Journal of Mahani Mathematical Research

Shahid Bahonar University of Kerman

Print ISSN: 2251-7952 Online ISSN: 2645-4505

WEAK CONVERGENCE OF FIXED POINT ITERATIONS IN S-METRIC SPACES

G. Siva ✉ and S. Loganathan

Article type: Research Article

(Received: 24 February 2023, Received in revised form 05 August 2023)

(Accepted: 27 October 2023, Published Online: 27 October 2023)

Abstract. This paper extends the notion of weak convergence in metric spaces to the case of S-metric spaces. Moreover, some results on the weak convergence of fixed point iterations of Banach's, Kannan's, Chatterjea's, Reich's, Hardy and Roger's types of contractions on S-metric spaces are obtained. In addition, an example is presented to demonstrate our primary result.

Keywords: Weak convergence, Semi S-metric space, Directed set.
2020 MSC: Primary 47H10, Secondary 54H25

1. Introduction

In both pure and applied mathematics, metric spaces are crucial. There have been so many attempts to find generalized metric spaces. Several authors have investigated fixed point results on various generalized metric spaces. Dhage introduced the concept of 2-metric space in [2]. The concept of D-metric space was introduced by G¨ahler in [4]. These two attempts have some drawbacks, see for example [12, 14]. So, G-metric space was introduced by Mustafa, and Sims in [11]. There are many articles on fixed point theory in G-metric spaces; see [13,19]. Sedghi et. al., modified the concept of D-metric spaces to D^*-metric spaces in [22].

The concept of S-metric space was introduced by Sedghi et al in [21]. A S-metric is a real valued mapping on N^3, for some set $N \neq \emptyset$, where the map represents the perimeter of the triangle. Also, giving examples to every G-metric is a D^* metric and every D^*-metric is a S-metric in [21]. There are many articles on fixed point theory in S-metric spaces; see [3, 15, 16, 20, 23].

The idea of weak convergence in normed spaces was expanded to metric spaces by Raj and Moorthy [17]. Some results regarding the weak convergence of fixed point iterations of contractions on cone metric spaces were proved by Moorthy and Siva [10].

Weak convergence in metric spaces has been expanded to S-metric spaces in this article. Also, we consider the S-metric of the type $S(\kappa, \varpi, \rho) = \sup\{S_i(\kappa, \varpi, \rho) : i \in I\}$ on a set $N \neq \emptyset$, where each S_i is a semi S-metric (i.e, $S_i(\kappa, \varpi, \rho) = 0$

✉ gsivamaths2012@gmail.com, ORCID: 0000-0002-4676-6883
DOI: 10.22103/jmmr.2023.21119.1407
Publisher: Shahid Bahonar University of Kerman
How to cite: G. Siva, S. Loganathan, *Weak convergence of fixed point iterations in S-metric spaces*, J. Mahani Math. Res. 2024; 13(1): 563 - 574.

need not imply $\kappa = \varpi = \rho$) on N, $\forall i$, in a directed set $(I, \leq)$, and when $S_i \leq S_j$ whenever $i \leq j$.

In the context of this article, weak convergence refers to the convergence of fixed point contraction iterations on S-metric spaces through each S_i.

2. S-Metric Spaces and Preliminaries

Definition 2.1. [21] Let $N \neq \emptyset$ be a set. The mapping $S : N^3 \to [0, \infty)$ is said to be a S-metric if

(1) $S(\kappa, \varpi, \rho) \geq 0$, for all $\kappa, \varpi, \rho \in N$ and
(2) $S(\kappa, \varpi, \rho) = 0$ if and only if(or,iff) $\kappa = \varpi = \rho$, for all $\kappa, \varpi, \rho \in N$; and
(3) $S(\kappa, \varpi, \rho) \leq S(\kappa, \kappa, \sigma) + S(\varpi, \varpi, \sigma) + S(\rho, \rho, \sigma)$, for all $\kappa, \varpi, \rho, \sigma \in N$.

Then (N, S) is called an S-metric space(or, SMS).
Moreover, The mapping S is said to be a semi S-metric on N if S satisfies (1), (3) and $\kappa = \varpi = \rho$ implies $S(\kappa, \varpi, \rho) = 0$ but $S(\kappa, \varpi, \rho) = 0$ need not imply $\kappa = \varpi = \rho$. Then (N, S) is called a semi S-metric space(or, SSMS).

Example 2.2. *Let d be an ordinary metric on $N \neq \emptyset$, then $S(\kappa, \varpi, \rho) = d(\kappa, \varpi) + d(\varpi, \rho) + d(\rho, \kappa)$ is a S-metric on N.*

Lemma 2.3. *[21] Let S be a S-metric on N, then $S(\kappa, \kappa, \varpi) = S(\varpi, \varpi, \kappa)$.*

Definition 2.4. [21] Suppose (N, S) is a SMS. Let $\{\kappa_n\}$ be a sequence in N.
(i) $\{\kappa_n\}$ is said to be convergent if for every $\epsilon > 0$, there exists a positive integer n_0 such that $\forall\ n > n_0$, $S(\kappa_n, \kappa_n, \kappa) < \epsilon$, for some $\kappa \in N$.
(ii) $\{\kappa_n\}$ is said to be a Cauchy sequence if for any $\epsilon > 0$, there exists a positive integer n_0 such that $\forall\ n, m > n_0$, $S(\kappa_n, \kappa_n, \kappa_m) < \epsilon$.

Remark 2.5. Let $\{\kappa_n\}$ be a sequence in N. Then $\{\kappa_n\}$ is said to be convergent to κ iff $S(\kappa_n, \kappa_n, \kappa) \to 0$ as $n \to \infty$, and $\{\kappa_n\}$ is said to be Cauchy iff $S(\kappa_n, \kappa_n, \kappa_m) \to 0$ as $n, m \to \infty$,

Definition 2.6. [21] An SMS (N, S) is said to be complete if every Cauchy sequence is convergent in N.

Remark 2.7. Definitions 2.4 and 2.6 can be extended for semi S-metrics and Remark 2.5 is also true in semi S-metric spaces(or, SSMSs).

Definition 2.8. If S_i is a semi S-metric on a $N \neq \emptyset$, then (N, S_i) is called complete if every Cauchy sequence is convergent in (N, S_i).

According to Remark 2.5, we have the next Remark 2.9.

Remark 2.9. Let a sequence $\{\kappa_n\}$ in N such that $S_i(\kappa_n, \kappa_n, \kappa_m) \to 0$ as $n, m \to \infty$, there exists a point κ in N such that $S_i(\kappa_n, \kappa_n, \kappa) \to 0$ as $n \to \infty$. Then (N, S_i) is called complete SSMS.

3. Fundamental Results

In order to obtain extensions for weak convergence of fixed point iterations of contractions on SMSs, the following theorems (3.1, 3.2, 3.3, 3.4, and 3.5) are applied.
Theorem 3.1 is a generalized version of Theorem 3.3 of [21], for SSMSs.

