USING FRAMES OF SUBSPACES IN GALERKIN AND RICHARDSON METHODS FOR SOLVING OPERATOR EQUATIONS

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Abstract. In this paper, two iterative methods are constructed to solve the operator equation $Lu = f$ where $L : H \to H$ is a bounded, invertible and self-adjoint linear operator on a separable Hilbert space H. By using the concept of frames of subspaces, which is a generalization of frame theory, we design some algorithms based on Galerkin and Richardson methods, and then we investigate the convergence and optimality of them.

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1. Introduction

During the recent decade, several new applications of frames have been developed. Potential of frame is an almost unexplored field in numerical analysis hereof. On the one hand, the redundancy of a frame can give the freedom to implement further properties, which would be mutually exclusive in the Riesz bases case, e.g. both high

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smoothness and small support. On the other hand, while working with a weaker concept, the concrete construction of a frame is usually much simpler than an stable multi-scale basis. Consequently, there is some hope that the frame approach might simplify the geometrical construction on bounded domains. To handle these emerging applications of frames, new methods have to be developed. One starting point is to first build frames *"locally"* and then piece them together to obtain global frames for the whole space. One advantage of this idea is that it will facilitate the construction of frames for special applications by initially constructing frames or choosing already known frames for smaller spaces. And in a second step one would construct a frame for the entire space from them. This arises the concept of *Frames of Subspaces*.

In this paper, we will use frames of subspaces to get some approximate solutions for operator equation

$$Lu = f, \tag{1.1}$$

where $L : H \to H$ is a bounded (so there exist two positive constants c_1 and c_2, such that

$$c_1 \parallel u \parallel_H \leq \parallel Lu \parallel_H \leq c_2 \parallel u \parallel_H, \ \forall u \in H. \tag{1.2}$$

), invertible and self-adjoint linear operator on a separable Hilbert space H. A natural approach to construct an approximate solution is to solve a finite dimensional analog of the problem (1.1). In [7, 8], you can see the development of the adaptive numerical methods for solving the problem (1.1) by using frames. To begin with, we first will briefly recall definitions and basic properties of frames and frames of subspaces. For detailed information, we refer the reader to the survey article by Cassaza and Gitta Kutyniok [5] and the book by Christensen [6].

2. Preliminaries

Throughout this paper H shall always denote an arbitrary separable Hilbert space. Furthermore, all subspaces are assumed to be closed. Moreover, Λ denotes a countable index set and I denotes the identity operator. Also, if W is a subspace of a Hilbert space H, we let π_W denote the orthogonal projection of H onto W.

Assume that H is a separable Hilbert space, Λ is a countable index set and $\Psi = (\psi_\lambda)_{\lambda \in \Lambda} \subset H$ is a frame for H. This means that there exist constants $0 <$

$A_\Psi \leq B_\Psi < \infty$ such that

$$A_\Psi \parallel f \parallel_H^2 \leq \sum_{\lambda\in\Lambda} |< f, \psi_\lambda >|^2 \leq B_\Psi \parallel f \parallel_H^2, \quad \forall f \in H. \tag{2.1}$$

For the frame Ψ, the *synthesis operator* $T : \ell_2(\Lambda) \to H$ is defined by

$$T((c_\lambda)_{\lambda\in\Lambda}) = \sum_{\lambda\in\Lambda} c_\lambda \psi_\lambda,$$

and the *analysis operator* $T^* : H \to \ell_2(\Lambda)$ is defined by

$$T^*(f) = (< f, \psi_\lambda >)_{\lambda\in\Lambda}.$$

Also, the operator $S := TT^* : H \to H$ defined by

$$S(f) = \sum_{\lambda\in\Lambda} < f, \psi_\lambda > \psi_\lambda,$$

is called the *frame operator*. Note that T is surjective, T^* is injective, T^* is the adjoint of T and in view of (2.1) T is bounded, in fact we have

$$\parallel T \parallel = \parallel T^* \parallel \leq \sqrt{B_\Psi}.$$

Since $\ker(T) = (Ran(T^*))^\perp$, we have $\ell_2(\Lambda) = Ran(T^*) \oplus \ker(T)$. It was shown in [4], for the frame $\Psi = (\psi_\lambda)_{\lambda\in\Lambda}$, S is a bounded, invertible and positive operator satisfying $A_\Psi I_H \leq S \leq B_\Psi I_H$ and $B_\Psi^{-1} I_H \leq S^{-1} \leq A_\Psi^{-1} I_H$. Also, the sequence

