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ON RM-ALGEBRAS WITH AN ADDITIONAL CONDITION

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Abstract. In this paper, we apply a new condition to RM-algebras. We obtain some relations among this condition with other axioms in some algebras of logic and some examples are given to illustrate them. We prove that the relation derived from this new algebra is a partial ordering. It is proved that RM-algebras with condition (I) are abelian group. Also, we present that the BI-algebras, BCK-algebras, L-algebras, KL-algebras, CL-algebras and BE-algebras satisfying (I) are trivial.

Keywords: Groupoid, RM-algebra, BE-algebra, BI-algebra, L-algebra.
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1. Introduction

In 1966, Imai and Iseki ([5,7]) introduced BCK and BCI algebras as algebras connected with some logics (also see, [6]). In 1997, Abbott ([1]) introduced the concept of implication algebra in the sake to formalize the logical connective implication in the classical propositional logic. In 2006, Chajda ([3]) generalized implication algebra to the pre-implication algebra and implication algebra based on orthosemilattices. In 2007, Kim and Kim ([10]) defined BE-algebras, which are a generalization of dual BCK-algebras. After, Rump ([13]) to a solution of the quantum Yang-Baxter equation defined L-algebras characterized the L-algebras with a natural embedding into the negative cone of an ℓ-group. In 2010, Meng ([11]) introduced the notions of CI-algebras as a generalization of BE-algebras and investigated its properties. In 2017, Borumand Saeid et al. ([2]) introduced *BI-algebras* as an extension of both a (dual) implication algebras and an implicative BCK-algebra, and they investigated some ideals and congruence relations. They showed that every implicative BCK-algebra is a BI-algebra but the converse is not true in general. All of the algebras mentioned above are contained in the class of RM-algebras were investigated some their properties by Walendziak (see [17]- [20]).

In this paper, we apply the identity $x \to (y \to z) = (z \to x) \to y$ to RM-algebras $(X, \to, 1)$ and investigate some relations between this condition with other axioms in some algebras of logic and some examples are given to illustrate

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them. We prove that the new algebra $(X, \leq)$ is a partially ordered set. The relation between RM-algebras and abelian groups are given.

2. Preliminaries

We recall the basic definitions and some elementary aspects that are necessary for this paper.

Recall that a *groupoid* (or, *magma*) $(X, \to)$ is a nonempty set X with a single binary operation "$\to$".

An groupoid $(X, \to)$ is called a *right-zero* (resp. *left-zero*) *semigroup* if $x \to y = y$ (resp. $x \to y = x$), for any $x, y \in X$ ([9]).

A RM-*algebra* ([17]) is an algebra $(X, \to, 1)$ of type $(2, 0)$ satisfying the following axioms: for all $x, y \in X$

(R) $x \to x = 1$,
(M) $1 \to x = x$.

Notice that some researchers named (R) as (B) and the notion of BI-algebra in this paper is indeed a dual form of the original definition in [2].

A BI-*algebra* ([2]) is an algebra $(X, \to, 1)$ of type $(2, 0)$ if satisfies (B) (=(R)) and the following axiom: for all $x, y \in X$

(BI) $(x \to y) \to x = x$.

A *implication algebra* is a groupoid $(X, \to)$ if it satisfies (R), (BI) and the following axioms: for all $x, y, z \in X$

(C) $(x \to y) \to y = (y \to x) \to x$,
(E) $x \to (y \to z) = y \to (x \to z)$.

An RM-algebra $(X, \to, 1)$ is called *commutative* if it satisfies (C) ([18]).

An RM-algebra $(X, \to, 1)$ is called *implicative* if it satisfies (BI) ([16]).

A L-*algebra* ([13]) is an algebra $(X, \to, 1)$ of type $(2, 0)$ if it satisfies (R), (M) and the following axioms: for all $x, y, z \in X$

(D) $x \to 1 = 1$,
(L) $(x \to y) \to (x \to z) = (y \to x) \to (y \to z)$,
(An) $x \to y = y \to x = 1 \implies x = y$.

