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SOME CODES AND DESIGNS INVARIANT UNDER THE GROUPS S_7 AND S_8

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Abstract. We use the Key-Moori Method 1 and examine 1-designs and codes from the representations of the alternating group A_7. It is shown that a self-dual symmetric 2-$(35, 18, 9)$ design and an optimal even binary $[21, 14, 4]$ LCD code are found such that they are invariant under the full automorphism groups S_8 and S_7, respectively. Moreover, designs with parameters 1-$(21, l, k_{1,l})$ and 1-$(35, l, k_{2,l})$ are obtained, where ω is a codeword, $l = \mathrm{wt}(\omega)$, $k_{1,l} = l|\omega^{S_7}|/21$ and $k_{2,l} = l|\omega^{S_7}|/35$. It is seen that there exist a 2-$(21, 5, 12)$ design with the full automorphism group S_7 among these 1-designs.

Keywords: Code, Design, Automorphism group, Alternating group, Primitive permutation representation.
2020 MSC: Primary 05B05, 94B05, 20D45.

1. Introduction

Key and Moori [17,18] considered the representations of the Janko groups J_1 and J_2, and then constructed 1-designs and codes invariant under the groups J_1, J_2 or $\bar{J}_2$, where $\bar{J}_2$ is the extension of J_2 by its outer automorphism. Darafsheh et al. [6, 7, 9] considered the primitive representations of the projective special linear groups $PSL_2(q)$, $q \leq 50$, and found designs and their automorphism groups. Furthermore, the binary codes and their automorphism groups from the groups $PSL_2(8)$, $PSL_2(9)$ and $PSL_2(13)$ are obtained [8, 16]. Also, Darafsheh [5] found designs with parameters 1-$\left(\binom{q}{2}, q+1, q+1\right)$, 1-$\left(\binom{q+1}{2}, q-1, q-1\right)$ and 1-$\left(\binom{q+1}{2}, 2(q-1), 2(q-1)\right)$ from the group $PSL_2(q)$, $q = 2^n$, such that the last design is invariant under the full automorphism group S_{q+1}. In [12,13,15], the current author considered the primitive permutation representations of $PSL_2(q)$, $q = 53, 59, 61, 64, 81, 89$ and found designs and their automorphism groups. Furthermore, Moori and Saeidi obtained designs and their codes from the Tits group ${}^2F_4(2)'$ and some 1-designs from the group $PSL_2(2^n)$ [21, 22].

Recently, Darafsheh et al. constructed an optimal ternary code from the group $PSL_2(9)$ invariant under the group S_6 [10]. It was seen that there is

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a 2-$(15, 7, 36)$ design with the automorphism group S_6. Moreover, the current author constructed a self-orthogonal even code over $GF(4)$ from $PSL_2(9)$ invariant under the automorphism group A_8 [14]. It was shown that $\mathrm{Supp}(\omega)^{A_8}$ forms a 2-$(15, l, \lambda)$ design, where $\mathrm{wt}(\omega) = l$ and $\lambda = \binom{l}{2}|\omega^{A_8}|/\binom{15}{2}$.

In this paper, motivated by the above works, designs and their binary codes from the primitive permutation representations of the alternating group A_7 are considered. It is shown that designs with parameters 1-$(35, 4, 4)$, 1-$(35, 12, 12)$, 2-$(35, 18, 9)$ and two non-isomorphic 1-$(21, 10, 10)$ designs are obtained. According to [3], this 2-(35,18,9) design is new. These designs give us three binary LCD codes with parameters $[35, 20, 4]$, $[21, 14, 4]$ and $[21, 6, 6]$ such that their duals are $[35, 15, 5]$, $[21, 7, 6]$ and $[21, 15, 3]$ codes, respectively. We show that these codes have the symmetric group S_7 as the full automorphism group. Moreover, for any codeword ω in the above codes, we examine the stabilizers $(S_7)_\omega$ and determine their structures. By taking $\mathrm{Supp}(\omega)$ and orbiting it under the group S_7, designs with parameters 1-$(21, l, l|\omega^{S_7}|/21)$ and 1-$(35, l, l|\omega^{S_7}|/35)$ are obtained, where $l = \mathrm{wt}(\omega)$. It is shown that two complementary designs with parameters 2-$(21, 5, 12)$ and 2-$(21, 16, 144)$ exist among these 1-designs and moreover, as far as we know, these designs are new.

2. **Preliminaries**

Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be an incidence structure. The disjoint sets $\mathcal{P}$ and $\mathcal{B}$ are called point and block sets, respectively. Also, the incidence relation $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{B}$ is a flag set. If $(p, B) \in \mathcal{I}$ then we write $p\mathcal{I}B$. A block B can be identified with the set of points incident with it, which in this case $\mathcal{I}$ is the membership relation $\in$. If we replace the blocks of $\mathcal{S}$ by their complement then $\overline{\mathcal{S}}$, the complement of $\mathcal{S}$, is obtained. Also, the incidence structure $\mathcal{S}^\top = (\mathcal{B}, \mathcal{P}, \mathcal{I}^\top)$ is called the dual of $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$, where $B\mathcal{I}^\top p \leftrightarrow p\mathcal{I}B$. A one-to-one correspondence $\varphi : \mathcal{P} \to \mathcal{P}'$ is an isomorphism between $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ and $\mathcal{S}' = (\mathcal{P}', \mathcal{B}', \mathcal{I}')$ if $p\mathcal{I}B \leftrightarrow \varphi(p)\mathcal{I}'\varphi(B)$ for all $p \in \mathcal{P}$ and $B \in \mathcal{B}$. If such an isomorphism exists then $\mathcal{S}$ and $\mathcal{S}'$ are isomorphic and we write $\mathcal{S} \cong \mathcal{S}'$. Now, $\mathcal{S}$ is a self-dual structure if $\mathcal{S}$ and $\mathcal{S}^\top$ are isomorphic. An automorphism of $\mathcal{S}$ is an isomorphism of $\mathcal{S}$ onto itself. The automorphism group $\mathrm{Aut}(\mathcal{S})$ is the group consisting of all the automorphisms of $\mathcal{S}$. A t-(v, k, λ) design is an incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ such that $|\mathcal{P}| = v$, $|B| = k$ for each block B and any t points of $\mathcal{P}$ are incident with precisely λ blocks. The number of blocks, denoted by b, is $\lambda\binom{v}{t}/\binom{k}{t}$. The design $\mathcal{D}$ is called symmetric if $v = b$. Also, $\mathcal{D}$ is trivial if the blocks are exactly all k-subsets of $\mathcal{P}$. We know that $\lambda_s = \lambda\binom{v-s}{t-s}/\binom{k-s}{t-s}$ is the number of blocks incident with exactly s points, where $s \leq t$. Each t-(v, k, λ) design is in fact an s-(v, k, λ_s) design. The replication number, denoted by r, is λ_1. In each 2-(v, k, λ) design, we have $r(k-1) = \lambda(v-1)$ and $bk = vr$. For a t-(v, k, λ) design $\mathcal{D}$, $\overline{\mathcal{D}}$ is a design with parameters t-$(v, v-k, \overline{\lambda})$, where $\overline{\lambda} = \sum_{s=0}^{t}(-1)^s\binom{t}{s}\lambda_s$. Hence, according to a standard convention, we mention new t-(v, k, λ) designs with $k \leq v/2$. We refer the reader to [1,3] for more details.