$$\widetilde{\Psi} = (\widetilde{\psi}_\lambda)_{\lambda\in\Lambda} = (S^{-1}\psi_\lambda)_{\lambda\in\Lambda},$$

is a frame (called the canonical dual frame) for H with bounds B_Ψ^{-1}, A_Ψ^{-1}. Every $f \in H$ has the expansion

$$f = \sum_{\lambda\in\Lambda} < f, \psi_\lambda > \widetilde{\psi}_\lambda = \sum_{\lambda\in\Lambda} < f, \widetilde{\psi}_\lambda > \psi_\lambda.$$

We note that a complete sequence $\Psi = (\psi_\lambda)_{\lambda\in\Lambda}$ in H is called a Riesz basis if there exist constants A_Ψ, $B_\Psi > 0$ such that

$$A_\Psi \parallel C \parallel_{\ell_2(\Lambda)}^2 \leq \parallel \sum_{\lambda\in\Lambda} c_\lambda \psi_\lambda \parallel_H^2 \leq B_\Psi \parallel C \parallel_{\ell_2(\Lambda)}^2,$$

holds for all $C = (C_\lambda)_{\lambda\in\Lambda} \in \ell_2(\Lambda)$. It could be seen that each Riesz basis for a Hilbert space H is a frame for H. For an index set $\widetilde{\Lambda} \subset \Lambda$, $(\psi_\lambda)_{\lambda\in\widetilde{\Lambda}}$ is called a frame sequence if it is a frame for its closed span.

Now we pass on to study *frames of subspaces*, let H be a separable Hilbert space and Λ be a countable index set. For a family of weights $\{\omega_\lambda\}_{\lambda\in\Lambda} \subset \mathbb{R}^+$, the family of subspaces $\{H_\lambda\}_{\lambda\in\Lambda}$ of the Hilbert space H is called the *frames of subspaces* with respect to $\{\omega_\lambda\}_{\lambda\in\Lambda}$ for H, if there exist constants $0 < A \leq B < \infty$ such that

$$A \parallel f \parallel^2 \leq \sum_{\lambda\in\Lambda} \omega_\lambda^2 \parallel \pi_{H_\lambda}(f) \parallel^2 \leq B \parallel f \parallel^2 \quad \forall f \in H. \tag{2.2}$$

The constants A and B are called the frame bounds of the frames of subspaces. If $A = B$ then the frames of subspaces $\{H_\lambda\}_{\lambda\in\Lambda}$ with respect to $\{\omega_\lambda\}_{\lambda\in\Lambda}$ is called A-tight frames of subspaces. It was proved in [5], the frames of subspaces $\{H_\lambda\}_{\lambda\in\Lambda}$ is complete, in the sense that $\overline{span}_{\lambda\in\Lambda}\{H_\lambda\} = H$. The following theorem [5], shows that how we can string together frames for each of the subspaces H_λ to obtain a frame for the whole H.

Theorem 2.1. *Let Λ be an index set, $\{\omega_\lambda\}_{\lambda\in\Lambda} \subset \mathbb{R}^+$, and $\{\psi_{\lambda_i}\}_{i\in I_\Lambda}$ be a frame sequence in H with frame bounds A_λ and B_λ. Define $H_\lambda = \overline{span}_{i\in I_\Lambda}\{\psi_{\lambda_i}\}$ for all $\lambda \in \Lambda$, and suppose that $0 < A = \inf_{\lambda\in\Lambda} A_\lambda \leq B = \sup_{\lambda\in\Lambda} B_\lambda < \infty$. Then $\{\omega_\lambda\psi_{i_\lambda}\}_{\lambda\in\Lambda, i\in I_\lambda}$ is a frame for H if and only if $\{H_\lambda\}_{\lambda\in\Lambda}$ is a frames of subspaces with respect to $\{\omega_\lambda\}_{\lambda\in\Lambda}$ for H.*