In a groupoid $(X, \to)$, an element $1 \in X$ is a *logical unit* ([13]) if it satisfies (R), (M) and (D).

If an L-algebra $(X, \to, 1)$ satisfies

(K) $x \to (y \to x) = 1$,

then it is called a KL-*algebra* ([4]). A CL-*algebra* ([4]) is an L-algebra $(X, \to, 1)$ satisfying

(WE) $(x \to (y \to z)) \to (y \to (x \to z)) = 1$.

A CI-*algebra* ([11]) is an algebra $(X, \to, 1)$ of type $(2, 0)$ if it satisfies (R), (M) and (E). If a CI-algebra $(X, \to, 1)$ satisfies (D), then it is called a BE-*algebra* ([10]).

Throughout this paper, let X be a nonempty set and we define a relation "$\leq$" on the algebra $(X, \to, 1)$ by

$$x \leq y \iff x \to y = 1.$$

Notice that in every RM-algebra $(X, \to, 1)$ the relation $\leq$ is only reflexive.

3. Some properties of RM-algebras

This section is a continuation of the [2], [8] and [17], where the property of niceness of groupoids, BI-algebras and RM-algebras were studied and some properties of them are investigated.

Notice that, let "$\to$" be a binary operation on a set X, the axiom

(I) $x \to (y \to z) = (z \to x) \to y$

in this paper is indeed a dual form $(xy)z = y(zx)$ of the original definition in [8]. It is obvious that if the binary operation $\to$ is both abelian (i.e., $x \to y = y \to x$) and associative (i.e., $(x \to y) \to z = x \to (y \to z)$), then (I) holds. If we take $z := x$ and $x := y$ in (I), then $y \to (y \to x) = (x \to y) \to y$. If we take $y = z := x$, then $x \to (x \to x) = (x \to x) \to x$. It is known that in some algebras of logic (e.g. implication algebra ([1,3]), L-algebra ([13]), etc.) a relation "$\leq$" can be introduced by $x \leq y$ if and only if $x \cdot y = 1$ where " $\cdot$ " is a binary operation on the universe X and 1 is an algebraic constant of X given by the derived identity $x \to x = y \to y$, for all $x, y \in X$. Hence in these algebras with (I) we get $x \to 1 = 1 \to x$, for all $x \in X$.

We see that if a right-zero semigroup (resp. left-zero semigroup) $(X, \to)$ satisfies (I), then $|X| = 1$, since

$$z = x \to z = x \to (y \to z) = (z \to x) \to y = x \to y = y,$$

for all $x, y, z \in X$. The proof is similar for the left-zero semigroup. There are only two RM-algebras $(X, \to_1, 1)$ and $(X, \to_2, 1)$ of order 2 with Cayley Tables 1 and 2. We can see that $(X, \to_2, 1)$ satisfies (I), but $(X, \to_1, 1)$ does not verify

Table 1. RM-algebra $(X, \to_1, 1)$

$\to_1$	x	1
x	1	1
1	x	1

Table 2. RM-algebra $(X, \to_2, 1)$

$\to_2$	x	1
x	1	x
1	x	1

(I), since

$$1 \rightarrow_1 (1 \rightarrow_1 x) = 1 \rightarrow_1 x = x \neq 1 = 1 \rightarrow_1 1 = (x \rightarrow_1 1) \rightarrow_1 1.$$

From now on, denote RM-algebra $(X, \rightarrow, 1)$ by X, unless otherwise stated. It is evident that X is a BI-algebra if it satisfies (BI) and a CI-algebra if it satisfies (E).

Example 3.1. *(i) Every abelian monoid (resp. group) $(X, \rightarrow, 1)$ satisfies (I).*

(ii) Consider the set $X = \{x, 1\}$ with Cayley Table 3:

Table 3. Groupoid $(X, \rightarrow_3)$

$\rightarrow_3$	x	1
x	x	1
1	1	1

Then $(X, \rightarrow_3)$ is an abelian groupoid satisfying (I), but it does not satisfy (R), since $x \rightarrow_3 x = x \neq 1 = 1 \rightarrow_3 1$. Hence $(X, \rightarrow_3, 1)$ and $(X, \rightarrow_3, x)$ are not RM-algebras.