Theorem 3.1. *[21] Let (N, S_i) be a complete SSMS. Suppose $H : N \to N$ is a given function such that $S_i(H\kappa, H\kappa, H\varpi) \leq lS_i(\kappa, \kappa, \varpi)$, $\forall \kappa, \varpi \in N$, for some $l \in (0, 1)$. Fix $\kappa_0 \in N$ and define $\kappa_1, \kappa_2, \kappa_3,...$ by $\kappa_{n+1} = H\kappa_n$, $\forall n = 0, 1, 2, 3, ...$ Then there exists a member κ^* in N such that $S_i(\kappa_n, \kappa_n, \kappa^*) \to 0$ as $n \to \infty$ and $S_i(H\kappa^*, H\kappa^*, \kappa^*) = 0$. Furthermore, if S_i is a S-metric, then H has a unique fixed point(or, UFP) in N.*

The Theorem 3.2 is a generalized version of Corollary 2.8 of [20], for SSMSs.

Theorem 3.2. *[20] Let (N, S_i) be a complete SSMS. Suppose $H : N \to N$ is a given function such that $S_i(H\kappa, H\kappa, H\varpi) \leq l(S_i(H\kappa, H\kappa, \kappa)+S_i(H\varpi, H\varpi, \varpi))$, $\forall \kappa, \varpi \in N$, for some $l \in (0, \frac{1}{2})$. Fix $\kappa_0 \in N$ and define $\kappa_1, \kappa_2, \kappa_3,...$ by $\kappa_{n+1} = H\kappa_n$, $\forall n = 0, 1, 2, 3, ...$ Then there exists a member κ^* in N such that $S_i(\kappa_n, \kappa_n, \kappa^*) \to 0$ as $n \to \infty$ and $S_i(H\kappa^*, H\kappa^*, \kappa^*) = 0$. Furthermore, if S_i is a S-metric, then H has a UFP in N.*

The Theorem 3.3 is a generalized version of Corollary 2.15 of [20], for SSMSs.

Theorem 3.3. *[20] Let (N, S_i) be a complete SSMS. Suppose $H : N \to N$ is a given function such that $S_i(H\kappa, H\kappa, H\varpi) \leq l(S_i(H\kappa, H\kappa, \varpi)+S_i(H\varpi, H\varpi, \kappa))$, $\forall \kappa, \varpi \in N$, for some $l \in (0, \frac{1}{3})$. Fix $\kappa_0 \in N$ and define $\kappa_1, \kappa_2, \kappa_3,...$ by $\kappa_{n+1} = H\kappa_n$, $\forall n = 0, 1, 2, 3, ...$ Then there exists a member κ^* in N such that $S_i(\kappa_n, \kappa_n, \kappa^*) \to 0$ as $n \to \infty$ and $S_i(H\kappa^*, H\kappa^*, \kappa^*) = 0$. Furthermore, if S_i is a S-metric, then H has a UFP in N.*

The Theorem 3.4 is a generalized version of theorem 2.17 of [20], for SSMSs.

Theorem 3.4. *[20] Let (N, S_i) be a complete SSMS. Suppose $H : N \to N$ is a given function such that $S_i(H\kappa, H\kappa, H\varpi) \leq pS_i(\kappa, \kappa, \varpi) + lS_i(\varpi, \varpi, H\kappa)$, $\forall \kappa, \varpi \in N$, for some $p, l \in (0, \frac{1}{3})$. Fix $\kappa_0 \in N$ and define $\kappa_1, \kappa_2, \kappa_3,...$ by $\kappa_{n+1} = H\kappa_n$, $\forall n = 0, 1, 2, 3, ...$ Then there exists a point κ^* in N such that $S_i(\kappa_n, \kappa_n, \kappa^*) \to 0$ as $n \to \infty$ and $S_i(H\kappa^*, H\kappa^*, \kappa^*) = 0$. Furthermore, if S_i is a S-metric, then H has a UFP in N.*

The Theorem 3.5 is a generalized version of theorem 2.19 of [20], for SSMSs.

Theorem 3.5. *[20] Let (N, S_i) be a complete SSMS. Suppose $H : N \to N$ is a given function such that $S_i(H\kappa, H\kappa, H\varpi) \leq l[S_i(\kappa, \kappa, \varpi) + S_i(H\kappa, H\kappa, \kappa) + S_i(H\varpi, H\varpi, \varpi) + S_i(H\kappa, H\kappa, \varpi) + S_i(H\varpi, H\varpi, \kappa)]$, $\forall \kappa, \varpi \in N$, for some $l \in (0, \frac{1}{3})$. Fix $\kappa_0 \in N$ and define $\kappa_1, \kappa_2, \kappa_3,...$ by $\kappa_{n+1} = H\kappa_n$, $\forall n = 0, 1, 2, 3, ...$ Then there exists a point κ^* in N such that $S_i(\kappa_n, \kappa_n, \kappa^*) \to 0$ as $n \to \infty$ and $S_i(H\kappa^*, H\kappa^*, \kappa^*) = 0$. Furthermore, if S_i is a S-metric, then H has a UFP in N.*

4. Main Results

In this section, we establish a few theorems regarding the weak convergence of fixed point iterations of various contraction types in an SMS.

Assumption 4.1. Let $N \neq \emptyset$ be an SMS with a S-metric S. Suppose $(S_i)_{i\in I}$ is a family of semi S-metrics on N such that $S(\kappa,\kappa,\varpi) = \sup_{i\in I} S_i(\kappa,\kappa,\varpi)$, $\forall$ $\kappa,\varpi \in N$. Suppose further that $(I,\leq)$ is a directed set such that $S_i(\kappa,\kappa,\varpi) \leq S_j(\kappa,\kappa,\varpi)$, $\forall$ $\kappa,\varpi \in N$, whenever $i \leq j$ in I.

Assumption 4.2. Consider a set of the form $Z_i = \{\varpi \in N : S_i(\kappa_i,\kappa_i,\varpi) = 0\} \neq \emptyset$, for some $\kappa_i \in N$. It is called an i-zero set. If $(Z_i)_{i\in I}$ is a collection of i-zero sets such that $Z_i \supseteq Z_j$, for $i \leq j$ in I, then $\bigcap_{i\in I} Z_i \neq \emptyset$

For the next five theorems, assumptions 4.1 and 4.2 are assumed to be true. The next result considers Banach's contraction [8] for SMSs.

