Shahid Bahonar University of Kerman
Journal of Mahani Mathematical Research
Print ISSN: 2251-7952 Online ISSN: 2645-4505

SOME SUPERCHARACTER THEORIES OF A CERTAIN GROUP OF ORDER $6n$

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Article type: Research Article
(Received: 02 September 2023, Received in revised form 14 November 2023)
(Accepted: 02 December 2023, Published Online: 02 December 2023)

Abstract. In this paper, we are going to obtain some normal supercharacter theories of a group of order $6n$ with the presentation $U_{6n} =< a, b : a^{2n} = b^3 = 1, a^{-1}ba = b^{-1} >$ in special cases. We will also prove that the automorphic supercharacter theories of this group can be computed with the other methods.

Keywords: Character theory, Supercharacter, Lattice of normal subgroups
2020 MSC: Primary 20C12, 20E15.

1. Introduction

Computing the irreducible characters of certain groups can be a challenging task. An example of such a group is, $U_n(\mathbb{F}_q)$, which presents the group of upper triangular matrices over a field of size q, with all diagonal entries equal to one. To overcome this problem, Carlos André [2] and Yan [11] presented the concept of supercharacter theory as an estimate of character table of the $U_n(\mathbb{F}_q)$. It was then developed by Diaconis and Isaacs in [5] as a generalization of the theory of irreducible characters of finite groups. Motivated by finding the character table of $U_n(\mathbb{F}_q)$, this article is an attempt to determine the supercharacter theories of a group of order $6n$ in special cases. In fact, we have applied the results of the paper [1] to this group.

The principal application of the supercharacter theory of the groups is to derive properties of the character table of groups when determining the character table is difficult or unachievable. For a finite group G, let $\mathrm{Irr}(G)$ and $\mathrm{Con}(G)$ denote the set of irreducible and conjugacy classes of G, respectively. 1_G denotes the identity character of G and the identity element of G is denoted by 1. The trivial subgroup $\{1\}$ of G is denoted by 1.

2. Preliminaries

Definition 2.1. [5] Let G be a finite group. Let $\mathcal{K}$ be a partition of G and $\mathcal{X}$ be a partition of $\mathrm{Irr}(G)$. A supercharacter theory for G is a pair $(\mathcal{K}, \mathcal{X})$ such that

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https://doi.org/10.22103/jmmr.2023.22138.1505
Publisher: Shahid Bahonar University of Kerman

How to cite: E. Armioun, *Some supercharacter theories of a certain group of order* $6n$, J. Mahani Math. Res. 2024; 13(2): 179 - 189.

(1) $|\mathcal{K}| = |\mathcal{X}|$;
(2) $\{1\} \in \mathcal{K}$;
(3) For each $X \in \mathcal{X}$ there exists a character σ_X with $\mathrm{supp}(\sigma_X) \subseteq X$ such that $\sigma_X(x) = \sigma_X(y)$ for all $x, y \in K$ and $K \in \mathcal{K}$.

The character σ_X is called supercharacter and every member of $\mathcal{K}$ is called superclass. The set of all supercharacter theories of G is denoted by $\mathrm{Sup}(G)$. For each subset $X \subseteq \mathrm{Irr}(G)$, we set $\sigma_X = \sum_{\chi \in X} \chi(1)\chi$. In [5] it is proved that $\{1_G\} \in \mathcal{X}$. Furthermore, for every X that belongs to $\mathcal{X}$, the supercharacter that is associated with X is a constant multiple of σ_X. For the sake of simplicity, it is taken into consideration by σ_X.
Every group with order more than two has two trivial supercharacter theories:

(1) Maximal supercharacter theory:
$$M(G) = \big(\{1\} \cup \{G - \{1\}\}, \{1_G\} \cup \{Irr(G) - \{1_G\}\}\big).$$
(2) Minimal supercharacter theory:
$$m(G) = \big(\text{The set of conjugacy classes of } G, \bigcup_{\chi \in \mathrm{Irr}(G)} \{\{\chi\}\}\big).$$

The set of supercharacter theories for a finite group G forms a lattice in the following natural way. $\mathrm{Sup}(G)$ is a poset by defining $(\mathcal{X}, \mathcal{K}) \preceq (\mathcal{Y}, \mathcal{L})$ if and only if $\mathcal{X} \preceq \mathcal{Y}$ in the sense that every part of $\mathcal{X}$ is a subset of some part of $\mathcal{Y}$. It is shown in [8] that this is equivalent to $\mathcal{K} \preceq \mathcal{L}$.

