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INDEX RANK-k NUMERICAL RANGE OF MATRICES

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Abstract. We introduce the $\alpha-$higher rank form of the matrix numerical range, which is a special case of the matrix polynomial version of the higher rank numerical range. We also, investigate some algebraic and geometrical properties of this set for general and nilpotent matrices. Some examples to confirm the results are brought.

Keywords: $\alpha-$higher rank numerical range, index higher rank numerical range, $\alpha-$numerical range, index numerical range.

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1. Introduction

Let $\mathbb{M}_n$ be the algebra of all $n\times n$ complex matrices and $A \in \mathbb{M}_n$. We denote by $\mathrm{col}(A)$ the column space of A, and by $\ker(A)$ the null space or the kernel of A. We know that $\{0\} = \ker(A^0) \subseteq \ker(A) \subseteq \ker(A^2) \subseteq \cdots$, and $\mathbb{C}^n = \mathrm{col}(A^0) \supseteq \mathrm{col}(A) \supseteq \mathrm{col}(A^2) \supseteq \cdots$, where by convention $A^0 = I_n$ is the $n \times n$ identity matrix. Let $k \in \mathbb{N}\cup\{0\}$ be the smallest number such that $\ker(A^k) = \ker(A^{k+1})$, or equivalently, $\mathrm{col}(A^k) = \mathrm{col}(A^{k+1})$. The number k is called the index of A and denoted by $\mathrm{ind}(A)$. Obviously, $\mathbb{C}^n = \ker(A^k) \oplus \mathrm{col}(A^k)$. If $\mathrm{rank}(A^k) = r$, then A is similar to a matrix $\begin{pmatrix} C & 0 \\ 0 & N \end{pmatrix}$, where $C \in \mathbb{M}_r$ is an invertible matrix and $N \in \mathbb{M}_{n-r}$ is a nilpotent matrix of index k. It is clear that $\mathrm{ind}(A) = 0$ for nonsingular matrix A, and for the case that A is singular, $\mathrm{ind}(A)$ is equal to the maximum size of Jordan blocks of A related to the zero eigenvalue.

Throughout the paper, k, m and n are considered as natural numbers, and $k \leq n$. Let

$$\mathcal{P}(\gamma) = A_m\gamma^m + A_{m-1}\gamma^{m-1} + \cdots + A_1\gamma + A_0, \tag{1}$$

where $A_i \in \mathbb{M}_n$ $(i = 0, 1, \ldots, m)$ and $A_m \neq 0$, be a matrix polynomial of degree m and order n. The numerical range of $\mathcal{P}(\gamma)$ is defined and denoted in [7] by

$$\mathcal{W}[\mathcal{P}(\gamma)] = \{z \in \mathbb{C} : x^*\mathcal{P}(z)x = 0 \text{ for some nonzero } x \in \mathbb{C}^n\}. \tag{2}$$

For the special case $\mathcal{P}(\gamma) = \gamma I - A$, where $A \in \mathbb{M}_n$, we see that $\mathcal{W}[\mathcal{P}(\gamma)] = \mathcal{W}(A) := \{x^*Ax : x \in \mathbb{C}^n, x^*x = 1\}$ which is the classical numerical range

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of A. For general information on the numerical range of matrices and matrix polynomials, we refer to [4, 5, 7].

Let $A \in M_n$. The matrix polynomial $(A - \lambda I)^{\alpha+1}$, in which λ is a complex variable, α is a nonnegative integer and $I \in M_n$ is the identity matrix, is in particular importance. The matrix polynomial $(A - \lambda I)^{\alpha+1}$ is useful in Krylove subspace methods to solve the equation systems $Ax = b$. Safarzadeh and Salemi in [8] used the joint numerical range of the powers of $(A - \lambda I)$ to find the polynomial numerical hulls of the matrix A and used this to find bounds on the number of needed iterations required to reach the solution. For every nonnegative integer α and $A \in \mathbb{M}_n$, the numerical range of the matrix polynomial $\mathcal{P}(\gamma) = (\gamma I - A)^{\alpha+1}$ is called the *$\alpha-$numerical range* of A. The $\alpha-$numerical range of A introduced in [8] as

$$\mathcal{W}^{(\alpha)}(A) := \mathcal{W}[\mathcal{P}(\gamma) = (\gamma I - A)^{\alpha+1}]. \tag{3}$$

