I}$ be a family of commutative ideals of X. Then $\bigcap_{i \in I} I_i \in CI(X)$.*

(ii) *Let X be right distributive and commutative,* $I_1, I_2 \in CI(X)$. *Then* $I_1 \cup I_2 \in I(X)$ *if and only if* $I_1 \subseteq I_2$ *or* $I_2 \subseteq I_1$.

The following example shows that the union of two commutative ideals may be not a commutative ideal in general.

Example 5.6. Let $X := \{0, x, y, z, t\}$ be a set with the following table.

$*_{23}$	0	x	y	z	t
0	0	0	0	0	0
x	x	0	x	x	x
y	y	y	0	x	y
z	z	z	t	0	y
t	t	t	t	x	0

Then $(X, *_{23}, 0)$ is a BI-algebra. Also, $I_1 = \{0, x, t\}$, $I_2 = \{0, x, z\}$, $I_3 = \{0, y, z\}$ and $I_4 = \{0, z, t\}$ are commutative ideals of X. Further, we can see that $I = I_2 \cup I_4 = \{0, x, z, t\}$ is not a/an (commutative) ideal of X, since $y *_{23} z = x \in I$ and $z \in I$, but $y \notin I$.

Theorem 5.7 (Extension property)**.** *Let X be a right distributive $BI(A)$-algebra, $I, G \in I(X)$ and $I \subseteq G$. If $I \in CI(X)$, then $G \in CI(X)$.*

Proof. Assume $I \in CI(X)$, $G \in I(X)$, $I \subseteq G$ and $x * y \in G$. Put $u := x * y$. Applying right distributive law, (p_3) and (p_1) we get $(x * u) * y = 0 \in I$. Since I is commutative, we obtain $(x * u) * (y * (y * (x * u))) \in I \subseteq G$, and so $(x * u) * (y * (y * (x * u))) \in G$. Using right distributive law, we have
$(x * u) * (y * (y * (x * u))) = (x * (y * (y * (x * u)))) * (u * (y * (y * (x * u)))) \in G$. Since G is an ideal and $u * (y * (y * (x * u))) \leq u$ and $u \in G$, it follows that $u * (y * (y * (x * u))) \in G$. This shows that $x * (y * (y * (x * u))) \in G$. Also, since $x * u \leq x$, applying (A) three times, we get $x * (y * (y * x)) \leq x * (y * (y * (x * u)))$. Since G is an ideal and $x * (y * (y * (x * u))) \in G$, we can get $x * (y * (y * x)) \in G$. Therefore, $G \in CI(X)$. □

The following example shows that the conditions right distributivity and (A) are necessary.

Example 5.8. Let $X := \{0, x, y, z, t\}$ be a set with the following table.

$*_{24}$	0	x	y	z	t
0	0	0	0	0	0
x	x	0	z	x	y
y	y	z	0	y	y
z	z	z	z	0	y
t	t	z	t	y	0

Then $(X, *_{24}, 0)$ is a BI-algebra, but not right distributive, since

$$(x *_{24} y) *_{24} z = z *_{24} z = 0 \neq z = x *_{24} y = (x *_{24} z) *_{24} (y *_{24} z).$$

Also, it does not satisfy (A), since $0 \leq t$, but $z *_{24} t = y \not\leq z *_{24} 0 = z$. Further, $I_1 = \{0, t\}$ is a commutative ideal, but $I_2 = \{0, z, t\}$ is not a commutative ideal, since $t *_{24} x = z \in I$, but $t *_{24} (x *_{24} (x *_{24} t)) = y \notin I_2$

Theorem 5.9. *Let X be commutative and right distributive and let $I \in I(X)$. Then for all $x, y \in X$*

(i) $x * y = x * (y * (y * x))$,
(ii) $I \in CI(X)$.

Proof. (i) Using (p_9) we can see that $x * (x * y) \leq y$. Applying Proposition 4.6 and (A) we have $x * y \leq x * (x * (x * y))$. Again, by ($p_9$), $x * (x * (x * y)) \leq x * y$. By Proposition 3.6(i) and commutative law, we get $x * y = x * (x * (x * y)) = x * (y * (y * x))$.

(ii) By (i) and (CI), the proof is clear. □

In [4], it was shown that for every ideal I of X, we can define a binary relation " $\sim_I$ " by

$$x \sim_I y \text{ if and only if } x * y \in I \text{ and } y * x \in I.$$

The set $\{y : x \sim_I y\}$ will be denoted by $[x]_I$. Also, if X is right distributive, then "$\sim_I$" is a right congruence relation on X (see [4, Th. 5.1, Th. 5.2]). Further, for every subset I of X with $0 \in I$, if I has the condition: if $x * y \in I$, then $(z * x) * (z * y) \in I$. Then $X = I$ (see [4, Prop. 5.4]). So, if "$\sim_I$ is a compatible relation, then $X = I$. Now, let X be commutative and right distributive and I be an ideal of X, if we take $\frac{X}{I} = \{[x]_I : x \in X\}$ and define a binary operation "$\star$" on $\frac{X}{I}$ by

$$[x]_I \star [y_I] = [x * y]_I,$$

then $\frac{X}{I} = \{0\}$.

6. Conclusions

In this paper, the concept of (*branchwise*) *commutative BI-algebras* is introduced and showed that the class of commutative BI-algebras is a subclass of commutative BH-algebras and every singular BI-algebra is a BH-algebra.
We proved that every commutative right distributive $BI(A)$-algebra is a meet-semilattice. We initially presented a few examples, and some basic properties of such algebras are investigated. Further, let DCH be the set of all dual commutative Hilbert algebras, DI be the set of all dual implication algebras, CBI be the set of all commutative BI-algebras and CBH be the set of all commutative BH-algebras. Also, we show that the *right distributive commutative BI-algebras* is equivalent to *dual commutative Hilbert algebras.* Further, we proved that a given commutative BI-algebra can be characterized by *commutative ideals.* In Figure 1, we show the relation between commutative, transitive, distributive, BI(A), singular BI-algebras, dual implication algebras, dual Hilbert algebras and BH-algebras.

In the future works, could be to introduce the concept of a *fuzzy* BI-algebra. Another topic of research could be to define some types of ideals in a BI-algebra and investigate the relationship between these ideals.

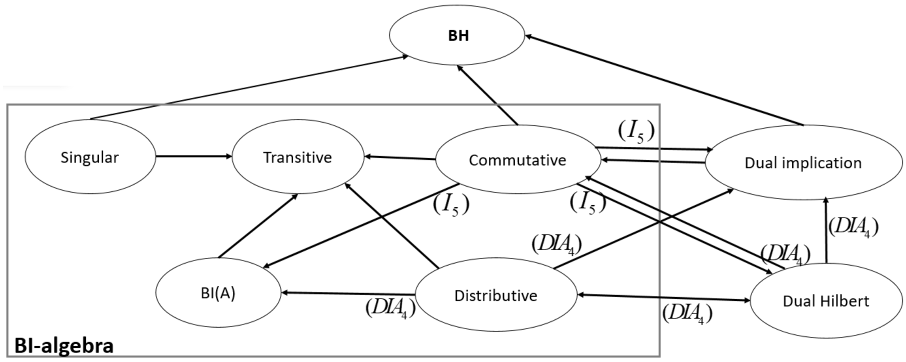

Figure 1. Relation between algebras discussed in the paper

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