(iii) Consider the set $X = \{x, y, 1\}$ with Cayley Table 4:

Table 4. RM-algebra $(X, \rightarrow_4, 1)$

$\rightarrow_4$	x	y	1
x	1	y	x
y	y	1	y
1	x	y	1

Then $(X, \rightarrow_4, 1)$ satisfies (I).

Proposition 3.2. *Let X satisfy* (I). *Then the following hold: for all $x, y \in X$*

(i) $x \rightarrow 1 = x$, (i.e., 1 is not a logical unit),
(ii) $x \rightarrow y = y \rightarrow x$, (i.e., $\rightarrow$ is abelian),
(iii) $(x \rightarrow y) \rightarrow x = y = x \rightarrow (y \rightarrow x)$,
(iv) $(y \rightarrow x) \rightarrow x = x \rightarrow (x \rightarrow y) = y$,
(v) $y \rightarrow (y \rightarrow (y \rightarrow x)) = x \rightarrow y$,
(vi) if $x \leq y$, then $y \rightarrow (y \rightarrow (y \rightarrow x)) = 1$ and $y \rightarrow ((x \rightarrow y) \rightarrow y) = 1$.

Proof. (i) Taking $x = y = z$ in (I) and by (M) and (R), we have

$$x = 1 \rightarrow x = (x \rightarrow x) \rightarrow x = x \rightarrow (x \rightarrow x) = x \rightarrow 1.$$

Thus, for all $x \in X$, $1 \rightarrow x = x \rightarrow 1 = x$, that means that 1 is not a logical unit.

(ii) Taking $x := 1$, $y := x$ and $z := y$ in (I) and using (i), we have

$$x \rightarrow y = 1 \rightarrow (x \rightarrow y) = (y \rightarrow 1) \rightarrow x = y \rightarrow x.$$

Hence (ii) holds.

(iii) Let $x, y \in X$ and $y \neq 1$. By above (iii), $x \to (y \to x) = y \neq 1$. Hence X does not satisfy (K). Consequently, X is not a KL-algebra.

(iv) By (ii) and (iii) the proof is obvious.

(v) Applying (I) and (iv), we obtain

$$y \to (y \to (y \to x)) = ((y \to x) \to y) \to y = x \to y \ (= y \to x \text{ by (ii)}).$$

(vi) Let $x \leq y$. Then $x \to y = 1$. Using (v), we conclude that (vi) is valid. □

Theorem 3.3. *Let X satisfy* (I)*. Then $(X, \leq)$ is a poset.*

Proof. By (R), the relation $\leq$ is reflexive. Assume $x \leq y$ and $y \leq x$. Then $x \to y = 1 = y \to x$. Using Proposition 3.2 (i), (I), (R) and (M), we obtain

$$x = x \to 1 = x \to (y \to x) = (x \to x) \to y = 1 \to y = y.$$

Thus, $\leq$ is antisymmetric.

Now, suppose $x \leq y$ and $y \leq z$. Then $x \to y = 1 = y \to z$. Using Proposition 3.2 (i) and (iii), we obtain

$$x \to z = (x \to z) \to 1 = (x \to z) \to (y \to z) = (z \to (x \to z)) \to y = x \to y = 1.$$

Hence $x \leq z$. Thus, $\leq$ is transitive. Therefore, the relation $\leq$ is a partial ordering on X, and so $(X, \leq)$ is a poset. □

The converse of Theorem 3.3 is not generally true as shown in the following example.