Proof. See [5]. □

For a frames of subspaces $\{H_\lambda\}_{\lambda\in\Lambda}$ with respect to $\{\omega_\lambda\}_{\lambda\in\Lambda}$ define

$$\left(\sum_{\lambda\in\Lambda} \oplus H_\lambda\right)_{\ell_2} = \left\{\{\psi_\lambda\}_{\lambda\in\Lambda} : \psi_\lambda \in H_\lambda,\ \sum_{\lambda\in\Lambda} \| \psi_\lambda \|^2 < \infty\right\},$$

with inner product given by

$$\langle\{\psi_\lambda\}_{\lambda\in\Lambda}, \{\varphi_\lambda\}_{\lambda\in\Lambda}\rangle = \sum_{\lambda\in\Lambda} \langle\psi_\lambda, \varphi_\lambda\rangle.$$

Now the *synthesis operator*

$$T_{H,\omega} : \left(\sum_{\lambda\in\Lambda} \oplus H_\lambda\right)_{\ell_2} \to H,$$

for the frames of subspaces $\{H_\lambda\}_{\lambda\in\Lambda}$, with respect to $\{\omega_\lambda\}_{\lambda\in\Lambda}$ is defined by

$$T_{H,\omega}(f) = \sum_{\lambda\in\Lambda} \omega_\lambda f_\lambda \quad \forall f = \{f_\lambda\}_{\lambda\in\Lambda} \in \left(\sum_{\lambda\in\Lambda} \oplus H_\lambda\right)_{\ell_2}.$$

Also, the adjoint $T^*_{H,\omega}$ of the synthesis operator is called the *analysis operator*. In fact, the concrete formula of $T^*_{H,\omega} : H \to \left(\sum_{\lambda\in\Lambda} \oplus H_\lambda\right)_{\ell_2}$ is given by

$$T^*_{H,\omega}(f) = \{\omega_\lambda \pi_{H_\lambda}(f)\}_{\lambda\in\Lambda}.$$

Similar to the case of frame theory, the synthesis operator $T_{H,\omega}$ is bounded, linear and onto operator, and the analysis operator $T^*_{H,\omega}$ is (possibly into) isomorphism. In an orderly fashion, here we define Frames of subspaces operator $S_{H,\omega}$ as follow

$$S_{H,\omega}(f) = T_{H,\omega}T^*_{H,\omega}(f) = \sum_{\lambda\in\Lambda} \omega^2 \pi_{H_\lambda}(f).$$

Likewise the frame situation, $S_{H,\omega}$ is a positive, self-adjoint and invertible operator on H with $AI_H \leq S_{H,\omega} \leq B_H$, where A and B are the bounds of the frames of subspaces. Further, the following reconstruction formula satisfies:

$$f = \sum_{\lambda\in\Lambda} \omega^2 S_{H,\omega}^{-1} \pi_{H_\lambda}(f) \ \ \forall f \in H.$$

It can be proved that $\{S_{H,\omega}^{-1} H_\lambda\}_{\lambda\in\Lambda}$ is a frames of subspaces with respect to $\{\omega_\lambda\}_{\lambda\in\Lambda}$. In fact, the following proposition holds true

Proposition 2.2. *Let $\{H_\lambda\}_{\lambda\in\Lambda}$ be a frames of subspaces with respect to $\{\omega_\lambda\}_{\lambda\in\Lambda}$, and let $L : H \to H$ be a bounded and invertible operator on H. Then $\{L(H_\lambda)\}_{\lambda\in\Lambda}$ is a frames of subspaces with respect to $\{\omega_\lambda\}_{\lambda\in\Lambda}$.*

Proof. See [5]. □

3. Galerkin Method by Using Frames of Subspaces

Galerkin method provides a general framework for approximation of operator equation, which includes the finite element method as a special case. Let H be a separable Hilbert space, and $a(.,.) : H \to H$ be a bounded bilinear form, i.e.

$$|a(u, v)| \leq M \|u\|_H \|v\|_H \quad \forall u, v \in H, \tag{3.1}$$

for some $M > 0$, and H-elliptic in the sense that

$$a(u, v) \geq C \|v\|_H^2 \quad \forall v \in H, \tag{3.2}$$

for some $C > 0$. Suppose $f \in H$, we consider solving the variational equation

$$a(u, v) =< f, v > \quad \forall v \in H, \tag{3.3}$$

of finding $u \in H$ such that satisfies in (3.3). Next theorem ensures the existence and uniqueness of solution u.