Let F_q be a finite field of order q. Each subspace of F_q^n is a linear code of length n over F_q. The elements of a code are called codewords. A linear code $\mathcal{C}$ over F_q is an $[n, k, d]_q$ code if its length, dimension and minimum distance are n, k and d, respectively. When d is unknown, $\mathcal{C}$ is called an $[n, k]_q$ code. If $\mathcal{C}$ is the singleton $\{0\}$, the whole space F_q^n, a subspace $\langle v \rangle$ of dimension 1 or a subspace $\langle v \rangle^\perp$ of dimension $n-1$ then $\mathcal{C}$ is trivial. A linear $[n, k, d]_q$ code $\mathcal{C}$ is optimal if $\mathcal{C}$ has the largest possible minimum distance d for given n, k and q. The support of $c = (c_1, \ldots, c_n) \in \mathcal{C}$ is $\mathrm{Supp}(c) = \{i \mid c_i \neq 0\}$ and moreover, the weight of c is $\mathrm{wt}(c) = |\mathrm{Supp}(c)|$. The diameter of $\mathcal{C}$, denoted by $\mathrm{diam}(\mathcal{C})$, is the largest weight of codewords of $\mathcal{C}$. The all-one word, denoted by $\jmath$, is a vector all of whose coordinate positions are one. The dual code $\mathcal{C}^\perp$ is defined to be the orthogonal subspace of $\mathcal{C} \leq F_q^n$. The hull of $\mathcal{C}$, denoted by $\mathrm{hull}(\mathcal{C})$, is $\mathcal{C} \cap \mathcal{C}^\perp$. The linear code $\mathcal{C}$ is called a linear complementary dual (shortly, LCD) code if $\mathrm{hull}(\mathcal{C}) = \{0\}$. If $\mathcal{C} \subseteq \mathcal{C}^\perp$ then $\mathcal{C}$ is said to be self-orthogonal. If $\mathcal{C} = \mathcal{C}^\perp$ then $\mathcal{C}$ is self-dual. The weight enumerator of $\mathcal{C}$ is $W_{\mathcal{C}}(x, y) = \sum_{l=0}^{n} A_l x^{n-l} y^l$, where A_l is the number of codewords of weight l in $\mathcal{C}$. A binary code $\mathcal{C}$ is even if $2 | \mathrm{wt}(c)$ for any codeword c. The linear codes $\mathcal{C}$ and $\mathcal{C}'$ are equivalent if we can obtain $\mathcal{C}'$ from $\mathcal{C}$ by permuting the coordinate positions and multiplying each coordinate position by the elements of $F_q \setminus \{0\}$. Furthermore, $\mathcal{C}$ and $\mathcal{C}'$ are isomorphic if we can obtain $\mathcal{C}'$ from $\mathcal{C}$ by permuting the coordinate positions. Each permutation of the coordinate positions that maps codewords to themselves is an automorphism of $\mathcal{C}$. The group of all the automorphisms of the code $\mathcal{C}$ is denoted by $\mathrm{Aut}(\mathcal{C})$. A weight class of $\mathcal{C}$ is a set such as $\mathcal{C}_{(l)} = \{c \in \mathcal{C} \mid \mathrm{wt}(c) = l\}$, where $0 \leq l \leq n$. Clearly, an automorphism preserves $\mathcal{C}_{(l)}$. For an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$, let $\mathcal{C}_q \leq F_q^{\mathcal{P}}$ be the linear code spanned by all the incidence vectors of the blocks over F_q. See [1,19] for more details.

Our notation for groups and their maximal subgroups is in accordance with $\mathbb{ATLAS}$ [4]. The semidirect and direct products of G and H are denoted by $G{:}H$ and $G \times H$, respectively. The elementary abelian group of order p^n, where p is prime, is denoted by p^n.

3. The designs and Binary codes from A_7

In this section, the construction of designs is based on the following theorem which is given by Key and Moori:

Theorem 3.1. *(Key-Moori Method 1) [18, Proposition 1] Let Ω be a set of size n and $\omega \in \Omega$. Let G be a finite primitive permutation group which acts on Ω. If $\Delta \neq \{\omega\}$ is an orbit of the stabilizer G_ω then the incidence structure $\mathcal{D} = (\Omega, \Delta^G, \in)$ is a symmetric 1-$(n, |\Delta|, |\Delta|)$ design, where $\Delta^G = \{\Delta^g \mid g \in G\}$. Furthermore, if the orbit Δ is self-paired then the design $\mathcal{D}$ is self-dual and G acts primitively as an automorphism group on $\mathcal{D}$.*

Theorem 3.2. *[17, Lemma 2] Let $\mathcal{D}$ be a symmetric 1-design obtained by a group G and the Key-Moori Method 1. Then, $G \leq \mathrm{Aut}(\mathcal{D})$.*

Note that if $\mathcal{C}$ is a design's code of $\mathcal{D}$ over the field F_q then $\mathrm{Aut}(\mathcal{D}) \leq \mathrm{Aut}(\mathcal{C})$. Now, we can construct binary codes from the primitive permutation representations of A_7 using a program in the software Magma [2] and the Key-Moori Method 1. Hence, we consider the action of A_7 on the set of the right cosets of its maximal subgroups. The alternating group A_7 is a simple group of order 2520 having five maximal subgroups up to conjugacy of orders 360, 168, 168, 120 and 72 such that they are isomorphic to A_6, $PSL_2(7)$, $PSL_2(7)$, S_5 and $(A_4 \times 3){:}2$, respectively [4]. By Magma, the actions of A_7 on the set of the right cosets of the first three subgroups are 2-transitive and thus the designs so obtained are trivial. Hence, we do not consider them. The last two subgroups correspond to the primitive representations of A_7 of degrees 21 and 35, respectively. The information about non-trivial designs and their binary codes obtained from A_7 is collected in Table 1 in the Appendix.