If $\alpha = \operatorname{ind}(A)$, then $\mathcal{W}^{(\alpha)}(A)$ is called the *index numerical range* of A, and is denoted by $\mathcal{IW}(A)$. $\alpha-$numerical range of the matrix A is a connected and compact subset of $\mathbb{C}$ which contains the spectrum of A [8, Theorem2]. Let $\mathcal{P}(\gamma)$ be a matrix polynomial as in (1). The *rank-k numerical range* of $\mathcal{P}(\gamma)$ is introduced by Aretaki and Maroulas (see [1]) as

$$\Lambda_k[\mathcal{P}(\gamma)] = \{\xi \in \mathbb{C} : Y^*\mathcal{P}(\xi)Y = 0_k \ \text{ for some } Y \in \mathbb{I}_{n\times k}\}, \tag{4}$$

where $\mathbb{I}_{n\times k} = \{Y \in \mathbb{M}_{n\times k} : Y^*Y = I_k\}$ is the set of all $n \times k$ isometry matrices and 0_k is the $k \times k$ zero matrix. It is known that $\Lambda_1[\mathcal{P}(\gamma)] = \mathcal{W}[\mathcal{P}(\gamma)]$. Also, for the special case $\mathcal{P}(\gamma) = \gamma I_n - A$, where $A \in \mathbb{M}_n$, we have $\Lambda_k[\mathcal{P}(\gamma)] = \Lambda_k(A) := \{\gamma \in \mathbb{C} : Y^*AY = \gamma I_k \text{ for some } Y \in \mathbb{I}_{n\times k}\}$ which is the *rank$-k$ numerical range* of A. Choi et.al., in [3, Proposition 1.1], showed equivalent descriptions of $\Lambda_k(A)$, such as $\lambda \in \Lambda_k(A)$ if and only if there exists a unitary matrix U such that the upper left $k \times k$ submatrix of U^*TU is λI_k. In this paper, after reviewing some preliminaries in Section 2, similar to the idea in [8] and using the concept of higher rank numerical range of matrix polynomials, we introduce the notion of the $\alpha-$rank$-k$ numerical range of matrices, see Section 3 below. In this section, we also study some of its algebraic and geometrical properties. In Section 4, we present the results of our study about the $\alpha-$rank$-k$ numerical range of nilpotent matrices.

2. Index higher rank numerical range

We use the concept of higher rank numerical range of matrix polynomials to introduce the notion of $\alpha-$rank$-k$ numerical range of matrices as follows:

Definition 2.1. Let $A \in \mathbb{M}_n$ and α be a nonnegative integer. The $\alpha-$rank$-k$ numerical range of A is defined and denoted by

$$\begin{aligned}\Lambda_k^{(\alpha)}(A) &= \{\xi \in \mathbb{C} : Y^*(A - \xi I)^{\alpha+1}Y = 0_k \text{ for some } Y \in \mathbb{I}_{n\times k}\}\\ &= \Lambda_k[(A - \xi I)^{\alpha+1}].\end{aligned}$$

In the special case $\alpha = \text{ind}(A)$, this set is called the index rank$-k$ numerical range of A and is denoted by $\mathcal{I}\Lambda_k(A)$.

When we speak about these sets in general and it is not important to emphasize k, we use $\alpha-$*higher rank numerical range* and *index higher rank numerical range* instead of $\alpha-$rank$-k$ numerical range and index rank$-k$ numerical range, respectively. In the following lemma, we summarize some elementary properties of the higher rank numerical range of matrix polynomials.

Lemma 2.2. *Let $\mathcal{P}(\gamma)$ be a matrix polynomial as in (1) with the numerical range $\mathcal{W}[\mathcal{P}(\gamma)]$, as in (2), and the rank$-k$ numerical range $\Lambda_k[\mathcal{P}(\gamma)]$, as in (4). Then the following assertions are true:*
(i) *[1, Proposition 1] $\Lambda_k[\mathcal{P}(\gamma)]$ is a closed subset in $\mathbb{C}$;*
(ii) *[1, Proposition 3] $\Lambda_k[\mathcal{P}(\gamma)] \subseteq \Lambda_{k-1}[\mathcal{P}(\gamma)] \subseteq \ldots \subseteq \Lambda_1[\mathcal{P}(\gamma)] = \mathcal{W}[\mathcal{P}(\gamma)]$;*
(iii) *[1, Propositionn 10] If $A_m \neq 0$ and $0 \notin \Lambda_k(A_m)$, then $\Lambda_k[\mathcal{P}(\gamma)]$ is bounded;*
(iv) *[1, Corollary 4] For any $k \leq n$, $\Lambda_k[\underbrace{\mathcal{P}(\gamma) \oplus \cdots \oplus \mathcal{P}(\gamma)}_{k-\ times}] = \mathcal{W}[\mathcal{P}(\gamma)]$;*
(v) *[1, Proposition 12] If $\xi \in \sigma[\mathcal{P}(\gamma)] \cap \partial\mathcal{W}[\mathcal{P}(\gamma)]$ with algebraic multiplicity k, then for $j = 1, \ldots, k$, $\xi \in \partial\Lambda_j[\mathcal{P}(\gamma)]$.*

In the following proposition we state some basic properties of $\alpha-$higher rank numerical range and index higher rank numerical range of matrices.