Theorem 4.3. *Suppose $(l_i)_{i\in I}$ is a collection of numbers in $(0,1)$. Let H be a function on N to itself such that $S_i(H\kappa,H\kappa,H\varpi) \leq l_i S_i(\kappa,\kappa,\varpi)$, $\forall\kappa,\varpi \in N$, $\forall i \in I$. Let's assume that every (N,S_i) is a complete SSMS, $\forall i \in I$. Then there exists a UFP κ^* of H in N. Furthermore, if $\kappa_0 \in N$ is fixed and $\kappa_1,\kappa_2,\kappa_3$,... are defined by $\kappa_{n+1} = H\kappa_n$, $\forall n = 0,1,2,3...$, then $S_i(\kappa_n,\kappa_n,\kappa^*) \to 0$ as $n \to \infty$, $\forall i \in I$.*

Proof. Let $\kappa_0 \in N$, and set $\kappa_1,\kappa_2,\kappa_3$,...in N by $\kappa_{n+1} = H\kappa_n$, $\forall n = 0,1,2,3,...$. Then, by Theorem 3.1, there is a point $\kappa_i^* \in N$ such that $S_i(\kappa_n,\kappa_n,{\kappa_i}^*) \to 0$ as $n \to \infty$ and $S_i(H{\kappa_i}^*,H{\kappa_i}^*,{\kappa_i}^*) = 0$, $\forall i \in I$. Take $Z_i = \{\kappa \in N : S_i({\kappa_i}^*,{\kappa_i}^*,\kappa) = 0\}$, an i-zero set, for every $i \in I$.
For $i \leq j$ in I, if $\kappa \in Z_j$, then we get

$$\begin{aligned} 0 \leq S_i({\kappa_i}^*,{\kappa_i}^*,\kappa) &\leq 2S_i({\kappa_i}^*,{\kappa_i}^*,\kappa_n) + S_i(\kappa,\kappa,\kappa_n) \\ &= 2S_i({\kappa_i}^*,{\kappa_i}^*,\kappa_n) + S_i(\kappa_n,\kappa_n,\kappa) \\ &\leq 2S_i({\kappa_i}^*,{\kappa_i}^*,\kappa_n) + 2S_j(\kappa_n,\kappa_n,{\kappa_j}^*) + S_j(\kappa,\kappa,{\kappa_j}^*), \\ \text{and } S_i({\kappa_i}^*,{\kappa_i}^*,\kappa) &\leq 2S_i(\kappa_n,\kappa_n,{\kappa_i}^*) + 2S_j(\kappa_n,\kappa_n,{\kappa_j}^*) + S_j({\kappa_j}^*,{\kappa_j}^*,\kappa). \end{aligned}$$

Since $2S_i(\kappa_n,\kappa_n,{\kappa_i}^*)+2S_j(\kappa_n,\kappa_n,{\kappa_j}^*) \to 0$ as $n \to \infty$, we get $S_i({\kappa_i}^*,{\kappa_i}^*,\kappa) = 0$. Hence, $Z_j \subseteq Z_i$, for $i \leq j$ in I, and by Assumption 4.2, we have $\bigcap_{i\in I} Z_i \neq \emptyset$. Let $\kappa^* \in \bigcap_{i\in I} Z_i$, then

$$0 \leq S_i(\kappa_n,\kappa_n,\kappa^*) \leq 2S_i(\kappa_n,\kappa_n,{\kappa_i}^*) + S_i(\kappa^*,\kappa^*,{\kappa_i}^*) = 2S_i(\kappa_n,\kappa_n,{\kappa_i}^*).$$

Since $S_i(\kappa_n, \kappa_n, \kappa_i{}^*) \to 0$ as $n \to \infty$, $\forall i \in I$, then we get $S_i(\kappa_n, \kappa_n, \kappa^*) \to 0$ as $n \to \infty$, $\forall i \in I$. Moreover,

$$\begin{aligned} 0 &\leq S_i(\kappa^*, \kappa^*, H\kappa^*) \\ &\leq 2S_i(\kappa^*, \kappa^*, \kappa_i{}^*) + S_i(H\kappa^*, H\kappa^*, \kappa_i{}^*) \\ &\leq 2S_i(\kappa^*, \kappa^*, \kappa_i{}^*) + S_i(\kappa_i{}^*, \kappa_i{}^*, H\kappa^*) \\ &\leq 2S_i(\kappa^*, \kappa^*, \kappa_i{}^*) + 2S_i(\kappa_i{}^*, \kappa_i{}^*, H\kappa_i{}^*) + S_i(H\kappa^*, H\kappa^*, H\kappa_i{}^*) \\ &= 2S_i(\kappa_i{}^*, \kappa_i{}^*, \kappa^*) + 2S_i(H\kappa_i{}^*, H\kappa_i{}^*, \kappa_i{}^*) + S_i(H\kappa_i{}^*, H\kappa_i{}^*, H\kappa^*) \\ &\leq l_i S_i(\kappa_i{}^*, \kappa_i{}^*, \kappa^*) = 0, \forall i \in I. \end{aligned}$$

Therefore, $S_i(\kappa^*, \kappa^*, H\kappa^*) = 0$, $\forall i \in I$. By Assumption 4.1, we have $H\kappa^* = \kappa^*$. Furthermore, if $\varpi^* = H\varpi^*$, for some $\varpi^* \in N$,

$$\begin{aligned} 0 \leq S_i(\kappa^*, \kappa^*, \varpi^*) = S_i(H\kappa^*, H\kappa^*, H\varpi^*) &\leq l_i S_i(\kappa^*, \kappa^*, \varpi^*), \forall i \in I \\ &\leq (l_i)^m S_i(\kappa^*, \kappa^*, \varpi^*), \forall m \in \mathbb{N}, \forall i \in I. \end{aligned}$$

Since $(l_i)^m K \to 0$ as $m \to \infty$, $S_i(\kappa^*, \kappa^*, \varpi^*) = 0$, $\forall i \in I$. By Assumption 4.1, we have $\kappa^* = \varpi^*$. □

Theorem 2.1 of [17] is generalized in Theorem 4.3.

Example 4.4. *Let N be the collection of all bounded continuous real valued functions on the real line $\mathbb{R}$, and $S : N^3 \to [0, \infty)$ be defined by $S(f, g, h) = \sup_{\kappa\in\mathbb{R}} |f(\kappa) - h(\kappa)| + \sup_{\kappa\in\mathbb{R}} |g(\kappa) - h(\kappa)|$, $\forall f, g, h \in N$. Then (N, S) is a complete SMS.*
For each $i = 1, 2, ...$, define $V_i = (-\infty, -1 - \frac{1}{3^i}] \cup [-1 + \frac{1}{3^i}, 1 - \frac{1}{3^i}] \cup [1 + \frac{1}{3^i}, \infty)$, and let $S_i(f, g, h) = \sup_{\kappa\in V_i}\{|f(\kappa) - h(\kappa)|\} + \sup_{\kappa\in V_i}\{|g(\kappa) - h(\kappa)|\}$, $\forall f, g, h \in N$.
Define $H : N \to N$ by $(H(f))(\kappa) = \kappa f(\kappa)$, if $|\kappa| \leq 1$, and $(H(f))(\kappa) = \frac{f(\kappa)}{\kappa}$ if $|\kappa| \geq 1$.
For $f, g \in N$, we have

$$S_i((H(f))(\kappa), (H(f))(\kappa), (H(g))(\kappa)) \leq l_i S_i(f(\kappa), f(\kappa), g(\kappa)),$$

where $l_i = \max\left\{\frac{1}{1+\frac{1}{3^i}}, 1 - \frac{1}{3^i}\right\}$, $\forall i \in I$.
Note that $S(f, g, h) = \sup_{i\in I} S_i(f, g, h)$, $\forall f, g, h \in N$, with $I = \{1, 2, ...\}$, which is a directed set under the usual ordering relation. Then, with the exception of Assumption 4.2, all of the conditions of Theorem 4.3 are fulfilled with $l_i = \max\left\{\frac{1}{1+\frac{1}{3^i}}, 1 - \frac{1}{3^i}\right\}$. Moreover, the zero function is UFP. Furthermore, the weak convergence to the zero function is guaranteed if the iteration process is started with the constant function 1.