Lemma 3.1 (Lax-Milgram Lemma)**.** *Assume H is a Hilbert space, $a(.,.)$ is a bounded, H-elliptic bilinear form on H, and $f \in H$. Then there is a unique solution of the problem*

$$u \in H, \ \ a(u, v) =< f, v > \quad \forall v \in H.$$

Proof. See [2]. □

In general, it is impossible to find the exact solution of (3.3) if the space H is infinite dimensional. A natural approach to construct an approximate solution is to a solve finite dimensional analog of (3.3). Thus, we let $H_N \subset H$ be an N-dimensional subspace. We project the problem (3.3) onto H_N,

$$u_N \in H_N, \ \ a(u_N, v) =< f, v > \quad \forall v \in H_N. \tag{3.4}$$

Under the assumption that the bilinear form $a(.,.)$ is bounded and H-elliptic, and $f \in H$, we can apply *Lax-Milgram Lemma* once more and conclude that the problem (3.4) has a unique solution u_N which, in general, approximates the exact solution u. To increase accuracy, it is natural to seek the approximate solution u_N in a large subspace H_N. Thus, for a sequence of subspaces $H_{N_1} \subset H_{N_2} \subset \cdots \subset H$, we compute the corresponding sequence of approximate solutions $u_{N_i} \in H_{N_i}$, $i \in \mathbb{N}$. This solution finding procedure is called *Galerkin Method*. In [1] by using frame, an adaptive algorithm based on Galerkin method is developed for solving the equation (1.1).

Proposition 3.2. *Assume H is a Hilbert space, $H_N \subset H$ is a subspace of H, $a(.,.)$ is a bounded, H-elliptic bilinear form on H, and $f \in H$. Let $u \in H$ be the solution of the problem (3.3), and $u_N \in H_N$ be the Galerkin approximation defined in (3.4). Then there is a constant c such that*

$$\|u - u_N\|_H \leq c \inf_{v \in H_N} \|u - v\|_H .$$

Proof. See [2]. □

Corollary 3.3. *We make the assumptions stated in proposition 3.2. Assume $H_{N_1} \subset H_{N_2} \subset \cdots$ is a sequence of finite dimensional subspaces of H with the property that*

$$\overline{\bigcup_{n \geq 1} H_n} = H. \tag{3.5}$$

Then the Galerkin method converges:

$$\|u - u_n\|_H \to 0 \ \ as \ \ n \to \infty, \tag{3.6}$$

where $u_n \in H_n$ is the Galerkin solution defined by (3.4).

Proof. See [9]. □

Assume that $\{H_i\}_{i \in \mathbb{N}}$ be a frames of subspaces. We consider the bilinear form

$$a(.,.) := < L.,. >,$$

on H, and the problem of finding u such that

$$a(u, v) = < f, v > \quad \forall v \in H. \tag{3.7}$$

We note that a is bounded and H-elliptic. Now, by projecting (3.7) onto H_i the problem takes the form of finding u_i such that

$$u_i \in H_i, \quad a(u_i, v) =< f, v > \quad \forall v \in H_i. \tag{3.8}$$

Note that by *Lax-Milgram Lemma*, (3.8) has a unique solution as u_i. Let x_i be the solution of the equation

$$a(x_i, v) =< f, v > \quad \forall v \in H_1 + H_2 + \cdots + H_i. \tag{3.9}$$

Since

$$H_1 \subseteq H_1 + H_2 \subseteq \cdots \subseteq H, \tag{3.10}$$

the sequence of solutions $\{x_i\}_{i\in\mathbb{N}}$ of equation (3.9), coincides with the sequence of approximate solutions produced by Galerkin method for (3.10). The following lemma expresses that we can obtain the Galerkin approximate solution of the equation (1.1) by considering smaller subspaces.

