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A GENERALIZED NOTION OF ORTHOGONALITY PRESERVING MAPPINGS ON INNER PRODUCT MODULES

A.R. Khoddami

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Abstract. In this paper, we define a new concept called “strongly orthogonality preserving mappings ” for inner product modules, which extends the existing notion of “orthogonality preserving mappings”. Also, we provide a condition that is both necessary and sufficient for a linear map between inner product modules to be strongly orthogonality preserving. Some examples related to the definition are given.

Keywords: Strongly orthogonality preserving map, Inner product module, C^*-algebra.
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1. Introduction

Recently, there has been research on mappings that preserve orthogonality in Hilbert C^*-modules [1, 2, 4, 5]. We present some terms that will be used to describe our findings. Let $\mathscr{E}$ be a left $\mathscr{C}-$module, where $\mathscr{C}$ is a C^*-algebra. It is important that the linear structures on both $\mathscr{C}$ and $\mathscr{E}$ are compatible, i.e., $\lambda(ay) = a(\lambda y)$ for every $\lambda \in \mathbb{C}, a \in \mathscr{C}$ and $y \in \mathscr{E}$. If there exists a mapping $\langle\cdot,\cdot\rangle : \mathscr{E} \times \mathscr{E} \to \mathscr{C}$ with the following properties

(1) $\langle y, y\rangle \geqslant 0$ for all $y \in \mathscr{E}$,
(2) $\langle y, y\rangle = 0$ if and only if $y = 0$,
(3) $\langle y, z\rangle = \langle z, y\rangle^*$ for all $y, z \in \mathscr{E}$,
(4) $\langle ay, z\rangle = a\langle y, z\rangle$ for all $a \in \mathscr{C}$, and $y, z \in \mathscr{E}$,
(5) $\langle \alpha y + \beta z, w\rangle = \alpha\langle y, w\rangle + \beta\langle z, w\rangle$ for every $y, z, w \in \mathscr{E}$ and $\alpha, \beta \in \mathbb{C}$,

then the pair $(\mathscr{E}, \langle\cdot,\cdot\rangle)$ is called a left pre-Hilbert $\mathscr{C}-$module. The map $\langle\cdot,\cdot\rangle$ is said to be a $\mathscr{C}-$valued inner product. If the pre-Hilbert $\mathscr{C}-$module $(\mathscr{E}, \langle\cdot,\cdot\rangle)$ is complete with respect to the norm $\|y\| = \|\langle y, y\rangle\|^{\frac{1}{2}}$, then it is called a Hilbert $\mathscr{C}-$module. It is well-known that the C^*-algebra $\mathscr{C}$ can be reorganized to become a Hilbert $\mathscr{C}-$module, if we define the inner product $\langle b, c\rangle = bc^*, b, c \in \mathscr{C}$. The corresponding norm is equivalent to the norm on $\mathscr{C}$ because,

$$\|b\| = \|\langle b, b\rangle\|_{\mathscr{C}}^{\frac{1}{2}} = \|bb^*\|_{\mathscr{C}}^{\frac{1}{2}} = (\|b\|_{\mathscr{C}}^2)^{\frac{1}{2}} = \|b\|_{\mathscr{C}}.$$

✉ khoddami.alireza@shahroodut.ac.ir, ORCID: 0000-0002-9428-8185
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If $\mathscr{E}$ and $\mathscr{F}$ are two inner product left $\mathscr{C}-$modules, a linear mapping $S:\mathscr{E}\to\mathscr{F}$ is called orthogonality preserving (OP) if, $\langle Sx, Sy\rangle = 0$ whenever $\langle x, y\rangle = 0$, for x and y in $\mathscr{E}$. It is important to note that an orthogonality preserving map (OPM) may not be continuous in general [5].
This article introduces and explores a generalized concept of orthogonality preserving mappings on inner product modules, inspired by the notion of strongly zero-product preserving maps on normed algebras [3]. We show that this new concept is different from the traditional notion of orthogonality preserving mappings. Furthermore, the article provides a necessary and sufficient condition for a linear map between inner product modules to be considered strongly orthogonality preserving (SOP). Finally, the article includes several examples to illustrate these concepts.

2. Main results

In this section, we generalize the notion of orthogonality preserving mappings. Some basic properties concerning the concept of strongly orthogonality preserving mappings are presented.

Definition 2.1. Let $\mathscr{C}$ be a C^*-algebra and $\mathscr{E}$, $\mathscr{F}$ be two left pre-Hilbert $\mathscr{C}-$modules. A linear mapping $S:\mathscr{E}\to\mathscr{F}$ is called SOP if for any two sequences $\{y_n\}_n, \{z_n\}_n$ in $\mathscr{E}$, $\langle Sy_n, Sz_n\rangle \longrightarrow 0$, as $n\longrightarrow\infty$ whenever $\langle y_n, z_n\rangle\longrightarrow 0$, as $n\longrightarrow\infty$. We use SOP for " strongly orthogonality preserving" and SOPM for " strongly orthogonality preserving map".