The next result considers Kannan's contraction [7] for SMSs.

Theorem 4.5. *Suppose $(l_i)_{i\in I}$ is a collection of numbers in $(0, \frac{1}{2})$. Let H be a function on N to itself such that $S_i(H\kappa, H\kappa, H\varpi) \leq l_i(S_i(H\kappa, H\kappa, \kappa) +$*

$S_i(H\varpi, H\varpi, \varpi))$, $\forall \kappa, \varpi \in N$, $\forall i \in I$. Let's assume that every (N, S_i) is a complete SSMS, $\forall i \in I$. Then there exists a UFP κ^* of H in N. Furthermore, if $\kappa_0 \in N$ is fixed and $\kappa_1, \kappa_2, \kappa_3,...$ are defined by $\kappa_{n+1} = H\kappa_n$, $\forall n = 0, 1, 2, 3, ...,$ then $S_i(\kappa_n, \kappa_n, \kappa^*) \to 0$ as $n \to \infty$, $\forall i \in I$.

Proof. Let $\kappa_0 \in N$, and set $\kappa_1, \kappa_2, \kappa_3$,...in N by $\kappa_{n+1} = H\kappa_n$, $\forall n = 0, 1, 2, 3, ...$ Then, by Theorem 3.2, there is a point $\kappa_i^* \in N$ such that $S_i(\kappa_n, \kappa_n, {\kappa_i}^*) \to 0$ as $n \to \infty$ and $S_i(H{\kappa_i}^*, H{\kappa_i}^*, {\kappa_i}^*) = 0$, $\forall i \in I$. Take $Z_i = \{\kappa \in N : S_i({\kappa_i}^*, {\kappa_i}^*, \kappa) = 0\}$, an i-zero set, for every $i \in I$.
For $i \leq j$ in I, if $\kappa \in Z_j$, then we get

$$\begin{aligned} 0 \leq S_i({\kappa_i}^*, {\kappa_i}^*, \kappa) &\leq 2S_i({\kappa_i}^*, {\kappa_i}^*, \kappa_n) + S_i(\kappa, \kappa, \kappa_n) \\ &= 2S_i({\kappa_i}^*, {\kappa_i}^*, \kappa_n) + S_i(\kappa_n, \kappa_n, \kappa) \\ &\leq 2S_i({\kappa_i}^*, {\kappa_i}^*, \kappa_n) + 2S_j(\kappa_n, \kappa_n, {\kappa_j}^*) + S_j(\kappa, \kappa, {\kappa_j}^*), \\ \text{and } S_i({\kappa_i}^*, {\kappa_i}^*, \kappa) &\leq 2S_i(\kappa_n, \kappa_n, {\kappa_i}^*) + 2S_j(\kappa_n, \kappa_n, {\kappa_j}^*) + S_j({\kappa_j}^*, {\kappa_j}^*, \kappa). \end{aligned}$$

Since $2S_i(\kappa_n, \kappa_n, {\kappa_i}^*) + 2S_j(\kappa_n, \kappa_n, {\kappa_j}^*) \to 0$ as $n \to \infty$, we get $S_i({\kappa_i}^*, {\kappa_i}^*, \kappa) = 0$. Hence, $Z_j \subseteq Z_i$, for $i \leq j$ in I, and by Assumption 4.2, we have $\bigcap_{i \in I} Z_i \neq \emptyset$. Let $\kappa^* \in \bigcap_{i \in I} Z_i$, then

$$S_i(\kappa_n, \kappa_n, \kappa^*) \leq 2S_i(\kappa_n, \kappa_n, {\kappa_i}^*) + S_i(\kappa^*, \kappa^*, {\kappa_i}^*) = 2S_i(\kappa_n, \kappa_n, {\kappa_i}^*).$$

Since $S_i(\kappa_n, \kappa_n, {\kappa_i}^*) \to 0$ as $n \to \infty$, $\forall i \in I$, then we get $S_i(\kappa_n, \kappa_n, \kappa^*) \to 0$ as $n \to \infty$, $\forall i \in I$. Moreover,

$$\begin{aligned} 0 &\leq S_i(\kappa^*, \kappa^*, H\kappa^*) 2S_i(\kappa^*, \kappa^*, {\kappa_i}^*) + S_i(H\kappa^*, H\kappa^*, {\kappa_i}^*) \\ &\leq 2S_i(\kappa^*, \kappa^*, {\kappa_i}^*) + S_i({\kappa_i}^*, {\kappa_i}^*, H\kappa^*) \\ &\leq 2S_i(\kappa^*, \kappa^*, {\kappa_i}^*) + 2S_i({\kappa_i}^*, {\kappa_i}^*, H{\kappa_i}^*) + S_i(H\kappa^*, H\kappa^*, H{\kappa_i}^*) \\ &= 2S_i({\kappa_i}^*, {\kappa_i}^*, \kappa^*) + 2S_i(H{\kappa_i}^*, H{\kappa_i}^*, {\kappa_i}^*) + S_i(H{\kappa_i}^*, H{\kappa_i}^*, H\kappa^*) \\ &\leq l_i[S_i(H{\kappa_i}^*, H{\kappa_i}^*, {\kappa_i}^*) + S_i(H\kappa^*, H\kappa^*, \kappa^*)]. \\ &\leq (l_i)^m S_i(\kappa^*, \kappa^*, H\kappa^*), \forall m \in \mathbb{N}, \forall i \in I. \end{aligned}$$

Since $(l_i)^m K \to 0$ as $m \to \infty$, $S_i(\kappa^*, \kappa^*, H\kappa^*) = 0$, $\forall i \in I$. By Assumption 4.1, we have $H\kappa^* = \kappa^*$. Furthermore, if $\varpi^* = H\varpi^*$, for some $\varpi^* \in S$, then

$$0 \leq S_i(\kappa^*, \kappa^*, \varpi^*) = S_i(H\kappa^*, H\kappa^*, H\varpi^*) \leq$$
$$l_i[S_i(H\kappa^*, H\kappa^*, \kappa^*) + S_i(H\varpi^*, H\varpi^*, \varpi^*)], \forall i \in I,$$

and $S_i(\kappa^*, \kappa^*, \varpi^*) = 0$, $\forall i \in I$. By Assumption 4.1, we have $\kappa^* = \varpi^*$. □

The next result considers Chatterjea's contraction [1] for SMSs.