Example 3.4. (Dual form of [12, Ex. 8]) *Let X be a poset with the greatest element 1. The operation $\to$ on X is defined by*

$$x \to y = \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{otherwise.} \end{cases}$$

Then $(X, \to 1)$ is a BCK-algebra, and so X is an RM-algebra. It does not verify (I), since for $x = y = 1$ and $z \neq= 1$, we obtain

$$x \to (y \to z) = 1 \to (1 \to z) = z \neq 1 = (z \to 1) \to 1 = (z \to x) \to y.$$

Proposition 3.5. *If X satisfies* (D) *and* (I)*, then $X = \{1\}$.*

Proof. Let $x \in X$. By Proposition 3.2 (i), $x \to 1 = x$. Since X satisfies (D), we have $x \to 1 = 1$. Consequently, $x = 1$, and hence $X = \{1\}$. □

Let $\mathbb{A}$ be the set of classes of BI-algebras, implicative BCK-algebras, L-algebras, KL-algebras, CL-algebras and BE-algebras.

Corollary 3.6. *If $X \in \mathbb{A}$ and satisfy* (I)*, then $X = \{1\}$.*

Proof. The proof is similar to the proof of Proposition 3.5. □

The following example shows that there is an abelian group which is not an RM-algebra.

Example 3.7. *Consider the set $X = \{x, y, 1\}$ with Cayley Table 5:*

TABLE 5. RM-algebra $(X, \to_5, 1)$

$\to_5$	x	y	1
x	y	1	x
y	1	x	y
1	x	y	1

Then $(X, \to_5, 1)$ is an abelian group (where $x^{-1} = y \neq x = y^{-1}$), but it does not satisfy (R), since

$$x \to_5 x = y \neq 1 \neq x = y \to_5 y.$$

Thus, $(X, \to_5, 1)$ is not an RM-algebra.

By (R) and Proposition 3.2 (i) and (ii), we have:

Theorem 3.8. *Let $(X, \to, 1)$ be an RM-algebra satisfying* (I).
Then $(X, \to, ^{-1}, 1)$ is an abelian group, where $x^{-1} = x$, for all $x \in X$.

Proof. Given $x, y, z \in X$. Applying Proposition 3.2 (ii) and (I) we get

$$\begin{aligned} x \to (y \to z) &= x \to (z \to y) \\ &= (y \to x) \to z \\ &= (x \to y) \to z. \end{aligned}$$

Hence the binary operation $\to$ is associative, and so $(X, \to)$ is a semigroup. By Proposition 3.2 (i), the element 1 is an identity in X. Thus, $(X, \to, 1)$ is a monoid. Now, using (R), since $x \to x = 1$, we have $x^{-1} = x$, for all $x \in X$. Therefore, $(X, \to, ^{-1}, 1)$ is a group and by Proposition 3.2 (ii) is abelian. □

The following example shows that condition (I) in Theorem 3.8 is necessary.

Example 3.9. (i) *Consider the set of all real numbers $\mathbb{R}$ and define the binary operation $\to$ on $\mathbb{R} \setminus \{0\}$ by $x \to y = \frac{y}{x}$, for all $x, y \in \mathbb{R} \setminus \{0\}$* (see [14]). *The algebra $(\mathbb{R} \setminus \{0\}, \to, 1)$ is an RM-algebra, but not an abelian group. It does not satisfy (I), since*

$$3 \to (5 \to 4) = 3 \to (\frac{4}{5}) = \frac{\frac{4}{5}}{3} = \frac{4}{15} \neq \frac{20}{3} = \frac{5}{\frac{3}{4}} = (\frac{3}{4}) \to 5 = (4 \to 3) \to 5.$$

Also, it is not associative, since

$$3 \to (5 \to 4) = \frac{4}{15} \neq \frac{12}{5} = \frac{4}{\frac{5}{3}} = (\frac{5}{3}) \to 4 = (3 \to 5) \to 4.$$

(ii) ([19]) *Consider the set $X = \{x, y, 1\}$ with Cayley Table 6:*

Table 6. RM-algebra $(X, \rightarrow_6, 1)$

$\rightarrow_6$	x	y	1
x	1	1	x
y	1	1	1
1	x	y	1

Then $(X, \rightarrow_6, 1)$ is an RM-algebra, but it does not satisfy (I), since

$$x = x \rightarrow_6 1 = x \rightarrow_6 (y \rightarrow_6 1) \neq (1 \rightarrow_6 x) \rightarrow_6 y = x \rightarrow_6 y = 1.$$