Lemma 3.4. *Let $\{H_i\}_{i\in\mathbb{N}}$ be a frames of subspaces with respect to $\{\omega_i\}_{i\in\mathbb{N}} \subset \mathbb{R}^+$, with bounds A, B, and with the property that $H_i \cap H_j = 0\,(i \neq j)$. Suppose that u_i is the solution of the equation (3.8) and x_i is the solution of the equation (3.9), then*

$$x_i = u_1 + u_2 + \cdots + u_i.$$

Also, the following inequality holds true

$$\frac{1}{B}\sum_{j=1}^{i} \omega_j^2 \left\| u_j \right\|^2 \leq \frac{1}{c_1^2} \left\| f \right\|^2.$$

Proof. We have

$$\begin{aligned} a(x_i - u_1 - u_2 - \cdots - u_i, v) &= a(x_i, v) - a(u_1, v) - a(u_2, v) - \cdots - a(u_i, v) \\ &= f(v) - a(u_1, v_1 + v_2 + \cdots + v_i) \\ &\qquad - a(u_2, v_1 + v_2 + \cdots + v_i) \\ &\qquad - \cdots - a(u_i, v_1 + v_2 + \cdots + v_i) \\ &= f(v) - a(u_1, v_1) - a(u_2, v_2) - \cdots - a(u_i, v_i) \\ &= f(v) - f(v_1) - f(v_2) - f(v_3) - \cdots - f(v_i) \\ &= f(v) - f(v_1 + v_2 + \cdots + v_i) \\ &= f(v) - f(v) = 0, \end{aligned}$$

that means $x_i = u_1 + u_2 + \cdots + u_i$. Now by the definition of frames of subspaces and since $\pi_{H_j} u = u_j$, we then have

$$A \|u\|_H^2 \leq \sum_j \omega_j^2 \|u_j\|^2 \leq B \|u\|_H^2 .$$

Combining this inequality with the inequality (1.2), we obtain

$$\frac{1}{B} \sum_{j=1}^{i} \omega_j^2 \|u_j\|^2 \leq \frac{1}{c_1^2} \|f\|^2 .$$

□

Now, we are ready to design an algorithm, by using frames of subspaces and based on the Galerkin method, that gives an approximate solution to the equation (1.1).

SOLVE $[\epsilon, f, L] \to u_\epsilon$

(i) Let $i := 1$, $u_0 := 0$, $\alpha_0 := 0$

(ii) $u_i :=$**GALERKIN** $[H_{\lambda_i}, L]$

(iii) $\alpha_i := \alpha_{i-1} + \omega_i^2 \|u_i\|^2$

(iv) $x_i := x_{i-1} + u_i$

(v) If $\frac{1}{c_1^2} \|f\|^2 - \frac{1}{B}\alpha_i \leq \epsilon^2$, Set $u_\epsilon = x_i$ and stop. Else

(vi) $i := i + 1$ go to (ii).

To investigate the convergence of the algorithm **SOLVE**, we present the following theorem.

Theorem 3.5. *The output of the algorithm **SOLVE** satisfies* $\|u - u_\epsilon\| \leq \epsilon$.

Proof. We have

$$\begin{aligned} \|u\|^2 &= \|u - x_i + x_i\|^2 \\ &= \|u - x_i\|^2 + \|x_i\|^2 \\ &\geq \|u - x_i\|^2 + \frac{1}{B} \sum_{j=1}^{i} \omega_j^2 \|u_j\|^2 . \end{aligned}$$

Therefore, by applying the inequality (1.2), we get

$$\begin{aligned} \|u - x_i\|^2 &\leq \|u\|^2 - \frac{1}{B} \sum_{j=1}^{i} \omega_j^2 \|u_j\|^2 \\ &\leq \frac{1}{c_1^2} \|f\|^2 - \frac{1}{B} \sum_{j=1}^{i} \omega_j^2 \|u_j\|^2 \leq \epsilon^2, \end{aligned}$$

as we desired. □

According to the algorithm **SOLVE**, the convergence rate of this algorithm originates from the algorithm **GALERKIN**, and it depends directly on it. But, by adding u_i to x_{i-1}, an improved convergence rate is achieved. The optimal output is produced when the sequence $\{H_i\}_{i \in I}$ consists of pairwise orthogonal subspaces.