Now we assume $\mathrm{Con}(G) = \{\mathcal{C}_1 = \{1\}, \mathcal{C}_2, \ldots, \mathcal{C}_h\}$. Suppose for each $\alpha \in A$ where $A \leq \mathbb{A}ut(G)$, we have $\mathcal{C}_i^\alpha = \mathcal{C}_j$ where $1 \leq i \leq h$, and $\chi_i^\alpha(g) = \chi_i(g^\alpha)$ for all $g \in G$, $\alpha \in A$. By a result due to Brauer [6], the number of conjugacy classes is fixed by α is equal to the number of irreducible characters of G is fixed by α. Furthermore, the number of orbits of action A on $\mathrm{Con}(G)$ is equal to the number of orbits of A acting on $\mathrm{Irr}(G)$. Now it is easy to see that the orbits of A on $\mathrm{Irr}(G)$ and $\mathrm{Con}(G)$ yield a supercharacter theory for G. These kinds of supercharacter theories for G is called automorphic. In [8] it is shown that all supercharacter theories of a cyclic group of order p, p prime, are automorphic. Let G and H be finite groups, where G acts on H by automorphism. $(\mathcal{K}, \mathcal{X}) \in \mathrm{Sup}(H)$ is called G-invariant if the action of G fixes all parts of $\mathcal{K}$. The set of all G-invariant supercharacter theories of H is denoted by $\mathrm{Sup}_G(H)$. If H is a normal subgroup of G, then the superclasses of G-invariant superchater theories are unions of conjugacy classes of G. Thus, by using normal subgroups of G, we can extend their supercharacter theories to the supercharacter theories of G.

Proposition 2.2. *([8], Theorem 4.3) Let N be a normal subgroup of G. Let $\mathcal{C} = (\mathcal{K}, \mathcal{X}) \in \mathrm{Sup}_G(N)$ and $\mathcal{D} = (\mathcal{L}, \mathcal{Y}) \in \mathrm{Sup}(G/N)$. Then there is a supercharacter theory for G in which the set of superclasses is*
$$\mathcal{M} = \mathcal{K} \cup \{\cup_{gN \in L} gN : L \in \mathcal{L} \setminus \{N\}\}$$

and the correspondence set of supercharacters is

$$\mathcal{Z} = \{X^G : X \in \mathcal{X} \setminus \{1_N\}\} \cup \mathcal{Y}$$

where X^G is the set of constituents of ϕ^G for all $\phi \in X$, where $X \in \mathcal{X}$. This supercharacter theory is called $\mathcal{C} * \mathcal{D}$ over N or $\mathcal{C} * \mathcal{D}$ factors over N.

Lemma 2.3. *([7]) Let G be a finite group, and let N be a normal subgroup of G. Then $\mathcal{C} \in \mathrm{Sup}(G)$ is $*$-product over N if and only if $\mathcal{C}$ is in the interval $[m_G(N)m(G/N)\ M(N)M(G/N)]$ where $m_G(N)$ is the minimal G-invariant supercharacter theory of N.*

Let S be a supercharacter theory of a finite group G and let N be a subgroup of G, If N is a union of superclasses of S, then we say N is S-normal subgroup.

Definition 2.4. ([8], Definition 6.3) Let S=$(\mathcal{K}, \mathcal{X}) \in \mathrm{Sup}(G)$ and let N be a S-normal subgroup of G. The restriction supercharacter theory S to N and the deflated supercharacter theory are defined as follows, respectively.

$$S_N = \Big(\{K \in \mathcal{K} : K \subseteq N\}, \{X_N : X \in \mathcal{X}, X \nsubseteq \mathrm{Irr}(G/N)\} \cup \{\{1_N\}\}\Big),$$

where $X_N = \{\phi \in \mathrm{Irr}(N) : \exists \chi \in X : \langle \phi, \chi_N \rangle \neq 0\}$.
$S^{G/N} = \Big(\{\{N\}\} \cup \{\{gN : g \in K\} : K \in \mathcal{K}, K \nsubseteq N\}, \{X \in \mathcal{X} : X \subseteq \mathrm{Irr}(G/N)\}\Big)$.