Theorem 4.6. *Suppose $(l_i)_{i \in I}$ is a collection of numbers in $(0, \frac{1}{3})$. Let H be a function on N to itself such that $S_i(H\kappa, H\kappa, H\varpi) \leq l_i(S_i(H\kappa, H\kappa, \varpi) + S_i(H\varpi, H\varpi, \kappa))$, $\forall \kappa, \varpi \in N$, $\forall i \in I$. Let's assume that every (N, S_i) is a complete SSMS, $\forall i \in I$. Then there exists a UFP κ^* of H in N. Furthermore, if*

$\kappa_0 \in N$ is fixed and $\kappa_1, \kappa_2, \kappa_3, \ldots$ are defined by $\kappa_{n+1} = H\kappa_n$, $\forall n = 0, 1, 2, 3, \ldots$, then $S_i(\kappa_n, \kappa_n, \kappa^*) \to 0$ as $n \to \infty$, $\forall i \in I$.

Proof. Let $\kappa_0 \in N$, and set $\kappa_1, \kappa_2, \kappa_3, \ldots$in N by $\kappa_{n+1} = H\kappa_n$, $\forall n = 0, 1, 2, 3, \ldots$. Then, by Theorem 3.3, there is a point $\kappa_i^* \in N$ such that $S_i(\kappa_n, \kappa_n, {\kappa_i}^*) \to 0$ as $n \to \infty$ and $S_i(H{\kappa_i}^*, H{\kappa_i}^*, {\kappa_i}^*) = 0$, $\forall i \in I$. Take $Z_i = \{\kappa \in N : S_i({\kappa_i}^*, {\kappa_i}^*, \kappa) = 0\}$, an i-zero set, for every $i \in I$.
For $i \leq j$ in I, if $\kappa \in Z_j$, then we get

$$\begin{aligned} 0 \leq S_i({\kappa_i}^*, {\kappa_i}^*, \kappa) &\leq 2S_i({\kappa_i}^*, {\kappa_i}^*, \kappa_n) + S_i(\kappa, \kappa, \kappa_n) \\ &= 2S_i({\kappa_i}^*, {\kappa_i}^*, \kappa_n) + S_i(\kappa_n, \kappa_n, \kappa) \\ &\leq 2S_i({\kappa_i}^*, {\kappa_i}^*, \kappa_n) + 2S_j(\kappa_n, \kappa_n, {\kappa_j}^*) + S_j(\kappa, \kappa, {\kappa_j}^*), \\ \text{and } S_i({\kappa_i}^*, {\kappa_i}^*, \kappa) &\leq 2S_i(\kappa_n, \kappa_n, {\kappa_i}^*) + 2S_j(\kappa_n, \kappa_n, {\kappa_j}^*) + S_j({\kappa_j}^*, {\kappa_j}^*, \kappa). \end{aligned}$$

Since $2S_i(\kappa_n, \kappa_n, {\kappa_i}^*) + 2S_j(\kappa_n, \kappa_n, {\kappa_j}^*) \to 0$ as $n \to \infty$, we get $S_i({\kappa_i}^*, {\kappa_i}^*, \kappa) = 0$. Hence, $Z_j \subseteq Z_i$, for $i \leq j$ in I, and by Assumption 4.2, we have $\bigcap_{i\in I} Z_i \neq \emptyset$. Let $\kappa^* \in \bigcap_{i\in I} Z_i$, then

$$0 \leq S_i(\kappa_n, \kappa_n, \kappa^*) \leq 2S_i(\kappa_n, \kappa_n, {\kappa_i}^*) + S_i(\kappa^*, \kappa^*, {\kappa_i}^*) = 2S_i(\kappa_n, \kappa_n, {\kappa_i}^*).$$

Since $S_i(\kappa_n, \kappa_n, {\kappa_i}^*) \to 0$ as $n \to \infty$, $\forall i \in I$, then we get $S_i(\kappa_n, \kappa_n, \kappa^*) \to 0$ as $n \to \infty$, $\forall i \in I$. Moreover,

$$\begin{aligned} 0 &\leq S_i(\kappa^*, \kappa^*, H\kappa^*) \\ &\leq 2S_i(\kappa^*, \kappa^*, {\kappa_i}^*) + S_i(H\kappa^*, H\kappa^*, {\kappa_i}^*) \\ &\leq 2S_i(\kappa^*, \kappa^*, {\kappa_i}^*) + S_i({\kappa_i}^*, {\kappa_i}^*, H\kappa^*) \\ &\leq 2S_i(\kappa^*, \kappa^*, {\kappa_i}^*) + 2S_i({\kappa_i}^*, {\kappa_i}^*, H{\kappa_i}^*) + S_i(H\kappa^*, H\kappa^*, H{\kappa_i}^*) \\ &= 2S_i({\kappa_i}^*, {\kappa_i}^*, \kappa^*) + 2S_i(H{\kappa_i}^*, H{\kappa_i}^*, {\kappa_i}^*) + S_i(H{\kappa_i}^*, H{\kappa_i}^*, H\kappa^*) \\ &\leq l_i[S_i(H{\kappa_i}^*, H{\kappa_i}^*, \kappa^*) + S_i(H\kappa^*, H\kappa^*, {\kappa_i}^*)]. \\ &\leq l_i[2S_i(H{\kappa_i}^*, H{\kappa_i}^*, {\kappa_i}^*) + 2S_i(H\kappa^*, H\kappa^*, \kappa^*) + 2S_i({\kappa_i}^*, {\kappa_i}^*, \kappa^*)] \\ &= (2l_i)^m S_i(\kappa^*, \kappa^*, H\kappa^*), \forall m \in \mathbb{N}, \forall i \in I. \end{aligned}$$

Since $(2l_i)^m \to 0$ as $m \to \infty$, $S_i(\kappa^*, \kappa^*, H\kappa^*) = 0$, $\forall i \in I$. By Assumption 4.1, we have $H\kappa^* = \kappa^*$. Furthermore, if $\varpi^* = H\varpi^*$, for some $\varpi^* \in S$, then

$$\begin{aligned} 0 \leq S_i(\kappa^*, \kappa^*, \varpi^*) &= S_i(H\kappa^*, H\kappa^*, H\varpi^*) \\ &\leq l_i[S_i(H\kappa^*, H\kappa^*, \varpi^*) + S_i(H\varpi^*, H\varpi^*, \kappa^*)], \\ &\leq 2l_i S_i(\kappa^*, \kappa^*, \varpi^*) \\ &\leq (2l_i)^m S_i(\kappa^*, \kappa^*, \varpi^*) \forall i \in I, \end{aligned}$$

Since $(2l_i)^m K \to 0$ as $m \to \infty$, $S_i(\kappa^*, \kappa^*, \varpi^*) = 0$, $\forall i \in I$. By Assumption 4.1, we have $\kappa^* = \varpi^*$. □

Theorem 4.7. *Suppose $(p_i)_{i\in I}$ and $(l_i)_{i\in I}$ are two collections of numbers in $(0, \frac{1}{3})$. Let H be a function on N to itself such that $S_i(H\kappa, H\kappa, H\varpi) \leq$*

$p_iS_i(\kappa,\kappa,\varpi)+l_iS_i(\varpi,\varpi,H\kappa)$, $\forall\kappa,\varpi\in N$, $\forall i\in I$. Let's assume that every (N,S_i) is a complete SSMS, $\forall i\in I$. Then there exists a UFP κ^* of H in N. Furthermore, if $\kappa_0\in N$ is fixed and $\kappa_1,\kappa_2,\kappa_3,\ldots$ are defined by $\kappa_{n+1}=H\kappa_n$, $\forall n=0,1,2,3,\ldots$, then $S_i(\kappa_n,\kappa_n,\kappa^*)\to 0$ as $n\to\infty$, $\forall i\in I$.