Proposition 2.3. *Let $A \in \mathbb{M}_n$. The following assertions are true:*
(i) *If $U \in \mathbb{M}_n$ is a unitary matrix, then $\Lambda_k^{(\alpha)}(U^*AU) = \Lambda_k^{(\alpha)}(A)$;*
(ii) *$\Lambda_k^{(\alpha)}(A) \subseteq \Lambda_{k-1}^{(\alpha)}(A) \subseteq \cdots \subseteq \Lambda_1^{(\alpha)}(A) = \mathcal{W}^{(\alpha)}(A)$;*
(iii) *If A is a nonsingular matrix, then $\mathcal{I}\Lambda_k(A) = \Lambda_k(A)$;*
(iv) *If $A \in \mathbb{M}_n(\mathbb{R})$, then $\Lambda_k^{(\alpha)}(A)$ is symmetric concerning real line;*
(v) *$\Lambda_k^{(\alpha)}(A)$ is a compact subset of $\mathbb{C}$;*
(vi) *If $t \in \mathbb{C}$, then $\Lambda_k^{(\alpha)}(tA) = t\Lambda_k^{(\alpha)}(A)$;*
(vii) *If $A, A^2, \ldots, A^{\alpha+1}$ have the same totally isotropic subspace $S = span\{x_1, \ldots, x_k\}$(i.e., $x_i s$ are orthonormal vectors for $i = 1, 2, \ldots k$, and $x_i^* A^l x_j = 0$ for $i, j = 1, \ldots, k$ and $l = 1, \ldots, \alpha + 1$), then $0 \in \Lambda_k^{(\alpha)}(A)$;*
$(viii)$ *If $\gamma \in \sigma(A) \cap \partial\mathcal{W}^{(\alpha)}(A)$ with algebraic multiplicity k, then $\gamma \in \partial\Lambda_k^{(\alpha)}(A)$.*

Proof. Let $X \in \mathbb{I}_{n\times k}$ be such that $X^*(A - \lambda I)^{\alpha+1}X = 0I_k$. If U is a unitary matrix, then using the matrix $U^*X \in \mathbb{I}_{n\times k}$, one can see that $\lambda \in \Lambda_k^{(\alpha)}(U^*AU)$. Also, if $X \in \mathbb{I}_{n\times k}$ such that $X^*(U^*AU - \lambda I)^{\alpha+1}X = 0I_k$, then using $UX \in \mathbb{I}_{n\times k}$, one can see that $\lambda \in \Lambda_k^{(\alpha)}(A)$ and the assertion in (i) holds. The result in (ii) is a consequence of Lemma 2.2(ii). If A is nonsingular, then $\text{ind}(A) = 0$ and this shows that $\mathcal{I}\Lambda_k(A) = \Lambda_k[A - \lambda I] = \Lambda_k(A)$, where $A - \lambda I$ is considered as a matrix polynomial. This proves the result in (iii). To see (iv), one can use the fact $A = \bar{A}$ and the definition of $\Lambda_k^{(\alpha)}(A)$. Since $(A - \lambda I)^{\alpha+1}$ is a monic matrix polynomial, using Lemma 2.2((i) and (iii)), the assertion in (v) also holds. To see (vi), suppose that $t \neq 0$. Then $\Lambda_k^{(\alpha)}(tA) = \{\lambda \in \mathbb{C} : 0 \in \Lambda_k(t^{\alpha+1}((\lambda/t)I - A)^{\alpha+1})\} = \{\lambda \in \mathbb{C} : 0 \in t^{\alpha+1}\Lambda_k((\lambda/t)I - A)^{\alpha+1})\}$. So, $\lambda \in \Lambda_k^{(\alpha)}(tA)$ if and only if $\lambda/t \in \Lambda_k^{(\alpha)}(A)$. This shows that $\Lambda_k^{(\alpha)}(tA) = t\Lambda_k^{(\alpha)}(A)$. For $t = 0$, the equality holds obviously. To prove the assertion in (vii), note that by the hypothesis, we have

$$\begin{aligned} x_i^*(A-\lambda I)^{\alpha+1}x_j &= x_i^*(-1)^{\alpha+1}\lambda^{\alpha+1}I_n x_j \\ &= (-1)^{\alpha+1}\lambda^{\alpha+1}x_i^*x_j \\ &= \begin{cases} (-1)^{\alpha+1}\lambda^{\alpha+1} & i = j, \\ 0 & i \neq j. \end{cases} \end{aligned}$$

If $X = (x_1, x_2, \ldots, x_n) \in \mathbb{I}_{n\times k}$, using the above equalities, the relation $X^*(A - \lambda I)^{\alpha+1}X = 0I_k$ holds for $\lambda = 0$ and so $0 \in \Lambda_k^{(\alpha)}(A)$. By $\partial\Lambda_k^{(\alpha)}(A)$ we mean the set of boundary points of the $\Lambda_k^{(\alpha)}(A)$. Since $\gamma \in \sigma(A)$, there exists an eigenvector $x \in \mathbb{C}^n$ such that $Ax = \gamma x$. So, $(Ax - \gamma I)^{\alpha+1}x = 0$. It follows that $\gamma \in \sigma(A - zI)^{\alpha+1}$. Now, by Lemma 2.2$(v)$ the result in $(viii)$ follows. □

In the following proposition, we prove subset relations between the intersection and union of $\alpha-$higher rank numerical range of matrices and $\alpha-$higher rank numerical range of a direct sum of matrices. Note that the identity matrix I, in the proof of the following proposition, is of appropriate size.