In [8] it is proved that S_N and $S^{G/N}$ are supercharacter theories of N and G/N, respectively. Now we can express Δ-product in the following way.

Proposition 2.5 ([8], Theorem. 7.2)**.** *Let G be a finite group with normal subgroups M and N such that $N \trianglelefteq M \trianglelefteq G$. Let S=$(\mathcal{K}, \mathcal{X}) \in \mathrm{Sup}_G(M)$ and T=$(\mathcal{L}, \mathcal{Y}) \in \mathrm{Sup}(G/N)$. Suppose*

(1) *N is S-normal;*
(2) *M/N is T-normal;*
(3) *$S^{M/N} = T_{M/N}$.*

Then there is a unique supercharacter theory of G in which its superclasses are

$$\mathcal{K} \cup \{\bigcup_{gN \in L} : L \in \mathcal{L}, L \nsubseteq M/N\},$$

and its supercharacters are

$$\mathcal{Y} \cup \{X^G : X \in \mathcal{X}, X \nsubseteq Irr(M/N)\}.$$

This supercharacter theory of G is called Δ-product of S and T, and is denoted by $S \Delta T$. Note that if $M = N$, then the Δ-product reduces to $*$-product.

Theorem 2.6. *([3], Theorem 1) If S is a supercharacter theory and H and K are S-normal subgroups of G such that $H \subseteq K$, then S is Δ-product of H and K if and only if $\mathcal{X}(g) = 0$ for every supercharacter $\mathcal{X}$ which its kernel does not contain H and $g \in G \setminus K$.*

Definition 2.7. Let G be a finite group and $S = (\mathcal{X}, \mathcal{K})$ be a supercharacter theory of G. Consider an ordering on the set of superclasses and an ordering on the set of supercharacters of S. The matrice in which its i,j-th entry is equal to the value of the jth supercharacter in the ith superclass is the supercharacter table corresponding to S.

Recently, in [1] a new method is presented to construct a supercharacter theory from an arbitrary set of normal subgroups. This is called a normal supercharacter theory.
Let G be a finite group, and let Norm(G) be the set of all the normal subgroups of G. Since the product of two normal subgroups of G is also a normal subgroup of G, the set Norm(G) has the structure of a semigroup.

Definition 2.8. ([1]) Let $S \subseteq \mathrm{Norm}(G)$. We define $A(S)$ to be the smallest subsemigroup generated by S, denoted by $A(S)$, has the obeying properties:

(1) $\{1\}, G \in A(S)$;
(2) $S \subseteq A(S)$;
(3) $A(S)$ is closed under intersection.

If we consider $S \subseteq \mathrm{Norm}(G)$ such that $A(S) = \{\{1\}, G\}$, then the corresponding normal supercharacter theory to $A(S)$ is equal to $M(G)$.
It is shown in [1] that every sublattice of the lattice of normal subgroups of G containing $\{1\}$ and G yields a normal supercharacter theory.

Example 2.9. *Let $G = U_{6p} =< a, b : a^{2p} = b^3, a^{-1}ba = b^{-1} >$, where p is a prime.*
Given the various cases of $S \subseteq \mathrm{Norm}(G)$, $A(S)$ can be one of the following:

(1) $\{\{1\}, G\}$,
(2) $\{\{1\}, G, \langle a^2, b\rangle\}$,
(3) $\{\{1\}, G, \langle a^p, b\rangle\}$,
(4) $\{\{1\}, G, \langle a^2\rangle\}$,
(5) $\{\{1\}, G, \langle a^2, b\rangle, \langle a^2\rangle\}$,
(6) $\{\{1\}, G, \langle b\rangle\}$,
(7) $\{\{1\}, G, \langle b\rangle, \langle a^p, b\rangle\}$,
(8) $\{\{1\}, G, \langle b\rangle, \langle a^2, b\rangle, \langle a^2\rangle\}$,
(9) $\{\{1\}, \langle b\rangle, \langle a^2, b\rangle, \langle a^p, b\rangle, G\}$,
(10) *Norm*(G).