In Table 1, the columns from left to right show maximal subgroups, the indices of these subgroups, the number of orbits of a stabilizer, the lengths of the orbits, the order of the automorphism group of each design, the parameters of the design's codes, the parameters of dual codes and the order of the full automorphism group of each constructed code, respectively. Also, the entry line $m(n)$ denotes that there are n orbits of length m.

Theorem 3.3. *(i) For A_7 of degree 21, two non-isomorphic 1-(21,10,10) designs $\mathcal{D}_{(10)_1}$ and $\mathcal{D}_{(10)_2}$ are obtained. These designs are self-dual and moreover, $\mathrm{Aut}(\mathcal{D}_{(10)_1}) = \mathrm{Aut}(\mathcal{D}_{(10)_2}) \cong S_7$.*

(ii) For A_7 of degree 35, a 1-(35,4,4) design $\mathcal{D}_4$, a 1-(35,12,12) design $\mathcal{D}_{12}$ and a 2-(35,18,9) design $\mathcal{D}_{18}$ are obtained. These designs are self-dual, $\mathrm{Aut}(\mathcal{D}_4) = \mathrm{Aut}(\mathcal{D}_{12}) \cong S_7$ and $\mathrm{Aut}(\mathcal{D}_{18}) \cong S_8$.

Proof. Consider the alternating group A_7 and its maximal subgroups $M_1 \cong S_5$ and $M_2 \cong (A_4 \times 3){:}2$.

(i) The action of A_7 on the set of the right cosets of M_1 gives us a primitive representation of A_7 of degree $[A_7 : M_1] = (7!/2)/5! = 21$. The point stabilizer $(A_7)_\omega$ have three orbits of lengths 1, 10 and 10. In fact, the group A_7 acts on the cosets of M_1 as a rank-3 primitive group. By Theorem 3.1, we obtain two symmetric 1-(21,10,10) designs $\mathcal{D}_{(10)_1}$ and $\mathcal{D}_{(10)_2}$. Magma shows that they are two non-isomorphic self-dual designs, but their automorphism groups are identical and of order 5040. Henceforth, these automorphism groups will be denoted by $\mathrm{Aut}(\mathcal{D}_{10})$. Magma implies that $\mathrm{Aut}(\mathcal{D}_{10}) \leq S_{21}$ is of order 5040 with a normal subgroup N of order 2520 such that $N \cong A_7$. By Magma, $1 \leq A_7 \leq \mathrm{Aut}(\mathcal{D}_{10})$ is a composition series for $\mathrm{Aut}(\mathcal{D}_{10})$ and $\mathrm{Aut}(\mathcal{D}_{10}) \cong A_7{:}2 \cong S_7$.

(ii) If we consider the action of A_7 on the set of the right cosets of M_2 then a primitive permutation representation of A_7 of degree $[A_7 : M_2] = (7!/2)/72 = 35$ is obtained. The point stabilizer $(A_7)_\omega$ has four orbits of lengths 1, 4, 12 and 18. By Theorem 3.1, the symmetric designs $\mathcal{D}_4$, $\mathcal{D}_{12}$ and $\mathcal{D}_{18}$ with parameters 1-(35,4,4), 1-(35,12,12) and 1-(35,18,18) are obtained, respectively. By Magma, these designs are self-dual and $\mathcal{D}_{18}$ is a 2-(35,18,9) design. Computations with Magma imply that $\mathrm{Aut}(\mathcal{D}_4) = \mathrm{Aut}(\mathcal{D}_{12}) \cong \mathrm{Aut}(\mathcal{D}_{10})$. Furthermore, $\mathrm{Aut}(\mathcal{D}_{18}) \leq S_{35}$ is a non-abelian group of order 40320 with a normal subgroup N of order 20160 such that $N \cong A_8$. By Magma, $1 \leq A_8 \leq \mathrm{Aut}(\mathcal{D}_{18})$ is a composition series for $\mathrm{Aut}(\mathcal{D}_{18})$ and $\mathrm{Aut}(\mathcal{D}_{18}) \cong A_8{:}2 \cong S_8$. □

Suppose that $\mathcal{C}_{(10)_1}$, $\mathcal{C}_{(10)_2}$, $\mathcal{C}_4$, $\mathcal{C}_{12}$ and $\mathcal{C}_{18}$ are binary codes constructed from the designs $\mathcal{D}_{(10)_1}$, $\mathcal{D}_{(10)_2}$, $\mathcal{D}_4$, $\mathcal{D}_{12}$ and $\mathcal{D}_{18}$, respectively. Our Magma computations show that $\mathcal{C}_4 = \mathcal{C}_{12}$, $\mathcal{C}_{18}$ is a trivial code with the co-dimension 1 and $\mathcal{C}_{(10)_1} \subseteq \mathcal{C}^{\perp}_{(10)_2}$.

