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COMMUTING CONJUGACY CLASS GRAPH OF THE FINITE 2−GROUPS $G_n(m)$ AND $G[n]$

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Abstract. Suppose G is a finite non-abelian group and $\Gamma(G)$ is a graph with non-central conjugacy classes of G as its vertex set. Two vertices L and K in $\Gamma(G)$ are adjacent if there are $a \in L$ and $b \in K$ such that $ab = ba$. This graph is called the commuting conjugacy class graph of G. The purpose of this paper is to compute the commuting conjugacy class graph of the finite 2−groups $G_n(m)$ and $G[n]$.

Keywords: Commuting conjugacy class graph, conjugacy class, center.
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1. Preliminaries

All groups in this paper are finite and we will consider only simple undirected graphs. Our graph theory notations are taken from [2] and we refer to [7] for group theory notions. Choose the set $X = \{\Delta_1, \ldots, \Delta_s\}$ of undirected graphs with mutually disjoint vertex sets. The *graph union* $\Delta_1 \cup \cdots \cup \Delta_s$ is a graph with vertex set $V(\Delta_1) \cup \cdots \cup V(\Delta_s)$ and edge set $E(\Delta_1) \cup \cdots \cup E(\Delta_s)$. In the case that all of these graphs are isomorphic together, we use the notation $s\Delta_1$ to denote $\Delta_1 \cup \cdots \cup \Delta_s$.

The commuting conjugacy class graph was first introduced by Herzog et al. [3]. in which the authors considered all non-identity conjugacy classes of a finite group G as the vertex set of the graph. But, in this paper we restricted our attention to the set of all non-central conjugacy classes of a finite non-abelian group G as the vertex set of this graph and two vertices K and L of this graph are adjacent if there are $a \in K$ and $b \in L$ such that $ab = ba$. In [6], the authors gave a classification of finite groups with triangle-free commuting conjugacy class graphs.

Suppose m and n are two positive integers such that $m \geq 2$ and $n \geq 3$. Suppose $G_n(m)$ is the group with generators $a_1, \ldots, a_n, b$ and the following

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relations:

$$\begin{array}{lll} a_1^2 = 1, & a_2^{2^m} = 1, & a_i^4 = 1 \quad (3 \le i \le n), \\ a_{n-1}^2 = b^2, & [a_i, a_j] = 1 \quad (1 \le i < j \le n), & \\ [a_1, b] = 1, & [a_n, b] = a_1, & [a_{i-1}, b] = a_i^2 \quad (3 \le i \le n). \end{array}$$

Theorem 1.1. ([4, Proposition 2.1]) *Suppose $G = G_n(m)$, where $m \ge 2$ and $n \ge 3$. Then the following conditions hold:*

(1) $|G| = 2^{2n+m-2}$;
(2) *Each element of G can be written uniquely in the form $a_1^{\alpha_1} a_2^{\alpha_2} \dots a_n^{\alpha_n} b^{\beta}$ where $\alpha_1, \beta \in \{0, 1\}$, $0 \le \alpha_2 \le 2^m - 1$ and $0 \le \alpha_i \le 3$ $(3 \le i \le n)$;*
(3) $Z(G) = \langle a_1 \rangle \times \langle a_2^2 \rangle \times \dots \times \langle a_n^2 \rangle \cong \mathbb{Z}_2^{n-1} \times \mathbb{Z}_{2^{m-1}}$ *and* $|Z(G)| = 2^{n+m-2}$;
(4) $G/Z(G) \cong \mathbb{Z}_2^n$;
(5) *The subgroup $\langle a_1, a_2, \dots, a_n \rangle$ is the unique abelian subgroup of G of index 2.*

It is an old question that which non-abelian groups have an abelian automorphism group. The first such a group was introduced by Miller [5] as $T = \langle x, y, z, u \rangle$ with defining relations $x^8 = y^2 = z^2 = u^2 = [x, z] = [x, u] = [z, u] = [z, y] = 1$, $yxy = x^5$ and $yuy = uz$. Miller proves that G is a non-abelian group with $G/Z(G) \cong {\mathbb{Z}_2}^3$ and $\mathrm{Aut(G)} \cong {\mathbb{Z}_2}^7$. Here, $\mathbb{Z}_n$ denotes the cyclic group of order n. Struik [8] generalized Miller's example by assuming that $n \ge 3$, $k = 2^{n-1}$, $G[n] = \langle x, y, z, u \rangle$ and defining relations $x^{2^n} = y^2 = z^2 = u^2 = [x, z] = [x, u] = [z, u] = [y, z] = 1$, $yxy = x^{k+1}$ and $yuy = zu$. Struik [8] proved that this group has order 2^{n+3} and its automorphism group is isomorphic to ${\mathbb{Z}_2}^6 \times \mathbb{Z}_{2^{n-2}}$.

2. Main Results

The purpose of this section is to obtain the structure of the commuting conjugacy class graph of $G_n(m)$ and $G[n]$.