Proposition 2.4. *Let $A, B \in \mathbb{M}_n$, k_1, k_2 and α be a positive integers such that $1 \leq k_1, k_2 \leq n$. Then, the following assertions are true:*
(i) $\Lambda_k^{(\alpha)}(A) \cup \Lambda_k^{(\alpha)}(B) \subseteq \Lambda_k^{(\alpha)}(A \oplus B)$;
(ii) $\Lambda_{k_1}^{(\alpha)}(A) \cap \Lambda_{k_2}^{(\alpha)}(B) \subseteq \Lambda_{k_1+k_1}^{(\alpha)}(A \oplus B)$.

Proof. To see part (i), let $\lambda \in \Lambda_k^{(\alpha)}(A)$, then we have $0 \in \Lambda_k[(A-\lambda I)^{(\alpha+1)}]$. So, there exists a unitary matrix $U \in \mathbb{M}_n$ such that $U^*(A - \lambda I)^{\alpha+1}U = \begin{bmatrix} 0_k & * \\ * & * \end{bmatrix}$, where 0_k denotes the $k \times k$ zero matrix. Let $V = U \oplus I \in \mathbb{M}_{2n}(C)$, then

$$\begin{aligned} V^*\big((A \oplus B) - \lambda I\big)^{\alpha+1}V &= V^*\big((A - \lambda I) \oplus (B - \lambda I)\big)^{\alpha+1}V \\ &= V^*\big((A - \lambda I)^{\alpha+1} \oplus (B - \lambda I)^{\alpha+1}\big)V \\ &= \begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix}^* \big((A - \lambda I)^{\alpha+1} \oplus (B - \lambda I)^{\alpha+1}\big) \begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} U^*(A - \lambda I)^{\alpha+1}U & 0 \\ 0 & (B - \lambda I)^{\alpha+1} \end{bmatrix} \\ &= \begin{bmatrix} 0_k & * \\ * & * \end{bmatrix} \in \mathbb{M}_{2n}, \end{aligned}$$

which means $\lambda \in \Lambda_k^{(\alpha)}(A \oplus B)$ and so, $\Lambda_k^{(\alpha)}(A) \subseteq \Lambda_k^{(\alpha)}(A \oplus B)$. By the same way, one can see that $\Lambda_k^{(\alpha)}(B) \subseteq \Lambda_k^{(\alpha)}(A \oplus B)$ and the result in (i) follows.

Let $\lambda \in \Lambda^{(\alpha)}_{k_1}(A) \cap \Lambda^{(\alpha)}_{k_2}(B)$, then $0 \in \Lambda_{k_1}(A-\lambda I)^{\alpha+1} \cap \Lambda_{k_2}(B-\lambda I)^{\alpha+1} \subseteq \Lambda_{k_1+k_2}\big((A-\lambda I)^{\alpha+1} \oplus (B-\lambda I)^{\alpha+1}\big)$, where the subset relation is a consequence of the relation $\Lambda_{k_1}(A) \cap \Lambda_{k_1}(B) \subseteq \Lambda_{k_1+k_2}(A \oplus B)$ (see [2, Page 830]). Such as in the proof of part (i), one can see that $(A-\lambda I)^{\alpha+1} \oplus (B-\lambda I)^{\alpha+1} = (A \oplus B - \lambda I)^{\alpha+1}$. So, $0 \in \Lambda_{k_1+k_2}\big((A-\lambda I)^{\alpha+1} \oplus (B-\lambda I)^{\alpha+1}\big) = \Lambda_{k_1+k_2}\big((A \oplus B - \lambda I)^{\alpha+1}\big)$ and so, $\lambda \in \Lambda^{(\alpha)}_{k_1+k_2}(A \oplus B)$. That is the result in part (ii).

□

The following proposition gives a connection between the $\alpha-$numerical range and the $\alpha-$higher rank numerical range of a matrix.

Proposition 2.5. *Let $A \in \mathbb{M}_n$. Then for any $k \leq n$,*

$$\Lambda^{(\alpha)}_k(\underbrace{A \oplus \cdots \oplus A}_{k-\ times}) = \mathcal{W}^{(\alpha)}(A).$$

In particular, $\mathcal{I}\Lambda_k(\underbrace{A \oplus \cdots \oplus A}_{k-\ times}) = I\mathcal{W}(A)$.