Theorem 3.4. *The binary code $\mathcal{C}_{(10)_1}$ is an even code with the parameters $[21, 14, 4]_2$ and 105 codewords of minimum weight. The dual code $\mathcal{C}^{\perp}_{(10)_1}$ is a $[21, 7, 6]_2$ code with seven codewords of minimum weight. Furthermore, $\jmath \in \mathcal{C}^{\perp}_{(10)_1}$, $\mathcal{C}_{(10)_1}$ is an optimal LCD code and $\mathrm{Aut}(\mathcal{C}_{(10)_1}) \cong S_7$.*

Proof. We know that $\mathcal{D}_{(10)_1}$ is a design with the even block size. Hence, the rows of the incidence matrix of $\mathcal{D}_{(10)_1}$ spans an even binary code $\mathcal{C}_{(10)_1}$ of length 21 and $\jmath = \omega_{21} \in \mathcal{C}^{\perp}_{(10)_1}$. The bijection $c \to c + \jmath$ on $\mathcal{C}^{\perp}_{(10)_1}$ implies the equalities $A_{21-l} = A_l$ between numbers of codewords. Magma shows that $\dim(\mathcal{C}_{(10)_1}) = 14$ and

$$\begin{aligned} W_{\mathcal{C}_{(10)_1}}(x,y) =& x^{21} + 105x^{17}y^4 + 805x^{15}y^6 + 3255x^{13}y^8 + 5481x^{11}y^{10} \\ &+ 4515x^9y^{12} + 1935x^7y^{14} + 252x^5y^{16} + 35x^3y^{18}, \\ W_{\mathcal{C}^{\perp}_{(10)_1}}(x,y) =& x^{21} + 7x^{15}y^6 + 35x^{12}y^9 + 21x^{11}y^{10} + 21x^{10}y^{11} + 35x^9y^{12} \\ &+ 7x^6y^{15} + y^{21}. \end{aligned}$$

Thus, $\mathcal{C}_{(10)_1}$ and $\mathcal{C}^{\perp}_{(10)_1}$ are $[21, 14, 4]$ and $[21, 7, 6]$ binary codes with 105 and 7 codewords of minimum weights, respectively. By Magma, $\dim(\mathrm{hull}(\mathcal{C}_{(10)_1})) = 0$. Hence, $\mathcal{C}_{(10)_1} \cap \mathcal{C}^{\perp}_{(10)_1} = \{0\}$ and $F_2^{21} = \mathcal{C}_{(10)_1} \oplus \mathcal{C}^{\perp}_{(10)_1}$. According to [11], $\mathcal{C}_{(10)_1}$ is an optimal code and $\mathcal{C}^{\perp}_{(10)_1}$ has a minimum distance 2 less than the optimal. By Theorems 3.2 and 3.3, $S_7 \cong \mathrm{Aut}(\mathcal{D}_{(10)_1}) \leq \mathrm{Aut}(\mathcal{C}_{(10)_1})$. Magma computations show that $|\mathrm{Aut}(\mathcal{C}_{(10)_1})| = 5040 = 7!$. So, $\mathrm{Aut}(\mathcal{C}_{(10)_1}) \cong S_7$. □

Theorem 3.5. *The code $\mathcal{C}_{(10)_2}$ is an even binary code with the parameters $[21, 6, 6]_2$ and seven codewords of minimum weight. The dual code $\mathcal{C}^{\perp}_{(10)_2}$ is a $[21, 15, 3]_2$ code with 35 codewords of minimum weight. Furthermore, $\mathcal{C}_{(10)_2}$ is an LCD code and $\mathrm{Aut}(\mathcal{C}_{(10)_2}) \cong S_7$.*

Proof. Since $\mathcal{D}_{(10)_2}$ is a design with the even block size, the associated binary code $\mathcal{C}_{(10)_2}$ is even and $\jmath = \omega_{21} \in \mathcal{C}^{\perp}_{(10)_2}$. Hence, the equalities $A_{21-l} = A_l$ are holden in $\mathcal{C}_{(10)_2}^{\perp}$. By Magma, $\dim(\mathcal{C}_{(10)_2}) = 6$ and

$$\begin{aligned} W_{\mathcal{C}_{(10)_2}}(x,y) =& x^{21} + 7x^{15}y^6 + 21x^{11}y^{10} + 35x^9y^{12}, \\ W_{\mathcal{C}_{(10)_2}^{\perp}}(x,y) =& x^{21} + 35x^{18}y^3 + 105x^{17}y^4 + 252x^{16}y^5 + 805x^{15}y^6 + 1935x^{14}y^7 \\ &+ 3255x^{13}y^8 + 4515x^{12}y^9 + 5481x^{11}y^{10} + 5481x^{10}y^{11} \\ &+ 4515x^9y^{12} + 3255x^8y^{13} + 1935x^7y^{14} + 805x^6y^{15} + 252x^5y^{16} \\ &+ 105x^4y^{17} + 35x^3y^{18} + y^{21}. \end{aligned}$$

Hence, $\mathcal{C}_{(10)_2}$ and $\mathcal{C}_{(10)_2}^{\perp}$ are $[21,6,6]$ and $[21,15,3]$ binary codes with 7 and 35 codewords of the minimum weights, respectively. By [11], $\mathcal{C}_{(10)_2}$ and $\mathcal{C}_{(10)_2}^{\perp}$ have minimum distance only 2 and 1 less than the optimal. Magma computations show that the dimension of hull$(\mathcal{C}_{(10)_2})$ is zero. Thus, $\mathcal{C}_{(10)_2} \cap \mathcal{C}_{(10)_2}^{\perp} = \{0\}$ and we have $F_2^{21} = \mathcal{C}_{(10)_2} \oplus \mathcal{C}_{(10)_2}^{\perp}$. By Magma, $|\operatorname{Aut}(\mathcal{C}_{(10)_2})| = 5040 = 7!$. On the other hand, Theorems 3.2 and 3.3 imply that $S_7 \cong \operatorname{Aut}(\mathcal{D}_{(10)_2}) \leq \operatorname{Aut}(\mathcal{C}_{(10)_2})$. So, $\operatorname{Aut}(\mathcal{C}_{(10)_2}) \cong S_7$. □

Theorem 3.6. *The linear code $\mathcal{C}_4$ is an even binary code with parameters $[35,20,4]$ and 35 codewords of minimum weight. The dual code $\mathcal{C}_4^{\perp}$ is a $[35,15,5]$ binary code with 21 codewords of minimum weight. Furthermore, $\mathcal{C}_4$ is an LCD code and $\operatorname{Aut}(\mathcal{C}_4) \cong S_7$.*