4. Richardson Iterative method by Using Frames of Subspaces

The most straight forward approach to an iterative solution of a linear system is to rewrite the equation (1.1) as a linear fixed-point iteration. One way to do this is the Richardson iteration. The abstract method reads as follows:

write $Lu = f$ as

$$u = (I - L)u + f.$$

For given $u_0 \in H$, define for $k \geq 0$,

$$u_{k+1} = (I - L)u_k + f. \tag{4.1}$$

Since $Lu - f = 0$, we can write

$$\begin{aligned} u_{k+1} - u &= (I - L)u_k + f - u - (f - Lu) \\ &= (I - L)u_k - u + Lu \\ &= (I - L)(u_k - u). \end{aligned}$$

Hence

$$\| u_{k+1} - u \|_H \leq \| I - L \|_{H\to H} \| u_k - u \|_H,$$

so that the sequence (4.1) converges if

$$\| I - L \|_{H\to H} < 1. \tag{4.2}$$

It is sometimes possible to precondition (1.1) by multiplying both sides by a matrix M,

$$MLu = Mf, \tag{4.3}$$

so that convergence of sequence (4.1) is improved. This is a very effective technique for solving differential equations, integral equations, and related problems [2, 3]. One way to do this, is to find a matrix M which approximates L^{-1} i.e. $M \approx L^{-1}$ or $ML \approx I$ [9], which in this case (4.2) would be satisfied as well i.e. $\| I - ML \|_{H\to H} < 1$. Here, we want to seek M by using frames of subspaces.

Let $\{H_\lambda\}_{\lambda\in\Lambda}$ be a frames of subspaces with respect to $\{\omega_\lambda\}_{\lambda\in\Lambda}$ for a separable Hilbert space H with the frames of subspaces operator $S_{H,\omega}$. By proposition 2.2, $\{L(H_\lambda)\}_{\lambda\in\Lambda}$ is a frames of subspaces with respect to $\{\omega_\lambda\}_{\lambda\in\Lambda}$. We denote the frames of subspaces operator of $\{L(H_\lambda)\}_{\lambda\in\Lambda}$ by $S'_{H,\omega}$, and we note that

$$\begin{aligned} S'_{H,\omega}(f) = \sum_{\lambda\in\Lambda} \omega_\lambda^2 L\pi_{H_\lambda} L^{-1} f &= L \sum_{\lambda\in\Lambda} \omega_\lambda^2 \pi_{H_\lambda} L^{-1} f \\ &= LS_{H,\omega} L^{-1} f, \end{aligned}$$

that means $S'_{H,\omega} = LS_{H,\omega}L^{-1}$.

Proposition 4.1. *Let* $\{H_\lambda\}_{\lambda\in\Lambda}$ *be a frames of subspaces with respect to* $\{\omega_\lambda\}_{\lambda\in\Lambda}$ *for* H*, and let* L *be a bounded, invertible and self-adjoint operator on* H*. If* $S'_{H,\omega}$ *denotes the frames of subspaces operator of* $\{L(H_\lambda)\}_{\lambda\in\Lambda}$ *with bounds* A *and* B*, then*

$$\left\| I - \frac{2}{c_1^2 A + c_2^2 B} L S'_{H,\omega} L \right\|_{H\to H} \leq \frac{c_2^2 B - c_1^2 A}{c_1^2 A + c_2^2 B}, \tag{4.4}$$

where c_1 *and* c_2 *are as in (1.2).*

Proof. For every $\nu \in H$ we have

$$\begin{aligned}
\left\langle \left(I - \frac{2}{c_1^2 A + c_2^2 B} L S'_{H,\omega} L \right) \nu, \nu \right\rangle &= \|\nu\|_H^2 - \frac{2}{c_1^2 A + c_2^2 B} \left\langle S'_{H,\omega} L\nu, L\nu \right\rangle \\
&= \|\nu\|_H^2 - \frac{2}{c_1^2 A + c_2^2 B} \left\langle \sum_{\lambda\in\Lambda} \omega_\lambda^2 \pi_{LH_\lambda}(L\nu), L\nu \right\rangle \\
&= \|\nu\|_H^2 - \frac{2}{c_1^2 A + c_2^2 B} \sum_{\lambda\in\Lambda} \omega_\lambda^2 \|\pi_{LH_\lambda}(L\nu)\|_H^2 \\
&\leq \|\nu\|_H^2 - \frac{2A}{c_1^2 A + c_2^2 B} \|L\nu\|_H^2 \\
&\leq \|\nu\|_H^2 - \frac{2A}{c_1^2 A + c_2^2 B} c_1^2 \|\nu\|_H^2 \\
&= \left(\frac{c_2^2 B - c_1^2 A}{c_1^2 A + c_2^2 B} \right) \|\nu\|_H^2 .
\end{aligned}$$