Proof. By Definition 2.1 and using Lemma 2.2(iv), we have:

$$\begin{aligned}
\Lambda^{(\alpha)}_k(\underbrace{A \oplus \cdots \oplus A}_{k-\text{ times}}) &= \Lambda_k[((A \oplus \cdots \oplus A) - \lambda I_{nk})^{\alpha+1}] \\
&= \Lambda_k[\underbrace{(A-\lambda I_n)^{\alpha+1} \oplus \cdots \oplus (A-\lambda I_n)^{\alpha+1}}_{k-\text{ times}}] \\
&= \mathcal{W}[(A-\lambda I_n)^{\alpha+1}] \\
&= \mathcal{W}^{(\alpha)}(A).
\end{aligned}$$

Hence, the proof is complete. □

The following two lemmas are useful in the last examples. Note that J_n is the $n \times n$ Jordan block with zero eigenvalue.

Lemma 2.6. *[8, Theorem 7 and Example 1] Let $A = J_n$. Then $\{z \in \mathbb{C} : |z| \leq (n+1)/2\} \subseteq \mathcal{I}\mathcal{W}(A)$. For the case $A = J_2$, the set equality holds.*

Lemma 2.7. *[8, Lemma 2] Let ξ be a nonzero real number and $A = diag(\xi, 0)$. Then, $\mathcal{I}\mathcal{W}(A) = \{z : |z - (\xi/2)| = |\xi|/2\}$.*

In the following two examples, we see that the $\alpha-$higher rank numerical range can be an empty or a nonempty set in $\mathbb{C}$.

Example 2.8. *Let J_2 be the 2×2 Jordan canonical matrix. One can see, by direct computation, that $\Lambda^{(2)}_2(J_2) = \mathcal{I}\Lambda_2(J_2) = \{0\}$. Also, by Proposition 2.5 and Lemma 2.6,*

$$\begin{aligned}
\mathcal{I}\Lambda_2(J_2 \oplus J_2) &= \mathcal{I}\mathcal{W}(J_2) \\
&= \mathcal{I}\Lambda_k(\underbrace{J_2 \oplus \cdots \oplus J_2}_{k-\ times}) = \{z \in \mathbb{C} : |z| \leq 1.5\},
\end{aligned}$$

which is a nonempty set.

Example 2.9. *If $\lambda \neq 0$ and $A = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}$, then by Lemma 2.7, $\Lambda_1^{(1)}(A) = \mathcal{IW}(A) = \{z \in \mathbb{C} : |z - (\lambda/2)| = \lambda/2\} \neq \emptyset$. If $k = 2$, to find $\Lambda_2^{(1)}(A)$, we must use $X \in \mathbb{I}_{2\times 2}$ such that $X^* \begin{pmatrix} (\lambda - z)^2 & 0 \\ 0 & z^2 \end{pmatrix} X = 0I_2$. This means that the matrix $\begin{pmatrix} (\lambda - z)^2 & 0 \\ 0 & z^2 \end{pmatrix}$ is unitarily similar to $0I_2$. Since the eigenvalues of unitarily similar matrices are equal, we have $\lambda = 0$ which contradicts the assumption $\lambda \neq 0$. This contradiction shows that $\Lambda_2^{(1)}(A) = \mathcal{I}\Lambda_2(A) = \emptyset$.*

Let $\mathcal{P}(\gamma)$ be a matrix polynomial as in (1) and $Y = [y_1 \dots y_k] \in \mathbb{I}_{n\times k}$. Then $Y^*\mathcal{P}(\gamma)Y$ is a $k\times k$ matrix polynomial whose entries are polynomials $y_i^*\mathcal{P}(\gamma)y_j$, where $i, j \in \{1, \dots k\}$. Let $\theta \leq m$ be the largest degree of these polynomial entries and $\beta \leq \theta$ is the largest degree of the remaining polynomials. Choose i_1 and j_1 such that $y_{i_1}^*\mathcal{P}(\gamma)y_{j_1}$ has degree θ. By [1, Page 811], *the generalized Sylvester matrix* related to $\mathcal{P}(\gamma)$ and Y is considered as

$$R_s(Y) = \begin{pmatrix} R_1(Y) \\ \vdots \\ R_{k^2}(Y) \end{pmatrix}, \tag{5}$$

where $R_1(Y) =$

$$\begin{pmatrix} y_{i_1}^* A_\theta y_{j_1} & y_{i_1}^* A_{\theta-1} y_{j_1} & & \cdots & y_{i_1}^* A_0 y_{j_1} & & 0 \\ & y_{i_1}^* A_\theta y_{j_1} & y_{i_1}^* A_{\theta-1} y_{j_1} & & & & \\ & \ddots & \ddots & & & & \ddots \\ 0 & & y_{i_1}^* A_{\theta-1} y_{j_1} & y_{i_1}^* A_{\theta-1} y_{j_1} & \cdots & y_{i_1}^* A_0 y_{j_1} & \end{pmatrix},$$

$R_1(Y) \in \mathbb{M}_{\beta\times(\theta+\beta)}$, and for $t = 2, \dots, k^2$,

$$R_t(Y) = \begin{pmatrix} 0 & & y_{i_t}^* A_\theta y_{j_t} & \cdots & y_{i_t}^* A_0 y_{j_t} \\ & y_{i_t}^* A_\theta y_{j_t} & \cdot & & \\ & \cdot & & \cdot & \\ y_{i_t}^* A_\theta y_{j_t} & \cdots & & y_{i_t}^* A_0 y_{j_t} & 0 \end{pmatrix},$$

$R_t(Y) \in \mathbb{M}_{\theta\times(\theta+\beta)}$, where $i_t, j_t \in \{1, \dots k\}$, $i_t \neq i_1$ and $j_t \neq j_1$. The discussions in [1, Page 811] show that

$$rank(R_s(Y)) = \theta + \beta - \delta(Y), \tag{6}$$

where $\delta(Y)$ is the degree of the greatest common divisor of the scalar polynomial $y_i^*\mathcal{P}(\gamma)y_j$, $i, j = 1, \dots, k$.