For N$\in A(S)$, we define

$$N^\circ = N \setminus \bigcup_{\substack{H \subset N \\ H \in A(S)}} H.$$

It follows that $\{1\}^\circ = \{1\}$. For each $N \in A(S)$ such that $N \trianglelefteq G$, we let $\mathcal{X}^N = \{\varphi \in \mathrm{Irr}(G) : N \leqslant ker\varphi\}$ and $\chi^N = \sum_{\varphi \in \mathcal{X}^N} \varphi(1)\varphi$. We also set $\mathcal{X}^{N^\circ} = \mathcal{X}^N \setminus \bigcup_{\substack{N \subset K \\ K \in A(S)}} \mathcal{X}^K$.

Theorem 2.10. *([1], Theorem 3. 4) For an arbitrary $S \subseteq \mathrm{Norm}(G)$,*

$$(\{N^\circ \neq \emptyset : N \in A(S)\}, \{\mathcal{X}^{N^\circ} \neq \emptyset : N \in A(S)\})$$

is a supercharacter theory for G. This supercharacter theory is called the normal supercharacter theory generated by S.

Lemma 2.11. *([1], Theorem 3. 5) Let G be a group and let $S \subseteq \mathrm{Norm}(G)$. Then for each $N \in A(S)$ we have*

$$\mathcal{X}^{N^\circ}(g) = \begin{cases} \sum_{\psi \in \mathcal{X}^{N^\circ}} \psi^2(1) & g \in N, \\ -\sum_{\substack{K \in A(S): \\ N \subset K}} \mathcal{X}^{K^\circ}(g) & g \notin N. \end{cases}$$

Now we consider the group U_{6n} which is defined by generators and relations as follows:

$$U_{6n} = \langle a, b | a^{2n} = b^3 = 1, a^{-1}ba = b^{-1} \rangle$$

Lemma 2.12. *([10]) If N is a normal subgroup of U_{6n}, then either $N = \langle a^i \rangle$ where i is an even divisor of $2n$ or $N = \langle a^i, b \rangle$ in which $i \mid 2n$ or $N = \langle b \rangle$.*

The group U_{6n} has $3n$ conjugacy classes:
$\{a^{2r}\}$, $\{a^{2r}b, a^{2r}b^2\}$, $\{a^{2r+1}, a^{2r+1}b, a^{2r+1}b^2\}$ such that $1 \leq r \leq n-1$. Using [9] we display the character table of $U_{6n} = \langle a, b : a^{2n} = b^3 = 1, a^{-1}ba = b^{-1} \rangle$ as follows:

Table 1. The character table of U_{6n} ($\omega = e^{\frac{2\pi i}{2n}}$, $\frac{G}{\acute{G}} = \langle \acute{G}a \rangle \cong \mathbb{Z}_{2n}$)

centralizer order	6n	3n	2n
class representation	a^{2r}	$a^{2r}b$	a^{2r+1}
$\chi_j (0 \leq j \leq 2n-1)$	ω^{2jr}	ω^{2jr}	$\omega^{j(2r+1)}$
$\psi_k (0 \leq k \leq n-1)$	$2\omega^{2kr}$	$-\omega^{2kr}$	0

3. Special cases of normal supercharacter theories of U_{6n}

In this part, we consider the normal supercharacter theories of U_{6n} where $A(S) = \{1, G, N\}$ such that $1 \subsetneqq N \subsetneqq G$.
We consider three cases as follows:
Case 1: $N = \langle a^d \rangle$ **where** d **is an even divisor of** $2n$**.**
In this case

$$\{1\}^\circ = \{1\},$$

$$N^\circ = \{a^d, a^{2d}, \ldots, a^{kd}\} \quad (1 \leq k \leq \frac{2n}{d} - 1),$$

$$G^\circ = G \setminus \langle a^d \rangle^\circ.$$

Also, the supercharacters are as the following:

$$\chi^{G^\circ} = 1_G,$$
$$\chi^{\langle a^2\rangle^\circ} = \{\chi_{\frac{2n}{d}i} : 1 \le i \le d-1\} \cup \{\psi_{\frac{2n}{d}j} : 0 \le j < d/2-1\},$$
$$\chi^{1^\circ} = \mathrm{Irr}(G) \setminus \chi^{\langle a^d\rangle^\circ} \cup \{1_G\}.$$