Example 2.2. *Let X be a locally compact Hausdorff space and $\mathscr{C}=\mathscr{E}=\mathscr{F}=C_0(X)$ be the C^*-algebra of all continuous complex valued functions that vanish at infinity on X. Define $S: C_0(X)\longrightarrow C_0(X)$ by $S(f)=fg$, where g is a non-zero function in $C_0(X)$. If the inner product on $C_0(X)$ is defined by $\langle h_1, h_2\rangle = h_1\overline{h_2}$, then one can easily check that S is SOP.*

Theorem 2.3. *Let $\mathscr{C}$ be a C^*-algebra and $\mathscr{E}$ and $\mathscr{F}$ be two left pre-Hilbert $\mathscr{C}-$modules. A linear mapping $S:\mathscr{E}\to\mathscr{F}$ is SOP if and only if there exists an $M>0$ such that*

$$\|\langle S(y), S(z)\rangle\| \leq M\|\langle y, z\rangle\|, \quad \forall y, z\in\mathscr{E}.$$

Proof. Let S be SOP. To obtain a contradiction, suppose there is no such M. Then for each $n\in\mathbb{N}$ there exist $y_n, z_n\in\mathscr{E}$ such that,

$$\|\langle S(y_n), S(z_n)\rangle\| > n\|\langle y_n, z_n\rangle\|.$$

So,

$$\left\|\langle \frac{y_n}{\|\langle S(y_n), S(z_n)\rangle\|}, z_n\rangle\right\| < \frac{1}{n}.$$

Let $y'_n = \frac{y_n}{\|\langle S(y_n), S(z_n)\rangle\|}$ and $z'_n = z_n$. Clearly, $\langle y'_n, z'_n\rangle\longrightarrow 0$, as $n\longrightarrow\infty$. So by supposition, we get $\langle S(y'_n), S(z'_n)\rangle\longrightarrow 0$, as $n\longrightarrow\infty$. That is a contradiction.

Indeed,

$$\begin{aligned}\|\langle S(y'_n), S(z'_n)\rangle\| &= \frac{\|\langle S(y_n), S(z_n)\rangle\|}{\|\langle S(y_n), S(z_n)\rangle\|} \\ &\longrightarrow 1,\end{aligned}$$

as $n \longrightarrow \infty$. The converse is trivial. □

Corollary 2.4. *Let $\mathscr{C}$ be a C^*-algebra and $\mathscr{E}$ and $\mathscr{F}$ be two left pre-Hilbert $\mathscr{C}-$modules and $S : \mathscr{E} \to \mathscr{F}$ be an SOPM. Then S is continuous. Moreover,*

$$\|S\| \leq \inf\left\{M^{\frac{1}{2}} \quad \Big| \quad \|\langle S(y), S(z)\rangle\| \leq M\|\langle y, z\rangle\|, \quad \forall y, z \in \mathscr{E}\right\}.$$

Proof. By Theorem 2.3, there exists an $M > 0$ such that,

$$\|\langle S(y), S(z)\rangle\| \leq M\|\langle y, z\rangle\|, \quad \forall y, z \in \mathscr{E}. \tag{1}$$

Upon substituting $z = y$ in (1), we conclude that

$$\|S(y)\|^2 \leq M\|y\|^2, \quad \forall y \in \mathscr{E}.$$

It follows that,

$$\|S(y)\| \leq M^{\frac{1}{2}}\|y\|, \quad \forall y \in \mathscr{E}.$$

Hence S is continuous and

$$\|S\| \leq \inf\left\{M^{\frac{1}{2}} \quad \Big| \quad \|\langle S(y), S(z)\rangle\| \leq M\|\langle y, z\rangle\|, \quad \forall y, z \in \mathscr{E}\right\}.$$

□

Remark 2.5. The converse of Corollary 2.4 is not the case in general. Indeed, let $\mathscr{C} = \mathscr{E} = \mathscr{F} = C([0, 1])$. Define $S : \mathscr{E} \to \mathscr{F}$ by $S(f) = \int_0^1 f(x)dx$. Clearly, S is a bounded linear map. We shall show that S is not an OPM. Let

$$f(x) = \begin{cases} 0 & 0 \leq x \leq \frac{1}{2} \\ x - \frac{1}{2} & \frac{1}{2} \lneq x \leq 1 \end{cases}$$

and

$$g(x) = \begin{cases} \frac{1}{2} - x & 0 \leq x \leq \frac{1}{2} \\ 0 & \frac{1}{2} \lneq x \leq 1. \end{cases}$$

It is obvious that $f, g \in C([0, 1])$. Also, $\langle f, g\rangle = f\overline{g} = fg = 0$. But

$$\begin{aligned}\langle S(f), S(g)\rangle =& \langle \int_0^1 f(x)dx, \int_0^1 g(x)dx\rangle \\ =& \langle \frac{1}{8}, \frac{1}{8}\rangle \\ =& \frac{1}{64} \neq 0.\end{aligned}$$

So S is not an OPM. Hence S is a bounded linear map that is not SOP.