Lemma 2.1. *Suppose G is a non-abelian finite group with center Z such that the quotient group G/Z is abelian. Then for every $x \in G \setminus Z$, $x^G = xH$ for some $H \le Z$ and $|x^G| \mid |Z|$.*

Proof. Since G/Z is abelian, G is nilpotent of class 2. Thus $[x, G] \le Z$ for any element $x \in G$. Let $H = [x, G]$, so that $x^G = xH$ and $|x^G| \mid |Z|$. Hence, the proof is complete. □

Theorem 2.2. *Suppose $m \ge 2$, $n \ge 3$ and $G = G_n(m)$. Then,* $\Gamma(G) = 2^{n-1} K_{2^{m-1}} \cup K_{(2^{n-1}-1)2^{n+m-3}}$.

Proof. By Theorem 1.1(4), $G/Z(G)$ is an abelian group. Then, by Lemma 2.1, for every $x \in G \setminus Z(G)$, $x^G = xH$ for some $H \le Z(G)$. For simplicity of our argument, we write $Z = Z(G)$. By Theorem 1.1(2-4), we have

$$\frac{G}{Z} \cong \langle a_2 Z \rangle \times \langle a_3 Z \rangle \times \dots \times \langle a_n Z \rangle \times \langle bZ \rangle,$$

in which for every $2 \leq i \leq n$, $o(a_iZ) = o(bZ) = 2$ and $a_iZbZ = bZa_iZ$. Thus, there exists $z \in Z$ which satisfies the following equation:

$$a_ib = ba_iz, \quad 2 \leq i \leq n. \tag{1}$$

Therefore, $G = \{b^{\alpha_1}a_2^{\alpha_2}\dots a_n^{\alpha_n}z \mid z \in Z, \alpha_i \in \{0,1\}, 1 \leq i \leq n\}$. By Equation 1, there exists $z \in Z$ such that

$$(b^{\alpha_1}a_2^{\alpha_2}\dots a_n^{\alpha_n})^2 = b^{2\alpha_1}a_2^{2\alpha_2}\dots a_n^{2\alpha_n}z \in Z.$$

We now assume that $x_{\alpha_2\dots\alpha_n} = a_2^{\alpha_2}\dots a_n^{\alpha_n}$ and $y_{\alpha_2\dots\alpha_n} = ba_2^{\alpha_2}\dots a_n^{\alpha_n}$. By definition of G and Theorem 1.1(5), $C(x_{\alpha_2\dots\alpha_n}) = \langle a_1, a_2, \dots, a_n\rangle$ which implies that $|C(x_{\alpha_2\dots\alpha_n})| = 2^{2n+m-3}$. This shows that $|(x_{\alpha_2\dots\alpha_n})^G| = 2$. Furthermore, $C(y_{\alpha_2\dots\alpha_n}) = Z \times \langle y_{\alpha_2\dots\alpha_n}\rangle$. This shows that $|C(y_{\alpha_2\dots\alpha_n})| = 2^{n+m-1}$ or equivalently $|(y_{\alpha_2\dots\alpha_n})^G| = 2^{n-1}$.

Since G/Z is an abelian group, by Lemma 2.1, we have $|(x_{\alpha_2\dots\alpha_n})^G| \mid |Z|$ and the number of conjugacy classes in the form of $(x_{\alpha_2\dots\alpha_n}z)^G$ with $z \in Z$ is as follows:

$$s = \frac{|Z|}{|(x_{\alpha_2\dots\alpha_n})^G|} = \frac{2^{n+m-2}}{2} = 2^{n+m-3}.$$

In the same way, $|(y_{\alpha_2\dots\alpha_n})^G| \mid |Z|$ and the number of conjugacy classes in the form of $(y_{\alpha_2\dots\alpha_n}z)^G$ with $z \in Z$ is as follows:

$$t = \frac{|Z|}{|(y_{\alpha_2\dots\alpha_n})^G|} = \frac{2^{n+m-2}}{2^{n-1}} = 2^{m-1}.$$

By the definition of G, we know that for every i and j, $[a_i, a_j] = 1$. On the other hand, $x_{\alpha_2\dots\alpha_n} = a_2^{\alpha_2}\dots a_n^{\alpha_n}$ in which $\alpha_i \in \{0,1\}$. Therefore, all of $x_{\alpha_2\dots\alpha_n}$'s commute pairwise and hence all conjugacy classes of the form $(x_{\alpha_2\dots\alpha_n})^G$ are adjacent in the commuting conjugacy class graph. It is easy to show that the number of $x_{\alpha_2\dots\alpha_n} = a_2^{\alpha_2}\dots a_n^{\alpha_n}$'s are $2^{n-1} - 1$. Thus, the subgraph induced by these elements is isomorphic to $K_{(2^{n-1}-1)2^{n+m-3}}$.