Now by using Lemma 2.11 the values of supercharacters on superclasses are obtained as follows:
If $g = 1$, then

$$\chi^{G^\circ}(g) = 1,$$
$$\chi^{\langle a^d\rangle^\circ}(g) = \sum_{i=1}^{d-1} \chi^2_{\frac{2n}{d}i}(1) + \sum_{j=1}^{d/2-1} \psi^2_{\frac{2n}{d}j}(1) = d-1+4(d/2) = 3d-1,$$
$$\chi^{1^\circ}(g) = \sum_{f \in \mathrm{Irr(G)} \setminus \chi^{\langle a^d\rangle}} f^2(1) = 6n-3d.$$

If $g \in \langle a^d\rangle^\circ$, then

$$\chi^{\langle a^d\rangle^\circ}(g) = \chi^{\langle a^d\rangle^\circ}(1) = 3d-1,$$
$$\chi^{1^\circ} = -\chi^{\langle a^d\rangle^\circ}(g) - \chi^{G^\circ}(g) = -3d$$

Therefore, the normal supercharacter table of U_{6p} corresponding to $A(S)$, is the same as Table 2, where $N = \langle a^d\rangle$.

Table 2. Normal Supercharacter table where $N = \langle a^d\rangle$

	$\{1\}$	N°	G°
χ_0	1	1	1
$\sum_{1\le i\le d} \chi_{\frac{2n}{d}i} + 2\sum_{0\le j<d/2-1} \psi_{\frac{2n}{d}j}$	$3d-1$	$3d-1$	-1
χ^{1°	$6n-3d$	$-3d$	0

Case 2: $N = \langle a^d, b\rangle$ **where d is a divisor of** $2n$. In this case, we have $\chi^{\langle a^d,b\rangle^\circ} = \{\chi_{\frac{2n}{d}i} : 1 \le i \le d-1\}$ and the normal supercharacter table corresponding to $A(S)$, is as shown in the Table 3.

Table 3. Normal Supercharacter table if $N = \langle a^d, b\rangle$

	$\{1\}$	N°	G°
χ_0	1	1	1
$\sum_{1\le i\le d-1} \chi_{\frac{2n}{d}i}$	$d-1$	$d-1$	-1
χ^{1°	$6n-d$	$-d$	0

Case 3: $N = \langle b\rangle$**.** The normal supercharacter table corresponding to $A(S)$, is shown in Table 4.

TABLE 4. Normal Supercharacter table if $N = \langle b \rangle$

	$\{1\}$	N°	G°
χ_0	1	1	1
$\sum_{1 \leq i \leq 2n-1} \chi_i$	$2n-1$	$2n-1$	-1
$2\sum_{0 \leq i \leq n-1} \psi_i$	$4n$	$-2n$	0

We now classify the normal supercharacter theories of $U_{6pq} = G$, p and q are distinct primes, where $S = \{N = \langle a^d \rangle\}$ (d is an even divisor of $2pq$). In this case $A(S) = \{1, N, G\}$.

If $N = \langle a^2 \rangle$, then $\chi^{\langle a^2 \rangle^\circ} = \{\chi_{pq}, \psi_0\}$. If $N = \langle a^{2p} \rangle$, then $\chi^{\langle a^{2p} \rangle^\circ} = \{\chi_q, \chi_{2q}, \ldots, \chi_{(2p-1)q}\} \cup \{\psi_0, \psi_q, \ldots, \psi_{pq-q}\}$. Similarly, If $N = \langle a^{2q} \rangle$, then $\chi^{\langle a^{2q} \rangle^\circ} = \{\chi_p, \chi_{2p}, \ldots, \chi_{(2q-1)p}\} \cup \{\psi_0, \psi_p, \ldots, \psi_{pq-p}\}$. The normal supercharacter tables of U_{6p} corresponding to $A(S)$ where $N = \langle a^2 \rangle$ and $N = \langle a^{2p} \rangle$ are given in the following (Tables 5, 6). If $N = \langle a^{2q} \rangle$, then the normal supercharacter table corresponding to $A(S)$, is similar to the Table 6, with the difference that p is replaced by q.