Also, the binary operation $\rightarrow_6$ is not associative, since

$$1 = y \rightarrow_6 1 = y \rightarrow_6 (x \rightarrow_6 x) \neq (y \rightarrow_6 x) \rightarrow_6 x = 1 \rightarrow_6 x = x.$$

Thus, $(X, \rightarrow_6, 1)$ is not an abelian group.

Theorem 3.10. *Let $(X, \rightarrow, 1)$ be an RM-algebra. If $\rightarrow$ is a binary associative operation, then $(X, \rightarrow, ^{-1}, 1)$ is an abelian group, where $x^{-1} = x$, for all $x \in X$.*

Proof. Taking $x = y = z$. Applying associative law, we get $x \rightarrow (x \rightarrow x) = (x \rightarrow x) \rightarrow x$, and using (R) and (M), we obtain $x \rightarrow 1 = 1 \rightarrow x = x$. Hence the element 1 is an identity in X. Thus, $(X, \rightarrow, 1)$ is a monoid. By (R), since $x \rightarrow x = 1$, we have $x^{-1} = x$, for all $x \in X$. Therefore, $(X, \rightarrow, ^{-1}, 1)$ is a group. We have to prove it is abelian. Given $x, y \in X$. Using (R), it follows that $(x \rightarrow y) \rightarrow (x \rightarrow y) = 1$. By multiplying both sides by x on the left, we have $x \rightarrow [(x \rightarrow y) \rightarrow (x \rightarrow y)] = x \rightarrow 1 = x$. Using associative law and (R), we get

$$\begin{aligned} [x \rightarrow (x \rightarrow y)] \rightarrow (x \rightarrow y) &= [(x \rightarrow x) \rightarrow y] \rightarrow (x \rightarrow y) \\ &= (1 \rightarrow y) \rightarrow (x \rightarrow y) \\ &= y \rightarrow (x \rightarrow y) \\ &= x. \end{aligned}$$

Hence $y \rightarrow (x \rightarrow y) = x$, for all $x, y \in X$. Now, by multiplying both sides by y on the left, we have $y \rightarrow [y \rightarrow (x \rightarrow y)] = y \rightarrow x$. Applying associative law, (R) and (M), we have $(y \rightarrow y) \rightarrow (x \rightarrow y) = 1 \rightarrow (x \rightarrow y) = x \rightarrow y = y \rightarrow x$. Therefore, $(X, \rightarrow, ^{-1}, 1)$ is an abelian group. □

Proposition 3.11. *If $(X, \rightarrow, 1)$ is an RM-algebra satisfying* (I)*, then* (E) *holds.*

Proof. Assuming $x, y, z \in X$. By (I) and Proposition 3.2 (ii), we have

$$\begin{aligned} x \rightarrow (y \rightarrow z) &= (z \rightarrow x) \rightarrow y \\ &= y \rightarrow (z \rightarrow x) \\ &= y \rightarrow (x \rightarrow z). \end{aligned}$$

Consequently, (E) is valid. □

The converse of Proposition 3.11 is not generally true as shown in the following example.

Example 3.12. ([20]) *(i) Consider the set* $X = \{x, y, z, 1\}$ *with Cayley Table 7:*

TABLE 7. RM-algebra $(X, \rightarrow_7, 1)$

$\rightarrow_7$	x	y	z	1
x	1	y	y	1
y	x	1	x	1
z	1	1	1	1
1	x	y	z	1

Then $(X, \rightarrow_7, 1)$ *is an RM-algebra verifying* (E)*, but it does not verify* (I)*, since*

$$1 = y \rightarrow_7 1 = y \rightarrow_7 (z \rightarrow_7 x) \neq (x \rightarrow_7 y) \rightarrow_7 z = y \rightarrow_7 z = x.$$