Similarly, we obtain

$$-\left(\frac{c_2^2 B - c_1^2 A}{c_1^2 A + c_2^2 B} \right) \|\nu\|_H^2 \leq \left\langle \left(I - \frac{2}{c_1^2 A + c_2^2 B} L S'_{H,\omega} L \right) \nu, \nu \right\rangle.$$

So, we conclude (4.4) □

Since $\frac{c_2^2 B - c_1^2 A}{c_1^2 A + c_2^2 B} < 1$, by proposition 4.1 we can put $M := \frac{2}{c_1^2 A + c_2^2 B} L S'_{H,\omega}$ to precondition (1.1). Thus, by performing the Richardson iteration (4.1) on preconditioned linear equation (4.3), we then get the following theorem

Theorem 4.2. *Let* $\{H_\lambda\}_{\lambda\in\Lambda}$ *be a frames of subspaces with respect to* $\{\omega_\lambda\}_{\lambda\in\Lambda}$ *for* H *with frames of subspaces operator* $S_{H,\omega}$*, and let* L *be as in (1.1). Put* $u_0 = 0$ *and for* $k \geq 1$*,*

$$u_k = u_{k-1} + \frac{2}{c_1^2 A + c_2^2 B} L S'_{H,\omega} (f - L u_{k-1}),$$

where $S'_{H,\omega}$ *denotes the frames of subspaces operator of* $\{L(H_\lambda)\}_{\lambda\in\Lambda}$ *with bounds* A *and* B*,and* c_1*,*c_2 *are as in (1.2), then*

$$\|u-u_k\|_H \le \left(\frac{c_2^2B-c_1^2A}{c_1^2A+c_2^2B}\right)^k \|u\|_H.$$

In particular the vectors u_k converge to u as $k\to\infty$.

Proof. By definition of u_k we obtain

$$\begin{aligned}
u-u_k &= u-u_{k-1}+\frac{2}{c_1^2A+c_2^2B}LS'_{H,\omega}\left(f-Lu_{k-1}\right)\\
&=\left(I-\frac{2}{c_1^2A+c_2^2B}LS'_{H,\omega}L\right)(u-u_{k-1})\\
&=\left(I-\frac{2}{c_1^2A+c_2^2B}LS'_{H,\omega}L\right)^2(u-u_{k-2})\\
&=\cdots\\
&=\left(I-\frac{2}{c_1^2A+c_2^2B}LS'_{H,\omega}L\right)^k(u-u_0).
\end{aligned}$$

Therefore

$$\|u-u_k\| \le \left\|I-\frac{2}{c_1^2A+c_2^2B}LS'_{H,\omega}L\right\|^k\|u\|.$$

Combining this inequality with (4.4) gives the result. □

Now, we are in a position to derive an algorithm in order to approximate the solution u of the equation (1.1). This algorithm depends on the knowledge of the bounds of the frames of subspaces $\{L(H_\lambda)\}_{\lambda\in\Lambda}$, and constants stated in (1.2). It is guaranteed that the convergence rate depends on them too.

RICHARDSON $[L, \epsilon, A, B, c_1, c_2] \to u_\epsilon$

(i) Let $\alpha_0 = \frac{c_2^2 B - c_1^2 A}{c_1^2 A + c_2^2 B}$

(ii) $k := 0,\ u_k := 0$

(iii) $k := k + 1$

(1) $u_k = u_{k-1} + \frac{2}{c_1^2 A + c_2^2 B} L S'_{H,\omega} (f - L u_{k-1})$

(2) $\alpha_k := (\alpha_0)^k \frac{\|f\|_H}{c_1}$

(iv) If $\alpha_k \leq \epsilon$ stop and set $u_\epsilon := u_k$, if else go to (iii).

Regarding to the parameter $\left(\frac{c_2^2 B - c_1^2 A}{c_1^2 A + c_2^2 B}\right)^k$ stated in theorem 4.2 , in the case when we have a tight frames of subspaces $\{L(H_\lambda)\}_{\lambda \in \Lambda}$ in hand, the algorithm **RICHARDSON** gains its highest rate of convergence of $\left(\frac{c_2^2 - c_1^2}{c_1^2 + c_2^2}\right)^k$.

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