TABLE 5. Normal Supercharacter table of U_{6pq} where $N = \langle a^2 \rangle$

	$\{1\}$	N°	G°
$\chi^{G^\circ} = \chi_0$	1	1	1
$\chi^{\langle a^2 \rangle^\circ} = \chi_{pq} + 2\psi_0$	5	5	-1
$\chi^{\langle 1 \rangle^\circ}$	$6pq - 6$	-6	0

TABLE 6. Normal Supercharacter table of U_{6pq} where $N = \langle a^{2p} \rangle$

	$\{1\}$	N°	G°
$\chi^{G^\circ} = \chi_0$	1	1	1
$\chi^{\langle a^{2p} \rangle^\circ} = \sum_{i=1}^{2p-1} \chi_{iq} + 2\sum_{j=0}^{p-1} \psi_{jq}$	$6p-1$	$6p-1$	-1
$\chi^{\langle 1 \rangle^\circ}$	$6pq - 6p$	$-6p$	0

4. **Automorphic supercharacter theories of U_{6p}**

In this part we let $n = p$ be an odd prime and then using the Brauer's theorem on character table to find some automorphic supercharacter theories of U_{6n}. Using [4] the automorphism group of U_{6n} is:
$K = \mathbb{A}ut(U_{6n}) = \{f_{k,l,r}(a) = a^k b^l, f_{k,l,r}(b) = b^r : (k, 2n) = 1, l = 0, \pm 1, r = \pm 1\} = A \times B$ where

$A = \{f_{k,0,1} : (k, 2n) = 1, 1 \leq k \leq 2n\}$ and $B = \{f_{1,l,r} : l = 0, \pm 1, r = \pm 1\}$.

Proposition 4.1. *The group U_{6p} has six K-invariant supercharacters and superclasses.*

Proof. It is enough to find orbits of K on $\mathrm{Con}(U_{6p})$ and $\mathrm{Irr}(U_{6p})$. By inspecting table (1) and considering elements of K we obtain the following orbits of K acting on $\mathrm{Con}(U_{6p})$:

(1) $K_1 = \{1\}$,
(2) $K_2 = \{a^p\}$,
(3) $K_3 = \mathrm{Class}(b)$,
(4) $K_4 = \mathrm{Class}(a) \cup \mathrm{Class}(a^3) \cup \cdots \cup \mathrm{Class}(a^{2p-1})$,
(5) $K_5 = \mathrm{Class}(a^2) \cup \mathrm{Class}(a^4) \cup \cdots \cup \mathrm{class}(a^{2p-2})$,
(6) $K_6 = \mathrm{Class}(a^2b) \cup \mathrm{Class}(a^4b) \cup \cdots \cup \mathrm{Class}(a^{2p-2}b)\}$.

The orbits of the action K on $\mathrm{Irr}(U_{6p})$ are:

(1) $X_1 = \{\chi_0\}$,
(2) $X_2 = \{\chi_p\}$,
(3) $X_3 = \{\psi_{2j} : 0 \leq j \leq \frac{p-1}{2}\}$,
(4) $X_4 = \{\chi_i : i \text{ is even}, 2 \leq i \leq 2p-2\}$,
(5) $X_5 = \{\psi_j : 1 \leq j \leq p-1 : j \text{ is odd}\}$,
(6) $X_6 = \{\chi_i : i \text{ is odd}, i \neq p\}$.

So the supercharacters are:

(1) $\sigma_1 = \chi_0$,
(2) $\sigma_2 = \chi_p$,
(3) $\sigma_3 = 2\sum_{j=0}^{j=\frac{p-1}{2}} \psi_{2j}$,
(4) $\sigma_4 = \sum_{1 \leq i \leq p-1} \chi_{2i}$,
(5) $\sigma_5 = 2\sum_{1 \leq j \leq \frac{p-1}{2}} \psi_{2j-1}$,
(6) $\sigma_6 = \sum_{\substack{i \text{ is odd} \\ i \neq p}} \chi_i$

□

Now we consider the automorphic supercharacter theories which are produced by subgroups A and B of K.