We apply a similar argument to show that all of the elements of the form $y_{\alpha_2,\dots,\alpha_n}$ do not commute pairwise and so all conjugacy classes of the form $(y_{\alpha_2\dots\alpha_n}z)^G$ are not adjacent in commuting conjugacy class graph. On the other hand, the number of $y_{\alpha_2\dots\alpha_n}$'s are 2^{n-1} and the subgraph induced by these elements is isomorphic to $2^{n-1}K_{2^{m-1}}$. The above arguments show that the commuting conjugacy class graph of G is $\Gamma(G) = 2^{n-1}K_{2^{m-1}} \cup K_{(2^{n-1}-1)2^{n+m-3}}$ that completes the proof. □

Theorem 2.3. *Suppose $n \geq 3$ and $G = G[n]$. Then, $\Gamma(G) = K_{3k} \cup 4K_{\frac{k}{2}}$ in which $k = 2^{n-1}$.*

Proof. By the presentation of G, it is easy to see that $yx = x^{k+1}y$ and so

$$yx^2 = (yx)x = x^{k+1}yx = x^{2k+2}y = x^2y. \tag{2}$$

Hence, $\langle x^2\rangle \subseteq Z(G)$. Since $z \in Z(G)$, $Z(G) = \langle x^2\rangle \times \langle z\rangle$. See also [8, p. 300]. Then the elements of G can be written as follows:

$\boxed{1}$	x	$\boxed{x^2}$	x^3	$\ldots$	x^{k-1}	$\boxed{x^k}$	x^{k+1}	$\ldots$	x^{2^n-1}
$\boxed{z}$	xz	$\boxed{x^2z}$	x^3z	$\ldots$	$x^{k-1}z$	$\boxed{x^kz}$	$x^{k+1}z$	$\ldots$	$x^{2^n-1}z$
u	xu	x^2u	x^3u	$\ldots$	$x^{k-1}u$	x^ku	$x^{k+1}u$	$\ldots$	$x^{2^n-1}u$
uz	xuz	x^2uz	x^3uz	$\ldots$	$x^{k-1}uz$	x^kuz	$x^{k+1}uz$	$\ldots$	$x^{2^n-1}uz$
y	xy	x^2y	x^3y	$\ldots$	$x^{k-1}y$	x^ky	$x^{k+1}y$	$\ldots$	$x^{2^n-1}y$
yz	xyz	x^2yz	x^3yz	$\ldots$	$x^{k-1}yz$	x^kyz	$x^{k+1}yz$	$\ldots$	$x^{2^n-1}yz$
yu	xyu	x^2yu	x^3yu	$\ldots$	$x^{k-1}yu$	x^kyu	$x^{k+1}yu$	$\ldots$	$x^{2^n-1}yu$
yuz	$xyuz$	x^2yuz	x^3yuz	$\ldots$	$x^{k-1}yuz$	x^kyuz	$x^{k+1}yuz$	$\ldots$	$x^{2^n-1}yuz$

We now compute the conjugacy classes of G. To do this, we consider the following cases:

(1) *Rows 1 and 2.* Suppose $i = 2t+1$ is an odd integer. Since $x^2 \in Z(G)$,

$$y^{-1}x^iy = yx^{2t+1}y = x^{2t}yxy = x^{2t}x^{k+1} = x^{k+2t+1} = x^{k+i}. \tag{3}$$

Thus for every $0 \leq t \leq \frac{k}{2}-1$, $(x^{2t+1})^G = \{x^{2t+1}, x^{k+2t+1}\}$. By this equality and the fact that $z \in Z(G)$, $(x^{2t+1}z)^G = (x^{2t+1})^G z = \{x^{2t+1}z, x^{k+2t+1}z\}$, for every $0 \leq t \leq \frac{k}{2}-1$.

(2) *Rows 3 and 4.* Applying Equations 2 and 3, for every $0 \leq i \leq 2^n - 1$, yields

$$(x^iu)^G = \begin{cases} \{x^iu, x^iuz\} & 2 \mid i \\ \{x^iu, x^{k+i}uz\} & 2 \nmid i \end{cases}.$$

(3) *Rows 5 and 6.* By Equations 2 and 3, for every $0 \leq i \leq k-1$, we get

$$(x^iy)^G = \{x^iy, x^iyz, x^{k+i}y, x^{k+i}yz\}.$$

(4) *Rows 7 and 8.* By Equations 2 and 3, for every $0 \leq i \leq k-1$, we get

$$(x^iyu)^G = \{x^iyu, x^iyuz, x^{k+i}yu, x^{k+i}yuz\}.$$

Since $[x, u] = 1$, all of the conjugacy classes in cases (1) and (2) are adjacent in the commuting conjugacy class graph. On the other hand, the number of the conjugacy classes contained in cases (1) and (2) is $2 \times \frac{k}{2} + 2^n$ and so, the restriction of the commuting conjugacy class graph to these conjugacy classes is the complete graph K_{3k}. Furthermore, by Equation 2, if $i+j$ is even, then $(x^iy)(x^jy) = (x^jy)(x^iy)$ and $(x^iyu)(x^jyu) = (x^jyu)(x^iyu)$. Therefore, the classes contained in case (3) and (4) can be partitioned into two parts and the subgraph induced on each part is the complete graph $K_{\frac{k}{2}}$. Hence, $\Gamma(G) = K_{3k} \cup 4K_{\frac{k}{2}}$ which completes the proof. □

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