Proof. The code $\mathcal{C}_4$ is even since the 1-design $\mathcal{D}_4$ has the even block size 4. Thus, $\jmath = \omega_{35} \in \mathcal{C}_4^{\perp}$ and we have the equalities $A_{21-l} = A_l$ on $\mathcal{C}_4^{\perp}$. Magma shows that $\dim(\mathcal{C}_4) = 20$ and

$$\begin{aligned} W_{\mathcal{C}_4}(x,y) =& x^{35} + 35x^{31}y^4 + 210x^{29}y^6 + 1750x^{27}y^8 + 10556x^{25}y^{10} \\ &+ 49700x^{23}y^{12} + 140540x^{21}y^{14} + 253925x^{19}y^{16} + 277200x^{17}y^{18} \\ &+ 190939x^{15}y^{20} + 91490x^{13}y^{22} + 28420x^{11}y^{24} + 3780x^9y^{26} \\ &+ 30x^7y^{28}, \\ W_{\mathcal{C}_4^{\perp}}(x,y) =& x^{35} + 21x^{30}y^5 + 105x^{27}y^8 + 70x^{26}y^9 + 105x^{25}y^{10} + 420x^{24}y^{11} \\ &+ 735x^{23}y^{12} + 1295x^{22}y^{13} + 2040x^{21}y^{14} + 2877x^{20}y^{15} \\ &+ 3990x^{19}y^{16} + 4725x^{18}y^{17} + 4725x^{17}y^{18} + 3990x^{16}y^{19} \\ &+ 2877x^{15}y^{20} + 2040x^{14}y^{21} + 1295x^{13}y^{22} + 735x^{12}y^{23} \\ &+ 420x^{11}y^{24} + 105x^{10}y^{25} + 70x^9y^{26} + 105x^8y^{27} + 21x^5y^{30} + y^{35}. \end{aligned}$$

Hence, $\mathcal{C}_4$ and $\mathcal{C}_4^{\perp}$ are binary codes with the parameters $[35,20,4]$ and $[35,15,5]$ containing 35 and 21 codewords of minimum weights, respectively. These codes are far from being optimal. By Magma, $\dim(\operatorname{hull}(\mathcal{C}_4)) = 0$ and $F_2^{35} = \mathcal{C}_4 \oplus \mathcal{C}_4^{\perp}$. Magma shows that $|\operatorname{Aut}(\mathcal{C}_4)| = 5040$ and moreover, by Theorems 3.2 and 3.3, $S_7 \cong \operatorname{Aut}(\mathcal{D}_4) \leq \operatorname{Aut}(\mathcal{C}_4)$. So, the assertion is implied. □

4. The designs from the codes $\mathcal{C}^{\perp}_{(10)_1}$, $\mathcal{C}^{\perp}_{(10)_2}$, $\mathcal{C}_4$ and $\mathcal{C}_4^{\perp}$

In this section, we use the following method given at the end of Section 4 of [20] to construct 1-designs:

Theorem 4.1. *[20] If $\mathcal{C}$ is a code of length n and $\omega \in \mathcal{C}$ then* $\mathrm{Supp}(\omega)^{\mathrm{Aut}(\mathcal{C})}$ *forms a 1-(n, l, k_l) design $\mathcal{D}_\omega$, where* $\mathrm{wt}(\omega) = l$ *and* $k_l = l|\omega^{\mathrm{Aut}(\mathcal{C})}|/n$.

Now, according to the Theorem 4.1, we consider the binary codes $\mathcal{C}^{\perp}_{(10)_1}$, $\mathcal{C}^{\perp}_{(10)_2}$, $\mathcal{C}_4$ and $\mathcal{C}_4^{\perp}$, and their automorphism group S_7. Note that the codes $\mathcal{C}_{(10)_1}$ and $\mathcal{C}_{(10)_2}$ will not be considered since $\mathcal{C}_{(10)_2} \subseteq \mathcal{C}^{\perp}_{(10)_1}$, $\mathcal{C}_{(10)_1} \subseteq \mathcal{C}^{\perp}_{(10)_2}$ and computations do not yield new designs. By Theorem 4.1, if ω is a codeword in $\mathcal{C}^{\perp}_{(10)_1}$ or $\mathcal{C}^{\perp}_{(10)_2}$ of weight l then $\mathrm{Supp}(\omega)^{S_7}$ forms a 1-$(21, l, l|\omega^{S_7}|/21)$ design. Again, if ω is a codeword in $\mathcal{C}_4$ or $\mathcal{C}_4^{\perp}$ of weight l then $\mathrm{Supp}(\omega)^{S_7}$ forms a 1-$(35, l, l|\omega^{S_7}|/35)$ design. These designs are constructed by a computer program in Magma. The information we get about the actions of S_7 on the mentioned codes is given in Tables 2-5 in the Appendix.

In Tables 2-5, codeword's weight is under the symbol 'wt'. If a weight class $\mathcal{C}_{(l)}$ splits into more than one orbit then the ith orbit is denoted by $\mathcal{C}_{(l)_i}$ and the related entry line is '$(l)_i$'. If the action is transitive then the entry line is written 'l'. The notation '$(S_7)_{(l)_i,\mathcal{C}}$' denotes the structure of the stabilizer of a codeword ω of weight l in the ith orbit of $\mathcal{C}_{(l)}$. These stabilizers can be determined by finding their normal or maximal subgroups in Magma. The maximality of stabilizers is written under the column 'Max.' and '$\mathcal{D}_{(l)_i,\mathcal{C}}$' shows the parameters of the constructed designs. The number of blocks is written under the column '# blocks'. In these tables, trivial designs will not be considered. Among these 1-designs, two 2-designs $\mathcal{D}_{5,\mathcal{C}^{\perp}_{(10)_2}}$ and $\mathcal{D}_{16,\mathcal{C}^{\perp}_{(10)_2}}$ are obtained and, in fact, they are complement to each other. Moreover, the following theorem is deduced:

Theorem 4.2. *If $\omega \in \mathcal{C}^{\perp}_{(10)_2}$ is of weight 5 then* $\mathrm{Supp}(\omega)^{S_7}$ *forms a* 2-$(21, 5, 12)$ *design $\mathcal{D}_{5,\mathcal{C}^{\perp}_{(10)_2}}$. Moreover,* $\mathrm{Aut}(\mathcal{D}_{5,\mathcal{C}^{\perp}_{(10)_2}}) \cong S_7$ *and* $(S_7)_\omega \cong 5{:}2^2$.

5. Conclusion

As it is shown, we can construct designs and their codes from the primitive permutation representations of a given group. Moreover, by computing the automorphism group of a code so obtained and orbiting the support of any codeword under this automorphism group, some designs can be constructed. One of our goals is to find t-designs with $t \geq 2$. In this manuscript, we see that there is a 2-$(21, 5, 12)$ design and, as far as I know, it is a new one. In order to find new designs, this process can be used for any group.