Proposition 4.2. *The group U_{6p} has six A-invariant supercharacters and superclasses.*

Proof. By inspecting Table (1) and considering the elements of A, we get the following orbits of A acting on $\mathrm{Con}(U_{6p})$:

(1) $K_1 = \{1\}$,
(2) $K_2 = \{a^p\}$,
(3) $K_3 = \mathrm{Class}(b)$,

(4) $K_4 = \text{Class}(a) \cup \text{Class}(a^3) \cup \cdots \cup \text{Class}(a^{p-2}) \cup \cdots \cup \text{Class}(a^{p+2}) \cup \text{Class}(a^{2p-1})$,
(5) $K_5 = \text{Class}(a^2) \cup \text{Class}(a^4) \cup \cdots \cup \text{Class}(a^{2p-2})$,
(6) $K_6 = \text{Class}(a^2b) \cup \text{Class}(a^4b) \cup \cdots \cup \text{Class}(a^{2p-2}b)$.

The orbits of A acting on $\text{Irr}(U_{6p})$ are:

(1) $X_1 = \{\chi_0\}$,
(2) $X_2 = \{\chi_p\}$,
(3) $X_3 = \{\chi_{2i-1} : 1 \leq i \leq p\}$,
(4) $X_4 = \{\chi_{2i} : 1 \leq i \leq p-1\}$,
(5) $X_5 = \{\psi_{2j} : 0 \leq j \leq \frac{p-1}{2}\}$,
(6) $X_6 = \{\psi_{2i-1} : 1 \leq i \leq \frac{p-1}{2}\}$.

So we have the following supercharacters for this supercharacter theory:

(1) $\sigma_1 = \chi_0$,
(2) $\sigma_2 = \chi_p$,
(3) $\sigma_3 = \sum_{i=1}^{p} \chi_{2i-1}$,
(4) $\sigma_4 = \sum_{1\leq i\leq p-1} \chi_{2i}$,
(5) $\sigma_5 = 2\sum_{j=0}^{\frac{p-1}{2}} \psi_{2j}$,
(6) $\sigma_6 = 2\sum_{i=1}^{\frac{p-1}{2}} \psi_{2i-1}$

□

Proposition 4.3. *The group U_{6p} has $3p$ B-invariant superclasses and supercharacers.*

Proof. We know that for $f_{1,l,r} \in B$, $f_{1,l,r}(a) \in \text{Class}(a)$ and $f_{1,l,r}(b) \in \text{Class}(b)$. Therefore, b is fixed by B.
The orbits of B acting on $\text{Con}(U_{6p})$ are:
$\{1\}$, $\{a\}$, $\{a^3\}, \ldots, \{a^{2p-1}\}$, $\{a^2\}$, $\{_{a^4}\}, \ldots, \{a^{2p-2}\}$, $\{a^2b\}$, $\{a^4b\}, \ldots,\{a^{2p-2}b\}$, $\{b\}$.
The constituents of corresponding supercharacters are:
$\{\chi_0\}$, $\{\chi_1\}$, $\ldots$, $\{\chi_{2p-1}\}$, $\{\psi_0\}$, $\ldots$, $\{\psi_{p-1}\}$.
As we see the action of B on U_{6p} generates the minimal supercharacter theory.

□

Notation: We say that a supercharater theory preserves parity, if every its superclass which contains odd powers of a, does not contain $\text{Class}(b), \text{Class}(a^{2k}b)$ or $\text{Class}(a^{2k})$ where $1 \leq k \leq p-1$.