Proof. Let $\kappa_0\in N$, and set $\kappa_1,\kappa_2,\kappa_3,\ldots$in N by $\kappa_{n+1}=H\kappa_n$, $\forall n=0,1,2,3,\ldots$. Then, by Theorem 3.4, there is a point $\kappa_i^*\in N$ such that $S_i(\kappa_n,\kappa_n,{\kappa_i}^*)\to 0$ as $n\to\infty$ and $S_i(H{\kappa_i}^*,H{\kappa_i}^*,{\kappa_i}^*)=0$, $\forall i\in I$. Take $Z_i=\{\kappa\in N: S_i({\kappa_i}^*,{\kappa_i}^*,\kappa)=0\}$, an i-zero set, for every $i\in I$.
For $i\leq j$ in I, if $\kappa\in Z_j$, then we get

$$\begin{aligned}0\leq S_i({\kappa_i}^*,{\kappa_i}^*,\kappa) &\leq 2S_i({\kappa_i}^*,{\kappa_i}^*,\kappa_n)+S_i(\kappa,\kappa,\kappa_n)\\ &= 2S_i({\kappa_i}^*,{\kappa_i}^*,\kappa_n)+S_i(\kappa_n,\kappa_n,\kappa)\\ &\leq 2S_i({\kappa_i}^*,{\kappa_i}^*,\kappa_n)+2S_j(\kappa_n,\kappa_n,{\kappa_j}^*)+S_j(\kappa,\kappa,{\kappa_j}^*),\\ \text{and } S_i({\kappa_i}^*,{\kappa_i}^*,\kappa) &\leq 2S_i(\kappa_n,\kappa_n,{\kappa_i}^*)+2S_j(\kappa_n,\kappa_n,{\kappa_j}^*)+S_j({\kappa_j}^*,{\kappa_j}^*,\kappa).\end{aligned}$$

Since $2S_i(\kappa_n,\kappa_n,{\kappa_i}^*)+2S_j(\kappa_n,\kappa_n,{\kappa_j}^*)\to 0$ as $n\to\infty$, we get $S_i({\kappa_i}^*,{\kappa_i}^*,\kappa)=0$. Hence, $Z_j\subseteq Z_i$, for $i\leq j$ in I, and by Assumption 4.2, we have $\bigcap_{i\in I} Z_i\neq\emptyset$. Let $\kappa^*\in\bigcap_{i\in I} Z_i$, then

$$0\leq S_i(\kappa_n,\kappa_n,\kappa^*) \leq 2S_i(\kappa_n,\kappa_n,{\kappa_i}^*)+S_i(\kappa^*,\kappa^*,{\kappa_i}^*)=2S_i(\kappa_n,\kappa_n,{\kappa_i}^*).$$

Since $S_i(\kappa_n,\kappa_n,{\kappa_i}^*)\to 0$ as $n\to\infty$, $\forall i\in I$, then we get $S_i(\kappa_n,\kappa_n,\kappa^*)\to 0$ as $n\to\infty$, $\forall i\in I$. Moreover,

$$\begin{aligned}0 &\leq S_i(\kappa^*,\kappa^*,H\kappa^*)\\ &\leq 2S_i(\kappa^*,\kappa^*,{\kappa_i}^*)+S_i(H\kappa^*,H\kappa^*,{\kappa_i}^*)\\ &\leq 2S_i(\kappa^*,\kappa^*,{\kappa_i}^*)+S_i({\kappa_i}^*,{\kappa_i}^*,H\kappa^*)\\ &\leq 2S_i(\kappa^*,\kappa^*,{\kappa_i}^*)+2S_i({\kappa_i}^*,{\kappa_i}^*,H{\kappa_i}^*)+S_i(H\kappa^*,H\kappa^*,H{\kappa_i}^*)\\ &= 2S_i({\kappa_i}^*,{\kappa_i}^*,\kappa^*)+2S_i(H{\kappa_i}^*,H{\kappa_i}^*,{\kappa_i}^*)+S_i(H{\kappa_i}^*,H{\kappa_i}^*,H\kappa^*)\\ &\leq p_iS_i({\kappa_i}^*,{\kappa_i}^*,\kappa^*)+l_iS_i(\kappa^*,\kappa^*,H{\kappa_i}^*),\\ &\leq l_iS_i(\kappa^*,\kappa^*,{\kappa_i}^*)+l_iS_i({\kappa_i}^*,{\kappa_i}^*,H{\kappa_i}^*)=0,\forall i\in I.\end{aligned}$$

Hence, $S_i(\kappa^*,\kappa^*,H\kappa^*)=0$, $\forall i\in I$. By Assumption 4.1, we have $H\kappa^*=\kappa^*$. Furthermore, if $\varpi^*=H\varpi^*$, for some $\varpi^*\in N$,

$$\begin{aligned}0\leq S_i(\kappa^*,\kappa^*,\varpi^*) &= S_i(H\kappa^*,H\kappa^*,H\varpi^*)\\ &\leq p_iS_i(\kappa^*,\kappa^*,\varpi^*)+l_iS_i(\varpi^*,\varpi^*,H\kappa^*),\forall i\in I\\ &\leq (p_i+l_i)^mS_i(\kappa^*,\kappa^*,\varpi^*),\forall m\in\mathbb{N},\forall i\in I.\end{aligned}$$

Since $(p_i+l_i)^m\to 0$ as $m\to\infty$, $S_i(\kappa^*,\kappa^*,\varpi^*)=0$, $\forall i\in I$. By Assumption 4.1, we have $\kappa^*=\varpi^*$. □

The next result considers Hardy and Roger's contraction [6] for SMSs.