6. Aknowledgement

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7. Conflict of interest

The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript, or in the decision to publish the results.

8. Appendix

The labelling of the columns of Table 1 is described in Section 3, above Theorem 3.3. Also, the labelling of the columns of Tables 2-5 is described in Section 4, above Theorem 4.2.

TABLE 1. Binary codes from A_7.

Max.	Deg.	#	Len.	$\vert \mathrm{Aut}(\mathcal{D})\vert$	Code	Dual	$\vert \mathrm{Aut}(\mathcal{C})\vert$
S_5	21	3	10(1)	5040	$[21, 14, 4]$	$[21, 7, 6]$	5040
			10(1)	5040	$[21, 6, 6]$	$[21, 15, 3]$	5040
$(A_4 \times 3){:}2$	35	4	4(1)	5040	$[35, 20, 4]$	$[35, 15, 5]$	5040
			12(1)	5040	$[35, 20, 4]$	$[35, 15, 5]$	5040
			18(1)	40320	—	—	—

TABLE 2. The stabilizers and designs from $\mathcal{C}^{\perp}_{(10)_1}$.

wt	$(S_7)_{(l)_i,\mathcal{C}^{\perp}_{(10)_1}}$	Max.	$\mathcal{D}_{(l)_i,\mathcal{C}^{\perp}_{(10)_1}}$	# blocks
6	S_6	Yes	1-(21, 6, 2)	7
9	$S_4 \times S_3$	Yes	1-(21, 9, 15)	35
10	$S_5 \times 2$	Yes	1-(21, 10, 10)	21
11	$S_5 \times 2$	Yes	1-(21, 11, 11)	21
12	$S_4 \times S_3$	Yes	1-(21, 12, 20)	35
15	S_6	Yes	1-(21, 15, 5)	7

TABLE 3. The stabilizers and designs from $\mathcal{C}^{\perp}_{(10)_2}$.

wt	$(S_7)_{(l)_i,\mathcal{C}^{\perp}_{(10)_2}}$	Max.	$\mathcal{D}_{(l)_i,\mathcal{C}^{\perp}_{(10)_2}}$	# blocks
3	$S_4 \times S_3$	Yes	1-(21, 3, 5)	35
4	$D_8 \times S_3$	No	1-(21, 4, 20)	105
5	$5{:}2^2$	No	2-(21, 5, 12)	252
$(6)_1$	$3^2{:}D_8$	No	1-(21, 6, 20)	70
$(6)_2$	D_{12}	No	1-(21, 6, 120)	420
$(6)_3$	$D_8 \times 2$	No	1-(21, 6, 90)	315
$(7)_1$	2^2	No	1-(21, 7, 420)	1260
$(7)_2$	D_{14}	No	1-(21, 7, 120)	360
$(7)_3$	$D_{12} \times 2$	No	1-(21, 7, 70)	210
$(7)_4$	$D_8 \times S_3$	No	1-(21, 7, 35)	105
$(8)_1$, $(8)_2$	2^2	No	1-(21, 8, 480)	1260
$(8)_3$	D_8	No	1-(21, 8, 240)	630
$(8)_4$	$S_4 \times 2$	No	1-(21, 8, 40)	105
$(9)_1$	S_3	No	1-(21, 9, 360)	840
$(9)_2$	D_8	No	1-(21, 9, 270)	630
$(9)_3$, $(9)_4$	2^2	No	1-(21, 9, 540)	1260
$(9)_5$	$S_4 \times 2$	No	1-(21, 9, 45)	105
$(9)_6$	D_{12}	No	1-(21, 9, 180)	420
$(10)_1$	D_8	No	1-(21, 10, 300)	630
$(10)_2$	2^3	No	1-(21, 10, 300)	630
$(10)_3$	D_{12}	No	1-(21, 10, 200)	420
$(10)_4$	2	No	1-(21, 10, 1200)	2520
$(10)_5$	$A_5{:}2^2$	Yes	1-(21, 10, 10)	21
$(10)_6$	2^2	No	1-(21, 10, 600)	1260
$(11)_1$	2	No	1-(21, 11, 1320)	2520
$(11)_2$	D_8	No	1-(21, 11, 330)	630
$(11)_3$	2^3	No	1-(21, 11, 330)	630
$(11)_4$	2^2	No	1-(21, 11, 660)	1260
$(11)_5$	D_{12}	No	1-(21, 11, 220)	420
$(11)_6$	$A_5{:}2^2$	Yes	1-(21, 11, 11)	21
$(12)_1$	D_8	No	1-(21, 12, 360)	630
$(12)_2$, $(12)_3$	2^2	No	1-(21, 12, 720)	1260
$(12)_4$	S_3	No	1-(21, 12, 480)	840
$(12)_5$	$S_4 \times 2$	No	1-(21, 12, 60)	105
$(12)_6$	D_{12}	No	1-(21, 12, 240)	420
$(13)_1$, $(13)_2$	2^2	No	1-(21, 13, 780)	1260
$(13)_3$	D_8	No	1-(21, 13, 390)	630
$(13)_4$	$S_4 \times 2$	No	1-(21, 13, 65)	105
$(14)_1$	D_{14}	No	1-(21, 14, 240)	360
$(14)_2$	2^2	No	1-(21, 14, 840)	1260
$(14)_3$	$D_{12} \times 2$	No	1-(21, 14, 140)	210
$(14)_4$	$D_8 \times S_3$	No	1-(21, 14, 70)	105
$(15)_1$	D_{12}	No	1-(21, 15, 300)	420
$(15)_2$	$3^2{:}D_8$	No	1-(21, 15, 50)	70
$(15)_3$	$D_8 \times 2$	No	1-(21, 15, 225)	315
16	$5{:}2^2$	No	2-(21, 16, 144)	252
17	$D_8 \times S_3$	No	1-(21, 17, 85)	105
18	$S_4 \times S_3$	No	1-(21, 18, 30)	35

TABLE 4. The stabilizers and 1-designs from $\mathcal{C}_4$.