Lemma 2.10. *[1, Proposition 11] Let $\mathcal{P}(\gamma)$ be a matrix polynomial as in (1) with $A_m \neq 0$ and let the higher rank numerical range of $\mathcal{P}(\gamma)$ be nonempty with τ connected components. For any $Y \in \mathbb{I}_{n\times k}$ such that $Y^*A_mY = \gamma I_k$ with*

$\gamma \in \Lambda_k(A_m) \setminus \{0\}$, *let* $R_s(Y)$ *be the Sylvester matrix with* $rank(R_s(Y)) < 2m$ *and let* r *be the minimum number of distinct roots of the equation* $Y^*\mathcal{P}(\gamma)Y = 0$. *If* $\Lambda_k(A_m) \setminus \{0\}$ *is connected, then,* $\tau \leq r \leq m$.

Let $A \in \mathbb{M}_n$ and $q \in \mathbb{C}$ with $|q| \leq 1$. The $q-$numerical range of A is defined (e.g., see [6]) by

$$\mathcal{W}_q(A) = \{x^*Ay : x, y \in \mathbb{C}^n, x^*x = 1 = y^*y, x^*y = q\}.$$

One can see easily that $\mathcal{W}_1(A) = \mathcal{W}(A)$. The following theorem gives an upper bound for the number of connected components of the $\alpha-$higher rank numerical range of matrices.

Theorem 2.11. *Let* $A \in \mathbb{M}_n$ *be a matrix such that* $\mathcal{W}_0(A^{\alpha+1}) = \{0\}$ *for nonnegative integer* α. *If* $\Lambda_k^{(\alpha)}(A) \neq \emptyset$ *has* τ *connected components, then* $\tau \leq r \leq \alpha+1$, *where* r *is the minimum number of distinct roots of the equation* $Y^*(A - \lambda I)^{\alpha+1}Y = 0$ *for any* $Y \in \mathbb{I}_{n\times k}$.

Proof. At first, we show that $\operatorname{rank}(R_s(Y)) < 2\alpha + 2$, where $R_s(Y)$, as in (5), is the Silvester matrix for any $Y \in \mathbb{I}_{n\times k}$ and let y_is $(i = 1, \ldots, k)$ be columns of Y. Let $\beta \leq \alpha + 1$, θ and $\delta(Y)$ are as mentioned before the Lemma 2.10. Looking at the main diagonal of $Y^*(A - \lambda I)^{\alpha+1}Y$, we have $\theta = \beta = \alpha + 1$. Since $\mathcal{W}_0(A^{\alpha+1}) = \{0\}$, all polynomials $y_i^*(A - \lambda I)^{\alpha+1}y_j = (-1)^{\alpha+1}\lambda^{\alpha+1}\delta_{ij} + (-1)^{\alpha}(\alpha+1)\lambda^{\alpha}y_i^*Ay_j + \cdots + (\alpha+1)y_i^*A^{\alpha+1}y_j$ are divisible to λ and so $\delta(Y) \geq 1$, for each $Y \in \mathbb{I}_{n\times k}$. Now by Equation (6), $\operatorname{rank}(R_s(Q)) < 2\alpha + 2$ and the result follows from Lemma 2.10. □

Let $(A_1, \ldots, A_t) \in \mathbb{M}_n^t, q \in \mathbb{C}$ with $|q| \leq 1$ and t be a positive integer. The joint $q-$numerical range of $(A_1, \ldots, A_t)$ is denoted by

$$J\mathcal{W}_q(A_1, \ldots, A_t) = \{(x^*A_1y, \ldots, x^*A_ty) : x, y \in \mathbb{C}^n, x^*x = 1 = y^*y, x^*y = q\}.$$

Using the notion of joint $q-$numerical range, we have the following corollary.

Corollary 2.12. *Let* α *be a nonnegative integer and* $A \in \mathbb{M}_n$ *with the property that* $0 \in J\mathcal{W}(A, A^2, \ldots, A^{\alpha})$ *and* $\mathcal{W}_0(A^{\alpha+1}) = \{0\}$. *Then,* $\Lambda_k^{(\alpha)}(A)$ *is connected.*

3. On the index higher rank numerical range of nilpotent matrices

At first, we state some relations for $\alpha-$rank$-k$ numerical range and $\alpha-$numerical range of zero square nilpotent matrices.