Theorem 4.8. *Suppose* $(l_i)_{i\in I}$ *is a collection of numbers in* $(0, \frac{1}{3})$. *Let* H *be a function on* N *to itself such that* $S_i(H\kappa, H\kappa, H\varpi) \leq l_i[S_i(\kappa, \kappa, \varpi) + S_i(H\kappa, H\kappa, \kappa) + S_i(H\varpi, H\varpi, \varpi) + S_i(H\kappa, H\kappa, \varpi) + S_i(H\varpi, H\varpi, \kappa)]$, $\forall \kappa, \varpi \in N$, $\forall i \in I$. *Let's assume that every* (N, S_i) *is a complete SSMS,* $\forall i \in I$. *Then there exists a UFP* κ^* *of* H *in* N. *Furthermore, if* $\kappa_0 \in N$ *is fixed and* $\kappa_1, \kappa_2, \kappa_3, ...$ *are defined by* $\kappa_{n+1} = H\kappa_n$, $\forall n = 0, 1, 2, 3, ...$, *then* $S_i(\kappa_n, \kappa_n, \kappa^*) \to 0$ *as* $n \to \infty$, $\forall i \in I$.

Proof. Let $\kappa_0 \in N$, and set $\kappa_1, \kappa_2, \kappa_3, ...$in N by $\kappa_{n+1} = H\kappa_n$, $\forall n = 0, 1, 2, 3, ...$ Then, by Theorem 3.5, there is a point $\kappa_i^* \in N$ such that $S_i(\kappa_n, \kappa_n, {\kappa_i}^*) \to 0$ as $n \to \infty$ and $S_i(H{\kappa_i}^*, H{\kappa_i}^*, {\kappa_i}^*) = 0$, $\forall i \in I$. Take $Z_i = \{\kappa \in N : S_i({\kappa_i}^*, {\kappa_i}^*, \kappa) = 0\}$, an i-zero set, for every $i \in I$.
For $i \leq j$ in I, if $\kappa \in Z_j$, then we get

$$\begin{aligned}
0 \leq S_i({\kappa_i}^*, {\kappa_i}^*, \kappa) &\leq 2S_i({\kappa_i}^*, {\kappa_i}^*, \kappa_n) + S_i(\kappa, \kappa, \kappa_n) \\
&= 2S_i({\kappa_i}^*, {\kappa_i}^*, \kappa_n) + S_i(\kappa_n, \kappa_n, \kappa) \\
&\leq 2S_i({\kappa_i}^*, {\kappa_i}^*, \kappa_n) + 2S_j(\kappa_n, \kappa_n, {\kappa_j}^*) + S_j(\kappa, \kappa, {\kappa_j}^*), \\
\text{and } S_i({\kappa_i}^*, {\kappa_i}^*, \kappa) &\leq 2S_i(\kappa_n, \kappa_n, {\kappa_i}^*) + 2S_j(\kappa_n, \kappa_n, {\kappa_j}^*) + S_j({\kappa_j}^*, {\kappa_j}^*, \kappa).
\end{aligned}$$

Since $2S_i(\kappa_n, \kappa_n, {\kappa_i}^*) + 2S_j(\kappa_n, \kappa_n, {\kappa_j}^*) \to 0$ as $n \to \infty$, we get $S_i({\kappa_i}^*, {\kappa_i}^*, \kappa) = 0$. Hence, $Z_j \subseteq Z_i$, for $i \leq j$ in I, and by Assumption 4.2, we have $\bigcap_{i\in I} Z_i \neq \emptyset$. Let $\kappa^* \in \bigcap_{i\in I} Z_i$, then

$$0 \leq S_i(\kappa_n, \kappa_n, \kappa^*) \leq 2S_i(\kappa_n, \kappa_n, {\kappa_i}^*) + S_i(\kappa^*, \kappa^*, {\kappa_i}^*) = 2S_i(\kappa_n, \kappa_n, {\kappa_i}^*).$$

Since $S_i(\kappa_n, \kappa_n, {\kappa_i}^*) \to 0$ as $n \to \infty$, $\forall i \in I$, then we get $S_i(\kappa_n, \kappa_n, \kappa^*) \to 0$ as $n \to \infty$, $\forall i \in I$. Moreover,

$$\begin{aligned}
0 &\leq S_i(\kappa^*, \kappa^*, H\kappa^*) \\
&\leq 2S_i(\kappa^*, \kappa^*, {\kappa_i}^*) + S_i(H\kappa^*, H\kappa^*, {\kappa_i}^*) \\
&\leq 2S_i(\kappa^*, \kappa^*, {\kappa_i}^*) + S_i({\kappa_i}^*, {\kappa_i}^*, H\kappa^*) \\
&\leq 2S_i(\kappa^*, \kappa^*, {\kappa_i}^*) + 2S_i({\kappa_i}^*, {\kappa_i}^*, H{\kappa_i}^*) + S_i(H\kappa^*, H\kappa^*, H{\kappa_i}^*) \\
&= 2S_i({\kappa_i}^*, {\kappa_i}^*, \kappa^*) + 2S_i(H{\kappa_i}^*, H{\kappa_i}^*, {\kappa_i}^*) + S_i(H{\kappa_i}^*, H{\kappa_i}^*, H\kappa^*) \\
&\leq l_i[S_i({\kappa_i}^*, {\kappa_i}^*, \kappa^*) + S_i(H{\kappa_i}^*, H{\kappa_i}^*, {\kappa_i}^*) + S_i(H\kappa^*, H\kappa^*, \kappa^*) \\
&\quad + S_i(H{\kappa_i}^*, H{\kappa_i}^*, \kappa^*) + S_i(H\kappa^*, H\kappa^*, {\kappa_i}^*)], \\
&\leq (l_i)[S_i(\kappa^*, \kappa^*, H\kappa^*) \\
&\quad + 2S_i(H{\kappa_i}^*, H{\kappa_i}^*, {\kappa_i}^*) + 2S_i(H\kappa^*, H\kappa^*, \kappa^*) + 2S_i({\kappa_i}^*, {\kappa_i}^*, \kappa^*)], \\
&\leq (3l_i)^m S_i(\kappa^*, \kappa^*, H\kappa^*), \forall m \in \mathbb{N}, \forall i \in I.
\end{aligned}$$

Hence, $S_i(\kappa^*, \kappa^*, H\kappa^*) = 0$, $\forall i \in I$. By Assumption 4.1, we have $H\kappa^* = \kappa^*$. Furthermore, if $\varpi^* = H\varpi^*$, for some $\varpi^* \in N$,

$$\begin{aligned} 0 &\leq S_i(\kappa^*, \kappa^*, \varpi^*) = S_i(H\kappa^*, H\kappa^*, H\varpi^*) \\ &\leq l_i[S_i(\kappa^*, \kappa^*, \varpi^*) + S_i(H\kappa^*, H\kappa^*, \kappa^*) + S_i(H\varpi^*, H\varpi^*, \varpi^*) \\ &\quad +S_i(H\kappa^*, H\kappa^*, \varpi^*) + S_i(H\varpi^*, H\varpi^*, \kappa^*)], \\ &\leq (3l_i)^m S_i(\kappa^*, \kappa^*, \varpi^*), \forall m \in \mathbb{N}, \forall i \in I. \end{aligned}$$

Since $(3l_i)^m K \to 0$ as $m \to \infty$, $S_i(\kappa^*, \kappa^*, \varpi^*) = 0$, $\forall i \in I$. By Assumption 4.1, we have $\kappa^* = \varpi^*$. □

The next corollary considers Reich's contraction [18] for SMSs.