wt	$(S_7)_{(l)_i,\mathcal{C}_4}$	Max.	$\mathcal{D}_{(l)_i,\mathcal{C}_4}$	# blocks
4	$S_4 \times S_3$	Yes	1-(35, 4, 4)	35
6	$D_{12} \times 2$	No	1-(35, 6, 36)	210
$(8)_1$	2^2	No	1-(35, 8, 288)	1260
$(8)_2$	$S_4 \times 2$	No	1-(35, 8, 24)	105
$(8)_3$	$3^2{:}D_8$	No	1-(35, 8, 16)	70
$(8)_4$	$D_8 \times 2$	No	1-(35, 8, 72)	315
$(10)_1, \ldots, (10)_4$	2^2	No	1-(35, 10, 360)	1260
$(10)_5$	D_{20}	No	1-(35, 10, 72)	252
$(10)_6$	2	No	1-(35, 10, 720)	2520
$(10)_7, (10)_8$	S_3	No	1-(35, 10, 240)	840
$(10)_9$	A_5	No	1-(35, 10, 24)	84
$(10)_{10}$	$S_3 \times S_3$	No	1-(35, 10, 40)	140
$(10)_{11}$	2^3	No	1-(35, 10, 180)	630
$(10)_{12}$	$S_3 \times 2^2$	No	1-(35, 10, 60)	210
$(12)_1, \ldots, (12)_9$	2	No	1-(35, 12, 864)	2520
$(12)_{10}, \ldots, (12)_{14}$	2^2	No	1-(35, 12, 432)	1260
$(12)_{15}, \ldots, (12)_{17}$	1	No	1-(35, 12, 1728)	5040
$(12)_{18}, (12)_{19}$	S_3	No	1-(35, 12, 288)	840
$(12)_{20}, \ldots, (12)_{22}$	D_8	No	1-(35, 12, 216)	630
$(12)_{23}$	2^3	No	1-(35, 12, 216)	630
$(12)_{24}, (12)_{25}$	$D_8 \times 2$	No	1-(35, 12, 108)	315
$(12)_{26}$	S_4	No	1-(35, 12, 72)	210
$(12)_{27}$	$S_4 \times S_3$	Yes	1-(35, 12, 12)	35
$(12)_{28}$	$S_4 \times 2$	No	1-(35, 12, 36)	105
$(12)_{29}$	D_{12}	No	1-(35, 12, 144)	420
$(14)_1, \ldots, (14)_{12}$	1	No	1-(35, 14, 2016)	5040
$(14)_{13}, \ldots, (14)_{36}$	2	No	1-(35, 14, 1008)	2520
$(14)_{37}, \ldots, (14)_{49}$	2^2	No	1-(35, 14, 504)	1260
$(14)_{50}, (14)_{51}$	S_3	No	1-(35, 14, 336)	840
$(14)_{52}$	2^3	No	1-(35, 14, 252)	630
$(14)_{53}$	D_{12}	No	1-(35, 14, 168)	420
$(14)_{54}$	$D_{12} \times 2$	No	1-(35, 14, 84)	210
$(14)_{55}$	$S_3 \times S_3$	No	1-(35, 14, 56)	140
$(14)_{56}$	$D_{14}{:}3$	Yes	1-(35, 14, 48)	120
$(16)_1, \ldots, (16)_{30}$	1	No	1-(35, 16, 2304)	5040
$(16)_{31}, \ldots, (16)_{57}$	2	No	1-(35, 16, 1152)	2520
$(16)_{58}, (16)_{59}$	3	No	1-(35, 16, 768)	1680
$(16)_{60}, \ldots, (16)_{77}$	2^2	No	1-(35, 16, 576)	1260
$(16)_{78}$	4	No	1-(35, 16, 576)	1260
$(16)_{79}, \ldots, (16)_{81}$	S_3	No	1-(35, 16, 384)	840
$(16)_{82}, \ldots, (16)_{85}$	D_8	No	1-(35, 16, 288)	630
$(16)_{86}$	2^3	No	1-(35, 16, 288)	630
$(16)_{87}, (16)_{88}$	D_{12}	No	1-(35, 16, 192)	420
$(16)_{89}$	$D_8 \times 2$	No	1-(35, 16, 144)	315
$(16)_{90}, (16)_{91}$	S_4	No	1-(35, 16, 96)	210
$(16)_{92}$	$S_4 \times 2$	No	1-(35, 16, 48)	105
$(16)_{93}$	$S_4 \times S_3$	Yes	1-(35, 16, 16)	35