We can observe that $\rightarrow_7$ *is not associative, since*

$$1 = y \rightarrow_7 y = y \rightarrow_7 (x \rightarrow_7 z) \neq (y \rightarrow_7 x) \rightarrow_7 z = x \rightarrow_7 z = y.$$

(ii) Consider the set $X = \{x, y, z, r, s, 1\}$ *with Cayley Table 8:*

TABLE 8. RM-algebra $(X, \rightarrow_8, 1)$

$\rightarrow_8$	x	y	z	r	s	1
x	1	y	z	r	y	1
y	x	1	z	r	z	1
z	x	y	1	1	x	1
r	x	y	1	1	x	1
s	1	1	1	1	1	1
1	x	y	z	r	s	1

Then $(X, \rightarrow_8, 1)$ *is an RM-algebra. It does not satisfy* (E)*, since*

$$z \rightarrow_8 (x \rightarrow_8 s) = z \rightarrow_8 y = y \neq 1 = x \rightarrow_8 x = x \rightarrow_8 (z \rightarrow_8 s).$$

Also, it does not verify (I)*, since*

$$y \rightarrow_8 (r \rightarrow_8 s) = y \rightarrow_8 x = x \neq r = 1 \rightarrow_8 r = (s \rightarrow_8 y) \rightarrow_8 r.$$

We can observe that $\rightarrow_8$ *is not associative, since*

$$s \rightarrow_8 (x \rightarrow_8 x) = s \rightarrow_8 1 = 1 \neq x = 1 \rightarrow_8 x = (s \rightarrow_8 x) \rightarrow_8 x.$$

The following corollary is a direct consequence of Proposition 3.11.

Corollary 3.13. *If* $(X, \rightarrow, 1)$ *is an RM-algebra satisfying* (I)*, then* $(X, \rightarrow, 1)$ *is a CI-algebra.*

Proof. By Proposition 3.11, we can see that (E) is valid, and so $(X, \to, 1)$ is a CI-algebra. □

In 2020, Smarandache et al. ([15, Th. 2.2.]) proved that in a CI-algebra $(X, \to, 1)$ the binary operation $\to$ is associative if and only if $x \to 1 = x$, for all $x \in X$. Hence CI-algebra $(X, \to, 1)$ with the condition (I) is an abelian group, but not a BE-algebra.

Proposition 3.14. *If* $(X, \to, 1)$ *is an RM-algebra satisfying* (C) *and* (I), *then* $X = \{1\}$.

Proof. By commutative law we have $(x \to y) \to y = (y \to x) \to x$, for all $x, y \in X$. Using Proposition 3.2 (ii) and (iii), we get

$$x = y \to (x \to y) = x \to (y \to x) = y.$$

Hence, $|X| = 1$, and so $X = \{1\}$. □

The following example shows that the conditions (C) and (I) in Proposition 3.14 are necessary.

Example 3.15. (i) *Consider the Example 3.9* (ii). *It does not verify* (C), *since*

$$(x \to_6 1) \to_6 1 = x \to_6 1 = x \neq 1 = x \to_6 x = (1 \to_6 x) \to_6 x.$$

(ii) *Consider the set* $X = \{x, y, 1\}$ *with Cayley Table 9:*

Table 9. RM-algebra $(X, \to_9, 1)$

$\to_9$	x	y	1
x	1	x	y
y	1	1	1
1	x	y	1

Then $(X, \to_9, 1)$ *is an RM-algebra satisfying* (C), *but it does not satisfy* (I), *since*

$$x \to_9 (y \to_9 1) = x \to_9 1 = y \neq x = x \to_9 y = (1 \to_9 x) \to_9 y.$$

(iii) *Consider the Example 3.1* (iii). *The algebra* $(X, \to_4, 1)$ *is an RM-algebra satisfying* (I), *but it does not satisfy* (C), *since*

$$(x \to_4 y) \to_4 y = y \to_4 y = 1 \neq y = y \to_4 x = (y \to_4 x) \to_4 x.$$