Proposition 4.4. *If S be is automorphic supercharacter theory of U_{6p}, then S preserves parity and also $\langle a^2, b\rangle$ and $\langle b\rangle$ are S-normal subgroups of U_{6p}.*

Proof. Every $f \in \mathbb{A}ut(U_{6p})$ maps a to $a^l b^{\pm 1}$ where l and $2p$ are coprime. Consequently, l is odd and $l \neq p$. This map permutes odd powers of a between themselves. Also b is sent to Class(b) under these automorphisms. Incidentally, if S is an arbitrary automorphic supercharacter theory, then $\langle a^2, b\rangle$ and $\langle b\rangle$ are S-normal. □

Theorem 4.5. *Automorphic supercharacter theories of U_{6p} can be obtained by Δ-product of $\langle a^2, b\rangle$ and $\langle b\rangle$.*

Proof. Since ψ_j's $(0 \leq j \leq p-1)$ are all irreducible characters of G which have value 0 on odd powers of a every subgroup of $\mathbb{A}ut(U_{6p})$ permutes ψ_j's among themselves. In addition, every $\psi_j(0 \leq j \leq p-1)$ does not contain $\langle b\rangle$ in its kernel, and also $\psi_j(g) = 0$ for every $g \in G \setminus \langle a^2, b\rangle$. So every supercharacter of an automorphic supercharacter theory satisfies the conditions of Theorem 2.6. Consequently, the set of automorphic supercharacter theories of U_{6p} is a subset of supercharacter theories obtained by the Δ-product of $\langle a^2, b\rangle$ and $\langle b\rangle$. □

Since in every automorphic supercharacter theory of $G = U_{6n}$, the superclass containing b is a singleton, by Lemma 2.3, every automorphic supercharacter theory of G can be obtained by $*$-product.

5. Conclusion

In this paper, we have found three automorphic supercharacter theories of U_{6n}. Then we have shown that the automorphic supercharacter theories of U_{6n} can be generated by Δ-product or $*$-product. Furthermore, motivated by the work of Aliniaei Fard [1] we have studied some normal supercharacter theories of U_{6n}. The structure of these supercharacter theories in all instances, indicates that they can be constructed by $*$-product. As future research, we may explicate the structure of the remaining supercharacter theories of U_{6n} and classify all the supercharacter theories associated with this group.

6. Aknowledgement

We would like to thank the reviewers for their thoughtful comments and efforts towards improving our manuscript.

References

[1] Aliniaeifard, F. (2017). Normal supercharacter theories and their supercharacters. Journal of Algebra, 469, 464-484. https://doi.org/10.1016/j.jalgebra.2016.09.005

[2] André, C. (1995). Basic characters of the unitriangular group. Journal of algebra, 175(1), 287-319. https://doi.org/10.1006/jabr.1995.1187

[3] Burkett, S. T., & Lewis, M. L. (2020). Vanishing-off subgroups and supercharacter theory products. International Journal of Algebra and Computation, 30(5), 1057-1072. https://doi.org/10.1142/S0218196720500307

[4] Darafsheh, M. R., & Yaghoobian, M. (2017). Tetravalent normal edge-transitive cayley graphs on a certain group of order 6n. Turkish Journal of Mathematics, 41(5), 1354-1359. https://doi.org/10.3906/mat-1504-39

[5] Diaconis, P., & Isaacs, I. (2008). Supercharacters and superclasses for algebra groups. Transactions of the American Mathematical Society, 360(5), 2359-2392. https://doi.org/10.1090/S0002-9947-07-04365-6
[6] Dornhoff, L. (1971). Group representation theory, part A. Marcel Dekker.
[7] Hendrickson, A. O. (2008). Supercharacter theories of finite cyclic groups [Unpublished PhD. thesis]. Wisconsin University.
[8] Hendrickson, A. O. (2012). Supercharacter theory constructions corresponding to schur ring products. Communications in Algebra, 40(12), 4420-4438. https://doi.org/10.1080/00927872.2011.602999
[9] James, G., & Liebeck, M. (2001). Representations and characters of groups (2nd ed.). Cambridge University Press.
[10] Shelash, H. B., & Ashrafi, A. R. (2021). The number of subgroups of a given type in certain finite groups. Iranian Journal of Mathematical Sciences and Informatics, 16(2), 73-87. https://doi.org/10.52547/ijmsi.16.2.73
[11] Yan, N. (2001). Representation theory of the finite unipotent linear [Unpublished PhD. thesis]. Pennsylvania University.

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