$(18)_1, \ldots, (18)_{30}$	1	No	1-(35, 18, 2592)	5040
$(18)_{31}, \ldots, (18)_{71}$	2	No	1-(35, 18, 1296)	2520
$(18)_{72}$	3	No	1-(35, 18, 864)	1680
$(18)_{73}, \ldots, (18)_{85}$	2^2	No	1-(35, 18, 648)	1260
$(18)_{86}$	6	No	1-(35, 18, 432)	840
$(18)_{87}, \ldots, (18)_{89}$	2^3	No	1-(35, 18, 324)	630
$(18)_{90}, \ldots, (18)_{93}$	D_{12}	No	1-(35, 18, 216)	420
$(18)_{94}$	2^3:3	No	1-(35, 18, 108)	210
$(20)_1, \ldots, (20)_{22}$	1	No	1-(35, 20, 2880)	5040
$(20)_{23}, \ldots, (20)_{46}$	2	No	1-(35, 20, 1440)	2520
$(20)_{47}$	3	No	1-(35, 20, 960)	1680
$(20)_{48}, \ldots, (20)_{54}$	2^2	No	1-(35, 20, 720)	1260
$(20)_{55}$	4	No	1-(35, 20, 720)	1260
$(20)_{56}, \ldots, (20)_{58}$	S_3	No	1-(35, 20, 480)	840
$(20)_{59}, \ldots, (20)_{63}$	D_8	No	1-(35, 20, 360)	630
$(20)_{64}$	D_{10}	No	1-(35, 20, 288)	504
$(20)_{65}, (20)_{66}$	D_{12}	No	1-(35, 20, 240)	420
$(20)_{67}$	$D_8 \times 2$	No	1-(35, 20, 180)	315
$(20)_{68}$	D_{10}:2	No	1-(35, 20, 144)	252
$(20)_{69}$	$D_{12} \times 2$	No	1-(35, 20, 120)	210
$(20)_{70}$	$S_5 \times 2$	Yes	1-(35, 20, 12)	21
$(20)_{71}$	S_6	Yes	1-(35, 20, 4)	3
$(22)_1, \ldots, (22)_8$	1	No	1-(35, 22, 3168)	5040
$(22)_9, \ldots, (22)_{22}$	2	No	1-(35, 22, 1584)	2520
$(22)_{23}, (22)_{24}$	3	No	1-(35, 22, 1056)	1680
$(22)_{25}, \ldots, (22)_{31}$	2^2	No	1-(35, 22, 792)	1260
$(22)_{32}, \ldots, (22)_{34}$	S_3	No	1-(35, 22, 528)	840
$(22)_{35}$	2^3	No	1-(35, 22, 396)	630
$(22)_{36}$	D_{12}	No	1-(35, 22, 264)	420
$(22)_{37}$	$S_3 \times S_3$	No	1-(35, 22, 88)	140
$(24)_1, (24)_2$	1	No	1-(35, 24, 3456)	5040
$(24)_3, \ldots, (24)_6$	2	No	1-(35, 24, 1728)	2520
$(24)_7, \ldots, (24)_9$	2^2	No	1-(35, 24, 864)	1260
$(24)_{10}$	4	No	1-(35, 24, 864)	1260
$(24)_{11}$	S_3	No	1-(35, 24, 576)	840
$(24)_{12}$	D_8	No	1-(35, 24, 432)	630
$(24)_{13}$	2^3	No	1-(35, 24, 432)	630
$(24)_{14}$	D_{12}	No	1-(35, 24, 288)	420
$(24)_{15}$	$D_8 \times 2$	No	1-(35, 24, 216)	315
$(24)_{16}$	S_4	No	1-(35, 24, 144)	210
$(24)_{17}$	$S_4 \times 2$	No	1-(35, 24, 72)	105
$(24)_{18}$	$(S_3 \times S_3)$:2	No	1-(35, 24, 48)	70
$(26)_1, (26)_2$	2^2	No	1-(35, 26, 936)	1260
$(26)_3$	S_3	No	1-(35, 26, 624)	840
$(26)_4$	$D_{12} \times 2$	No	1-(35, 26, 156)	210
$(26)_5$	2^3:3	No	1-(35, 26, 156)	210
28	$PSL_2(7)$	No	1-(35, 28, 24)	30

Table 5. The stabilizers and designs from $\mathcal{C}_4^{\perp}$.

wt	$(S_7)_{(l)_i,\mathcal{C}_4^{\perp}}$	Max.	$\mathcal{D}_{(l)_i,\mathcal{C}_4^{\perp}}$	# blocks
5	$S_5 \times 2$	Yes	1-(35, 5, 3)	21
8	$S_4 \times 2$	No	1-(35, 8, 24)	105
9	$(S_3 \times S_3){:}2$	No	1-(35, 9, 18)	70
10	$D_8 \times S_3$	No	1-(35, 10, 30)	105
11	D_{12}	No	1-(35, 11, 132)	420
$(12)_1$	D_8	No	1-(35, 12, 216)	630
$(12)_2$	$D_8 \times S_3$	No	1-(35, 12, 36)	105
$(13)_1$	D_8	No	1-(35, 13, 234)	630
$(13)_2$	2^3	No	1-(35, 13, 234)	630
$(13)_3$	$S_4 \times S_3$	Yes	1-(35, 13, 13)	35
$(14)_1$	2^2	No	1-(35, 14, 504)	1260
$(14)_2$	D_{12}	No	1-(35, 14, 168)	420
$(14)_3$	D_{14}	No	1-(35, 14, 144)	360
$(15)_1, (15)_2$	2^2	No	1-(35, 15, 540)	1260
$(15)_3$	D_{20}	No	1-(35, 15, 108)	252
$(15)_4$	$S_4 \times 2$	No	1-(35, 15, 45)	105
$(16)_1, (16)_2$	2^2	No	1-(35, 16, 576)	1260
$(16)_3$	S_3	No	1-(35, 16, 384)	840
$(16)_4$	D_8	No	1-(35, 16, 288)	630
$(17)_1$	2	No	1-(35, 17, 1224)	2520
$(17)_2$	2^2	No	1-(35, 17, 612)	1260
$(17)_3$	D_{12}	No	1-(35, 17, 204)	420
$(17)_4$	$D_8 \times 2$	No	1-(35, 17, 153)	315
$(17)_5$	$D_{12} \times 2$	No	1-(35, 17, 102)	210
$(18)_1$	2	No	1-(35, 18, 1296)	2520
$(18)_2$	2^2	No	1-(35, 18, 648)	1260
$(18)_3$	D_{12}	No	1-(35, 18, 216)	420
$(18)_4$	$D_8 \times 2$	No	1-(35, 18, 162)	315
$(18)_5$	$D_{12} \times 2$	No	1-(35, 18, 108)	210
$(19)_1, (19)_2$	2^2	No	1-(35, 19, 684)	1260
$(19)_3$	S_3	No	1-(35, 19, 456)	840
$(19)_4$	D_8	No	1-(35, 19, 342)	630
$(20)_1, (20)_2$	2^2	No	1-(35, 20, 720)	1260
$(20)_3$	D_{20}	No	1-(35, 20, 144)	252
$(20)_4$	$S_4 \times 2$	No	1-(35, 20, 60)	105
$(21)_1$	2^2	No	1-(35, 21, 756)	1260
$(21)_2$	D_{12}	No	1-(35, 21, 252)	420
$(21)_3$	D_{14}	No	1-(35, 21, 216)	360
$(22)_1$	2^3	No	1-(35, 22, 396)	630
$(22)_2$	D_8	No	1-(35, 22, 396)	630
$(22)_3$	$S_4 \times S_3$	Yes	1-(35, 22, 22)	35
$(23)_1$	D_8	No	1-(35, 23, 414)	630
$(23)_2$	$D_8 \times S_3$	No	1-(35, 23, 69)	105
24	D_{12}	No	1-(35, 24, 288)	420
25	$D_8 \times S_3$	No	1-(35, 25, 75)	105
26	$(S_3 \times S_3){:}2$	No	1-(35, 26, 52)	70
27	$S_4 \times 2$	No	1-(35, 27, 81)	105
30	$S_5 \times 2$	Yes	1-(35, 30, 18)	21

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