Corollary 2.6. *Let $\mathscr{C}$ be a C^*-algebra and $\mathscr{E}$ and $\mathscr{F}$ be two left pre-Hilbert $\mathscr{C}-$modules. Also, let $S : \mathscr{E} \to \mathscr{F}$ be a bijective linear map such that S and S^{-1} are SOP. Then there exist $\alpha, \beta \in (0, \infty)$ such that for all $y, z \in \mathscr{E}$,*

$$\alpha\|\langle y, z\rangle\| \leq \|\langle S(y), S(z)\rangle\| \leq \beta\|\langle y, z\rangle\|.$$

Moreover,

$$\alpha^{\frac{1}{2}}\|y\| \leq \|S(y)\| \leq \beta^{\frac{1}{2}}\|y\|, \quad \forall y \in \mathscr{E},$$

and

$$\langle y, z\rangle = 0 \Longleftrightarrow \langle S(y), S(z)\rangle = 0, \quad y, z \in \mathscr{E}.$$

Proof. As S and S^{-1} are SOP, there exist $\beta, \alpha > 0$ such that

$$\|\langle S(y), S(z)\rangle\| \leq \beta\|\langle y, z\rangle\|, \quad \forall y, z \in \mathscr{E}, \tag{2}$$

and

$$\|\langle S^{-1}(v), S^{-1}(w)\rangle\| \leq \frac{1}{\alpha}\|\langle v, w\rangle\|, \quad \forall v, w \in \mathscr{F}. \tag{3}$$

Upon substituting $v = S(y)$ and $w = S(z)$ in (3), we can conclude that,

$$\|\langle y, z\rangle\| \leq \frac{1}{\alpha}\|\langle S(y), S(z)\rangle\|, \quad \forall y, z \in \mathscr{E}. \tag{4}$$

By (2) and (4), we have

$$\alpha\|\langle y, z\rangle\| \leq \|\langle S(y), S(z)\rangle\| \leq \beta\|\langle y, z\rangle\|, \quad \forall y, z \in \mathscr{E}. \tag{5}$$

Letting $z = y$ in (5), we can conclude that,

$$\alpha^{\frac{1}{2}}\|y\| \leq \|S(y)\| \leq \beta^{\frac{1}{2}}\|y\|, \quad \forall y \in \mathscr{E}.$$

Finally (5) implies,

$$\langle y, z\rangle = 0 \Longleftrightarrow \langle S(y), S(z)\rangle = 0, \quad y, z \in \mathscr{E}.$$

□

It is clear that all SOPMs are OPs. Because if $\langle x, y\rangle = 0$, one can simply select $x_n = x$ and $y_n = y$. Since $\langle x_n, y_n\rangle \longrightarrow 0$, as $n \longrightarrow \infty$, it follows that $\langle S(x), S(y)\rangle = \langle S(x_n), S(y_n)\rangle \longrightarrow 0$, as $n \longrightarrow \infty$. The following example shows that the converse is not the case in general.

Recall that if x and y are elements of a Hilbert space H, then the operator $x \otimes y : H \longrightarrow H$ is defined by $(x \otimes y)(z) = \langle z, y\rangle x$, for all $z \in H$. Clearly, $\|x \otimes y\| = \|x\|\|y\|$. It is well-known that the operator $x \otimes x$ is a rank-one projection if and only if $\langle x, x\rangle = 1$. Also, every rank-one projection is of the form $x \otimes x$ for some unit vector $x \in H$.

We denote by $F(H)$ the set of finite-rank operators on H. Also, we denote by $K(H)$ the set of compact operators on H. By [6, Theorem 2.4.5], $F(H)$ is dense in $K(H)$. Also, by [6, Theorem 2.4.6], $F(H)$ is linearly spanned by the rank-one projections.