Theorem 3.1. *Let* A *be a nilpotent matrix with* $A^2 = 0$. *Then the following assertions are true:*
(i) $\Lambda_k^{(\alpha)}(A) \setminus \{0\} \subseteq (\alpha + 1)\Lambda_k(A)$;
(ii) If $0 \notin \Lambda_k(A)$, *then* $\Lambda_k^{(\alpha)}(A) \setminus \{0\} = (\alpha + 1)\Lambda_k(A)$;
(iii) $\mathcal{W}^{(\alpha)}(A) = (\alpha + 1)\mathcal{W}(A)$.

Proof. Let $\lambda \in \Lambda_k^{(\alpha)}(A)$ and $\lambda \neq 0$. Then there exists $X \in \mathbb{I}_{n\times k}$ such that $X^*(A-\lambda I)^{\alpha+1}X = 0$. Since $A^2 = 0$, it follows that $\lambda I_k = (\alpha+1)X^*AX$. So, $\Lambda_k^{(\alpha)}(A) \setminus \{0\} \subseteq (\alpha+1)\Lambda_k(A)$. Hence, the result in (i) holds.

To prove the assertion in (ii), it is easy to see that $(\alpha+1)\Lambda_k(A) \subseteq \Lambda_k^{(\alpha)}(A)$. Now, if $0 \notin \Lambda_k(A)$, then $(\alpha+1)\Lambda_k(A) = (\alpha+1)\Lambda_k(A) \setminus \{0\} \subseteq \Lambda_k^{(\alpha)}(A) \setminus \{0\}$, and hence, by the previous case, we see that the equality holds.

Finally, to prove the assertion in (iii), since A is a nilpotent matrix, it follows that $0 \in \sigma(A)$, and so, $0 \in \mathcal{W}(A)$ and $0 \in \mathcal{W}^{(\alpha)}(A)$. Now by setting $k = 1$ in (ii), the result holds. So, the proof is complete. □

In the following examples, we have two matrices one of them satisfies in Theorem 3.1(i) and (ii) and the other one is just satisfied in Theorem 3.1(i) and so, it shows that if $0 \in \Lambda_k(A)$, then the equality in Theorem 3.1(ii) may not hold.

Example 3.2. *Direct computation shows that $\Lambda_2(J_2) = \emptyset$. To find $\Lambda_2^{(2)}(J_2)$, by using Definition 2.1 we have*

$$\begin{aligned}\Lambda_2^{(2)}(J_2) &= \{\lambda \in \mathbb{C} : U^*(J_2-\lambda I)^3U = 0I_2, \textit{ for some unitary matrix } U \in \mathbb{M}_2\}\\ &= \{\lambda \in \mathbb{C} : (J_2-\lambda I)^3 = 0I_2\}\\ &= \{\lambda \in \mathbb{C} : 3\lambda^2 J_2 - \lambda^3 I = 0I_2\}.\end{aligned}$$

If $\lambda \neq 0$, then the last relation shows that $J_2 = \lambda/3I$, which is a contradiction. So, $\lambda = 0$ and hence $\Lambda_2^{(2)}(J_2) = \{0\}$. Now, it is obvious that $\Lambda_2^{(2)}(J_2) \setminus \{0\} = 3\Lambda_2(J_2) = \emptyset$.

Example 3.3. *Let $A = J_2 \oplus 0$, where 0 is a 1×1 zero matrix. Then the geometrical multiplicity of the zero eigenvalue is two and so, $0 \in \Lambda_2(A)$. Therefore, $\Lambda_2^{(2)}(A) \setminus \{0\} \neq \Lambda_2(A)$.*

In the following example, we find $\alpha-$higher rank numerical ranges of a 3×3 nilpotent matrix by using Theorem 3.1.

Example 3.4. *Let $A = \begin{pmatrix} 5 & -3 & 2\\ 15 & -9 & 6\\ 10 & -6 & 4\end{pmatrix}$. Then, $A^2 = 0$, $dim(ker(A)) = 2$ and $dim(ker(A^2)) = dim(ker(A^3)) = 3$. So, $ind(A) = 2$. In case $k = 1$, $\Lambda_1(A) = W(A)$ is a circular disk plotted in Figure 1 (using a MATLAB program). Since $\alpha = 2$, using Theorem 3.1(iii), we have $\Lambda_1^{(2)}(A) = W^{(2)}(A) =$*

Figure 1

$W[(\gamma I - A)^3] = 3W(A)$, which is a circular disk with three-time radius of the circular disk in Figure 1.