(iv) *Consider the set of all real numbers* $\mathbb{R}$ *and define the binary operation* $\to$ *on* $\mathbb{R}$ *by* $x \to y = y - x$, *for all* $x, y \in \mathbb{R}$. *The algebra* $(\mathbb{R}, \to, 1)$ *is an RM-algebra, but does not satisfy* (I), *since*

$$2 \to (7 \to 4) = 2 \to 3 = 1 \neq 9 = 7 - (-2) = -2 \to 7 = (4 \to 2) \to 7.$$

Also, it is not associative, since

$$3 \to (1 \to 9) = 3 \to 8 = 5 \neq 11 = 9 - (-2) = -2 \to 9 = (3 \to 1) \to 9.$$

We see that

$$(4 \to 7) \to 7 = 3 \to 7 = 4 \neq 7 = 4 - (-3) = -3 \to 4 = (7 \to 4) \to 4,$$

and so it does not satisfy (C).

Proposition 3.16. *If* $(X, \to, 1)$ *is an RM-algebra satisfying* (BI) *and* (I)*, then* $X = \{1\}$.

Proof. By (BI) we have $(x \to y) \to x = x$, for all $x, y \in X$. Using Proposition 3.2 (iii), we get $y = (x \to y) \to x = x$, for all $x, y \in X$. Hence, $|X| = 1$, and so $X = \{1\}$. □

The following example shows that the conditions (BI) and (I) in Proposition 3.16 are necessary.

Example 3.17. (i) ([17]) *Consider the set* $X = \{x, y, 1\}$ *with the following Cayley Table 10:*

TABLE 10. RM-algebra $(X, \to_{10}, 1)$

$\to_{10}$	x	y	1
x	1	y	x
y	x	1	x
1	x	y	1

Then $(X, \to_{10}, 1)$ *is an RM-algebra.* (I) *does not hold, since*

$$y \to_{10} (x \to_{10} y) = y \to_{10} y = 1 \neq x = 1 \to_{10} x = (y \to_{10} y) \to_{10} x.$$

Also, it does not verify (BI)*, since*

$$(x \to_{10} 1) \to_{10} x = x \to_{10} x = 1 \neq x.$$

(ii) ([20]) *Consider the set* $X = \{x, y, z, r, 1\}$ *with Cayley Table 11:*

TABLE 11. RM-algebra $(X, \to_{11}, 1)$

$\to_{11}$	x	y	z	r	1
x	1	1	z	z	1
y	r	1	1	r	1
z	x	x	1	x	1
r	y	y	y	1	1
1	x	y	z	r	1

Then $(X, \to_{11}, 1)$ *is an RM-algebra satisfying* (BI)*, but it does not verify* (I)*, since*

$$z \to_{11} (r \to_{11} y) = z \to_{11} y = x \neq r = 1 \to_{11} r = (y \to_{11} z) \to_{11} r.$$

(iii) *Consider the Example 3.1* (iii)*. The algebra* $(X, \to_4, 1)$ *is an RM-algebra satisfying* (I)*, but it does not satisfy* (BI)*, since*

$$x \to_4 (y \to_4 x) = x \to_4 y = y \neq x.$$

4. Conclusion

We discussed a new algebra as RM-algebra with an additional condition (I) and obtain some relations between (I), (E), (C) and (BI) as axioms in some algebras of logic and some examples are given to illustrate them. It is proved that the RM-algebras with associative law are abelian groups. Further, it is shown that an RM-algebra with condition (I) induced a partial ordering on X, and so the new algebra $(X, \leq)$ is a poset. Applied (I) to the BI-algebras, BCK-algebras, L-algebras, KL-algebras, CL-algebras and BE-algebras and proved that these are trivial. As a direction of research, one can investigate and extend these results to the other algebraic structures.

5. Data Availability Statement

“Not applicable”.

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8. Conflict of interest

The author declare no conflict of interest.

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