Example 2.7. *Let* $(e_n)_{n=1}^{\infty}$ *be an orthonormal basis for a separable Hilbert space* H *and let* $\mathscr{E} = F(H)$ *and* $\mathscr{C} = \mathscr{F} = K(H)$*. Define* $S : \mathscr{E} \longrightarrow \mathscr{F}$ *by*

$$S(\sum_{k=1}^{n} \lambda_k e_k \otimes e_k) = \sum_{k=1}^{n} 2^k \lambda_k e_k \otimes e_k.$$

We shall show that S *is an OPM. Indeed, let*

$$\langle \sum_{i=1}^{n} \lambda_i e_i \otimes e_i, \sum_{j=1}^{n} \mu_j e_j \otimes e_j \rangle = 0.$$

It follows that,

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_i \bar{\mu_j} (e_i \otimes e_i)(e_j \otimes e_j) &= \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_i \bar{\mu_j} \langle e_j, e_i \rangle e_i \otimes e_j \\ &= \sum_{i=1}^{n} \lambda_i \bar{\mu_i} e_i \otimes e_i = 0.\end{aligned}$$

So,

$$\begin{aligned}0 &= \sum_{i=1}^{n} \lambda_i \bar{\mu_i} (e_i \otimes e_i)(e_j) \\ &= \sum_{i=1}^{n} \lambda_i \bar{\mu_i} \langle e_j, e_i \rangle e_i = \lambda_j \bar{\mu_j} e_j, \quad 1 \leq j \leq n.\end{aligned}$$

It follows that

$$\lambda_j \bar{\mu_j} = 0, \quad 0 \leq j \leq n.$$

Hence,

$$\begin{aligned}\langle S(\sum_{i=1}^{n} \lambda_i e_i \otimes e_i), S(\sum_{j=1}^{n} \mu_j e_j \otimes e_j) \rangle &= \langle \sum_{i=1}^{n} 2^i \lambda_i e_i \otimes e_i, \sum_{j=1}^{n} 2^j \mu_j e_j \otimes e_j \rangle \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} 2^i 2^j \lambda_i \bar{\mu_j} \langle e_j, e_i \rangle e_i \otimes e_j \\ &= \sum_{i=1}^{n} 4^i \lambda_i \bar{\mu_i} e_i \otimes e_i \\ &= 0.\end{aligned}$$

This shows that S *is an OPM. Clearly,* S *is not continuous. Indeed,*

$$\begin{aligned}\|S(e_n \otimes e_n)\| &= \|2^n e_n \otimes e_n\| = 2^n \|e_n \otimes e_n\| \\ &= 2^n \|e_n\| \|e_n\| \\ &= 2^n.\end{aligned}$$

So by Corollary 2.4, S *is not SOP.*

Proposition 2.8. *Let $\mathscr{C}$ be a C^*-algebra and $\mathscr{E}, \mathscr{F}$, and $\mathscr{H}$ be left pre-Hilbert $\mathscr{C}-$modules. Also, let $S : \mathscr{E} \longrightarrow \mathscr{F}$ and $T : \mathscr{F} \longrightarrow \mathscr{H}$ be SOP. Then $T \circ S : \mathscr{E} \longrightarrow \mathscr{H}$ is SOP.*

Proof. As S and T are SOP, there exist $M, N > 0$ such that,

$$\|\langle S(a), S(c)\rangle\| \leq M\|\langle a, c\rangle\|, \quad \forall a, c \in \mathscr{E},$$

and

$$\|\langle T(b), T(d)\rangle\| \leq N\|\langle b, d\rangle\|, \quad \forall b, d \in \mathscr{F}.$$

So,

$$\begin{aligned} \|\langle T \circ S(a), T \circ S(c)\rangle\| &= \|\langle T(S(a)), T(S(c))\rangle\| \\ &\leq N\|\langle S(a), S(c)\rangle\| \\ &\leq MN\|\langle a, c\rangle\|, \quad \forall a, c \in \mathscr{E}. \end{aligned}$$

Hence by Theorem 2.3, $T \circ S$ is SOP. □

Remark 2.9. It is important to note that the direct product of two strongly orthogonality preserving maps is not necessarily strongly orthogonality preserving. Indeed, let H be a Hilbert space and let $S : H \longrightarrow H$ be the identity map and $T : H \longrightarrow H$ be the zero function. Obviously, S and T are SOP. But $S \oplus T : H \oplus H \longrightarrow H \oplus H$ is not SOP. Indeed, let $e_1 \in H$ be an element such that $\langle e_1, e_1\rangle = 1$. So,

$$\langle (e_1, e_1), (e_1, -e_1)\rangle = 0.$$

But,

$$\begin{aligned} \langle S \oplus T((e_1, e_1)), S \oplus T((e_1, -e_1))\rangle &= \langle (S(e_1), T(e_1)), (S(e_1), T(-e_1))\rangle \\ &= \langle S(e_1), S(e_1)\rangle \\ &= \langle e_1, e_1\rangle = 1 \neq 0. \end{aligned}$$

This shows that $S \oplus T$ is not OP. So $S \oplus T$ is not SOP.

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Ali Reza Khoddami
Orcid number: 0000-0002-9428-8185
Faculty of Mathematical Sciences
Shahrood University of Technology
P. O. Box 3619995161-316, Shahrood, Iran
Email address: khoddami.alireza@shahroodut.ac.ir