Corollary 4.9. *Suppose $(l_i)_{i\in I}$ is a collection of numbers in $(0, \frac{1}{3})$. Let H be a function on N to itself such that $S_i(H\kappa, H\kappa, H\varpi) \leq l_i[S_i(\kappa, \kappa, \varpi) + S_i(H\kappa, H\kappa, \kappa) + S_i(H\varpi, H\varpi, \varpi)]$, $\forall \kappa, \varpi \in N$, $\forall i \in I$. Let's assume that every (N, S_i) is a complete SSMS, $\forall i \in I$. Then there exists a UFP κ^* of H in N. Furthermore, if $\kappa_0 \in N$ is fixed and $\kappa_1, \kappa_2, \kappa_3,...$ are defined by $\kappa_{n+1} = H\kappa_n$, $\forall n = 0, 1, 2, 3, ...$, then $S_i(\kappa_n, \kappa_n, \kappa^*) \to 0$ as $n \to \infty$, $\forall i \in I$.*

5. Conclusion

A method to extend the concept of weak topology in normed spaces of functional analysis to metric spaces has been observed in this article. The convergence of the fixed point iteration procedure of Banach's, Kannan's, Chatterjea's, Reich's, and Hardy-Roger's contraction theorem with respect to this weak topology has been discussed in this article. Malviya and Fisher [9] introduced the concept of N-cone metric space by replacing $\mathbb{R}$ with a real Banach space in S-metric space. The N-cone metric space is a generalization of S-metric space. So we can generalize all the fixed point results of this article to N-cone metric space. The convergence of iteration methods was examined in this article with respect to a family of semi-S-metrics instead of a S-metric. In order to solve equations like differential equations, algebraic equations and integral equations, fixed point iteration methods should be used to find all feasible fixed point results, even if convergence is weak in nature.

Acknowledgement. The Council of Scientific and Industrial Research (CSIR), India is supporting the first author's current work.

References

[1] Chatterjea, S. K. (1972). *Fixed point theorems*, Rend. Acad. Bulgare. Sci., **25**, 727-730.

[2] Dhage, B. C. (1992). *Generalized metric spaces mappings with fixed point*, Bull. Calcutta Math. Soc., **84**, 329-336.

[3] Dung, N. V., Hieu, N. T., & Radojevic, S. (2014). *fixed point theorems for g-monotone maps on partilaly ordered S-metric spaces*, Filomat, **28**, 1885-1898. DOI:10.2298/FIL1409885D.

[4] G¨ahler, S. (1963). *2-metrische R¨aume und iher topoloische Struktur*, Math. Nachr., **26**, 115-148.
[5] Gupta, A. (2013). *Cyclic contraction on S-metric space*, Int. J. Anal. Appl. **3**, 119-130.
[6] Hardy, G. E., & Rogers, T. D. (1973). A generalization of a fixed point theorem of Reich Canad. Math. Bull., **16**, 201-206.
[7] Kannan, R. (1968). *Some results on fixed points*, Bull. Calcutta Math. Soc., **60**, 71-76.
[8] Khamsi, M. K., & Kirk, W. A. (2001). *An introduction to metric spaces and fixed point theory*, John Wiley, New York.
[9] Malviya, N., & Fisher, B. (2013). *N-Cone metric space and fixed points of asymptotically regular maps*, Filomat, 1-11.
[10] Moorthy, C. G., & Siva, G. (2021). *Weak order convergence of fixed point iterations of contraction mappings on cone metric spaces*, Journal of Physics:Conference Series, **1850**, 1-10. doi:10.1088/1742-6596/1850/1/012023
[11] Mustafa, Z., & Sims, B. (2006). *A new approach to generalized metric spaces*, J. Nonlinear Convex Anal., **7**, 289–297.
[12] Mustafa, Z., & Sims, B. (2003). *Some results concerning D-metric spaces*, Proc. Internat. Conf. Fixed Point Theory and Applications, pp. 189–198, Valencia, Spain.
[13] Mustafa, Z., Parvaneh, V., Abbas, M., & Roshan, J. R. (2013). *Some coincidence point results for generalized (ψ, φ)-weakly contractive mappings in ordered G-metric spaces*, Fixed Point Theory and Applications, **326**. doi: 10.1186/1687-1812-2013-326.
[14] Naidu, S. V. R., Rao, K. P. R., & Srinivasa Rao, N. (2005). *On the concepts of balls in a D-metric space*, Internat. J. Math. Math. Sci., 133–141. https://doi.org/10.1155/IJMMS.2005.133
[15] Ozgur, N. Y., & Ta¸s, N. (2018). *The Picard Theorem on S-Metric Spaces*, Acta Mathematica Scientia, **38**, 1245-1258. DOI:10.1016/S0252-9602(18)30811-7.
[16] Prudhvi, K. (2015). *Fixed point theorems in S-metric spaces*, Univers. J. Comput. Math., **3**, 19-21. DOI: 10.13189/ujcmj.2015.030201.
[17] Raj, S. I., & Moorthy, C. G. (2013). *Weak convergence of fixed point iterations in metric spaces*, J. Nonlinear Analysis and optimization, **4**, 189-192.
[18] Reich, S. (1971). *Some remarks concerning contraction mappings*, Canad. Math. Bull., **14**, 121-124.
[19] Roshan, J. R., Shobkolaei, N., Sedghi, S., Parvaneh, V., & Radenovic, S. (2014). *Common fixed point theorems for three maps in discontinuous G_b-metric spaces*, Acta Mathematica Scientia, **34**, 1643-1654. https://doi.org/10.1016/S0252-9602(14)60110-7.
[20] Sedghi, S., & Dung, N. V. (2014). *Fixed point theorems on S-metric spaces*, Mat. Vesnik, **66**, 113-124.
[21] Sedghi, S., Shobe, N., & Aliouche, A. (2012). *A generalization of fixed point theorems in S-metric spaces*, Mat Vesnik, **64**, 258-266.
[22] Sedghi, S., Shobe, N., & Zhou, H. (2007). *A common fixed point theorem in D^*-metric spaces*, Fixed Point Theory Appl., Article ID 27906, 13 pages. doi:10.1155/2007/27906.
[23] Siva, G. (2022). *Fixed points of contraction mappings with variations in S-metric space domains*, The Mathematics Student, **91**, 173-185.

G. Siva
orcid number: 0000-0002-4676-6883
Department of Mathematics
Alagappa University
Karaikudi-630 003, India
Email address: gsivamaths2012@gmail.com

S. Loganathan
orcid number: 0000-0001-8811-0738
PG and Research Department of Mathematics
Sethupathy Government Arts College
Ramanathapuram-623502, India
Email address: logaamaths2010@gmail.com