In case $k = 2$, using [2, Proposition 2.2], $\Lambda_2(A)$ is the empty set or a singleton set (since $2k > n$). Using $X = \begin{pmatrix} 3/\sqrt{34} & -5/\sqrt{323} \\ 5/\sqrt{34} & 3/\sqrt{323} \\ 0 & 17/\sqrt{323} \end{pmatrix} \in \mathbb{I}_{3\times 2}$, one can see that $0 \in \Lambda_2(A)$ and so, $\Lambda_2(A) = \{0\}$. Hence, by Theorem 3.1(i), we have $\Lambda_2^{(2)}(A)\setminus\{0\} \subseteq 3\Lambda_2(A) = \{0\}$ and $\Lambda_2^{(2)}(A) \subseteq \{0\}$. By the way, using Definition 2.1, $\Lambda_2^{(2)}(A) = \Lambda_2[(\gamma I - A)^3] = \Lambda_2[\gamma^3 I - 3\gamma^2 A] = \{\lambda \in \mathbb{C} : X^*(\lambda^3 I - 3\lambda^2 A)X = 0I_2, X \in \mathbb{I}_{3\times 2}\} = \{0\}$.
In case $k = 3$, $\Lambda_3(A) = \{\lambda \in \mathbb{C} : U^*AU = \lambda I, U$ is a unitary matrix$\}$. Hence, A is unitarily similar to a diagonal matrix, which is not true. To avoid contradiction, $\Lambda_3(A) = \emptyset$. Now, Theorem 3.1(i) shows that

$$\Lambda_3^{(2)}(A) \setminus \{0\} = \emptyset. \tag{7}$$

By using Definition 2.1, one can see that $0 \in \Lambda_3^{(2)}(A)$. So, by (7), we have $\Lambda_3^{(2)}(A) = \{0\}$.

For nilpotent matrices, one can see that zero is a member of the α-rank-k numerical range. In the next proposition, we see that the α-rank-k numerical range of nilpotent matrices is connected.

Proposition 3.5. *Let $A \in \mathbb{M}_n$ be a nilpotent matrix and ind$(A) = \alpha$. Then $\mathcal{I}\Lambda_k(A)$ is connected.*

Proof. Since A is nilpotent, by Schur decomposition Theorem, without loss of generality, the first column of A is zero and so is the first column of each power of A. One can see that $e_2^* A^i e_1 = 0, i = 1, \ldots, \alpha$, where $e_1 = (1, 0, \ldots, 0)$ and $e_2 = (0, 1, 0, \ldots, 0)$. So, $0 \in JW_0(A, A^2, \ldots A^\alpha)$. Since A is nilpotent with ind$(A) = \alpha$, we have $A^{\alpha+1} = 0$ and hence $\mathcal{W}_0(A^{\alpha+1}) = \{0\}$. Now, the result follows from Corollary 2.12. □

Recall that a set S in $\mathbb{C}$ has circular property if $\lambda \in S$ implies that $e^{i\theta}\lambda \in S$ for every $\theta \in \mathbb{R}$. We know that the Jordan blocks are special types of nilpotent matrices. In the following theorem, we prove circular property for the higher rank numerical range of this matrices.

Theorem 3.6. *Let J_n be the $n \times n$ Jordan matrix with zero eigenvalue. Then $\Lambda_k^{(\alpha)}(J_n)$ has the circular property. In particular, $I\mathcal{W}(J_n)$ is a closed disk around the origin.*

Proof. Let $U = \mathrm{diag}(1, e^{-i\theta}, e^{-2i\theta}, \ldots, e^{-(n-1)i\theta})$, where $\theta \in \mathbb{R}$. Obviously, U is a unitary matrix and $U^* J_n U = e^{i\theta} J_n$. So, by Proposition 2.3(vi), $\Lambda_k^{(\alpha)}(e^{i\theta} J_n)$ $= e^{i\theta}\Lambda_k^{(\alpha)}(J_n)$ for all $\theta \in \mathbb{R}$. Since by Proposition 3.5, $\Lambda_k^{(\alpha)}(J_n)$ is connected and compact (Proposition 2.3(v)), this means that $\Lambda_k^{(\alpha)}(J_n)$ is a closed disk around the origin. The special case is derived by choosing $k = 1$ and using the fact that $0 \in \sigma(J_n) \subseteq \mathcal{I}\mathcal{W}(J_n)$. □

In Lemma 2.6, we saw that $\mathcal{IW}(J_2) = \{z \in \mathbb{C} : |z| \leq 3/2\}$, i.e., in case $n = 2$, the radius of the mentioned disk in the above theorem is exactly $(2+1)/2$. In the following example, we show that if $n \neq 2$, then the mentioned disk radius may be bigger than $(n+1)/2$.

Example 3.7. *Let $z \in \mathcal{IW}(J_3)$. Then, there exists $x \in \mathbb{C}^3$ such that $\|x\| = 1$ and $x^*(J_3 - zI)^4 x = 0$. So, $z^4 - 4z^3 x^* J_3 x + 6z^2 x^* J_3^2 x = 0$. Now, let $x = (1/2, 1/\sqrt{2}, 1/2)^t$. Then $x^* J_3 x = \sqrt{2}/2$ and $x^* J_3^2 x = 1/4$. So, $z = \sqrt{2}+1/\sqrt{2} \in \mathcal{IW}(J_3)$, while $|z| > (3+1)/2$.*

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