Journal of Mahani Mathematical Research
Shahid Bahonar University of Kerman
Print ISSN: 2251-7952 Online ISSN: 2645-4505

LINEAR PRESERVERS OF ACU-MAJORIZATION ON $\mathbb{R}^3$ AND $M_{3,m}$

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Article type: Research Article
(Received: 12 August 2023, Received in revised form 19 December 2023)
(Accepted: 28 December 2023, Published Online: 30 December 2023)

Abstract. In this note, we present an equivalent condition for linear preservers of group majorization induced by closed subgroup G of $O(\mathbb{R}^n)$. Moreover, a new concept of majorization is defined on $\mathbb{R}^3$ as acu-majorization and this is extended for $3\times m$ matrices. Then we characterize all its linear preservers on $\mathbb{R}^3$ and $M_{3,m}$.

Keywords: majorization, group majorization, circulant majorization, linear preservers.
2020 MSC: 15A86, 15A39, 15B51

1. Introduction

For $x, y \in \mathbb{R}^n$, we say that y majorizes x and write $x \prec y$, if

$$\sum_{i=1}^{k} x_i^{\downarrow} \leqslant \sum_{i=1}^{k} x_i^{\downarrow}$$

for $k = 1, \ldots, n-1$ and equality holds for $k = n$, where $x^{\downarrow} = (x_1^{\downarrow}, \ldots, x_n^{\downarrow})$ is arrangement of x in non-increasing order (see [3]).

An $n \times n$ non-negative matrix D is called doubly stochastic if $De = e$ and $D^t e = e$, where $e = (1, \ldots, 1)^t$. We know that $x \prec y$ if and only if $x = Dy$ for some doubly stochastic matrix D [3, Theorem II.1.10]. Let $\mathbb{P}_n$ be the set of all $n \times n$ permutation matrices. Birkhoff theorem [3, Theorem II.2.3] says that the set of all $n \times n$ doubly stochastic matrices is the convex hull of $\mathbb{P}_n$. In the other words, $x \prec y$ if and only if $x \in \mathrm{conv}\{Py : P \in \mathbb{P}_n\}$. With this view, a new concept of majorization can be defined which is called group majorization, [8]. In the following, $O(V)$ means the set of all linear operators g on inner product space V such that $\langle gv, gw\rangle = \langle v, w\rangle$ for every $v, w \in V$.

Definition 1.1. [7] Let V be a finite dimensional inner product space and G be a subgroup of orthogonal group $O(V)$. We say that x is group majorized by y, write $x \prec_G y$, if $x \in \mathrm{conv}\{gy : g \in G\}$.

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https://doi.org/10.22103/jmmr.2023.22019.1490
Publisher: Shahid Bahonar University of Kerman
How to cite: M. Soleymani, *Linear preservers of acu-majorization on* $\mathbb{R}^3$ *and* $M_{3,m}$, J. Mahani Math. Res. 2024; 13(2): 199 - 208.

The concept of majorization has been studied in connection with vectors and matrices. Its motivation comes from mathematical statistics. One of the interesting things for mathematicians is the study of linear preservers of these preordering relations. Let $\mathcal{R}$ be a relation on $\mathbb{R}^n$. We say that a linear operator $A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ preservers $\mathcal{R}$, if $Ax\mathcal{R}Ay$ whenever $x\mathcal{R}y$. Ando characterized all linear preserver of majorization, [1, Corollary 2.7]. The equivalent condition for linear preservers of group majorization was proved by Niezgoda as follows. Assume that $\prec_G$ and $\prec_H$ are group majorizations induced by closed subgroups $G \subset O(V)$ and $H \subset O(W)$ respectively.

Theorem 1.2. *[7, Theorem 2.2] Let $T : V \to W$ be a linear map. Suppose that H is finite or countable. Then the following conditions are equivalent:*

(i) $x \prec_G y \Rightarrow T(x) \prec_H T(y)$

(ii) *For any $g \in G$ there exists $h \in H$ such that $Tg = hT$.*

In Section 2, we extend Theorem 1.2 for every closed subgroup H of $O(\mathbb{R}^n)$. In Section 3, we introduce an uncountable subgroup of $O(\mathbb{R}^3)$ and define a new concept of majorization. Also, its linear preservers are characterized. In Section 4, we provide a necessary and sufficient condition for matrix representation of linear preservers $T : M_{n,m} \to M_{n,m}$ of G-majorizations on matrices. We also characterize linear preservers of the new majorization on $M_{3,m}$.

2. A necessary and sufficient condition for linear preservers of group majorization

In this section, we present a method which has an essential role to characterize linear preservers of various types of majorizations. To verify linear preservers of group majorization, we deal with $x \sim_G y$ means $x \prec_G y$ and $y \prec_G x$. The following theorem gives an equivalent condition for $\sim_G$.

Theorem 2.1. *Let V be an inner product space, G be a subgroup of $O(V)$ and $x, y \in V$. Then $x \sim_G y$ if and only if $x = gy$ for some $g \in G$.*

Proof. By the definition of group majorization, $x \prec_G y$ means that $x = \sum_{t=1}^{k} \alpha_t g_t y$. Since $g_t \in O(V)$,

$$\|x\| = \|\sum_{t=1}^{k} \alpha_t g_t y\| \leq \sum_{t=1}^{k} \alpha_t \|g_t y\| = \sum_{t=1}^{k} \alpha_t \|y\| = \|y\|. \tag{1}$$

On the other hand, $y \prec_G x$ and then $\|y\| \leq \|x\|$. Hence, equality holds in (1). If $\alpha_{t'} \neq 0$ for some $1 \leq t' \leq k$, then

$$\|\alpha_{t'} g_{t'} y + z\| = \|\alpha_{t'} g_{t'} y\| + \|z\|, \tag{2}$$

where $z = \sum_{t=1, t\neq t'}^{k} \alpha_t g_t y$. Since equality holds in triangle inequality(cauchy-schwarz inequality), $z = \lambda \alpha_{t'} g_{t'} y$ for some $\lambda \in \mathbb{R}$. Therefore, $x = (1+\lambda)\alpha_{t'} g_{t'} y$. Since $\|x\| = \|y\|$, $|(1 + \lambda)\alpha_{t'}| = 1$. By equation (2), $|1 + \lambda| = 1 + |\lambda|$ and then $(1 + \lambda)\alpha_{t'} = 1$. $\square$

In the following, we will prove Theorem 1.2 for every group majorization induced by closed subgroup H of $O(\mathbb{R}^m)$.

Theorem 2.2. *Let G be a subgroup of $O(\mathbb{R}^n)$ and H be a closed subgroup of $O(\mathbb{R}^m)$. Let $\prec_G, \prec_H$ be the group majorizations induced by G, H, respectively. Then the following conditions are equivalent:*

(i) *A linear map $A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ preserves $\prec_G$ and $\prec_H$.*
(ii) *For every $g \in G$ there exists a matrix $h \in H$ such that $hA = Ag$.*

Proof. The proof of $(ii) \to (i)$ is obvious. $(i) \to (ii)$ Let $g \in G$ be arbitrary. Define $f_A(h) := \min\{\|(hA - Ag)e_i\|_2 : (hA - Ag)e_i \neq 0, i = 1, \dots, n\}$ provided that $hA \neq Ag$ and $f_A(h) = 0$ whenever $hA = Ag$. It is clear that f_A is a continuous function. We define $\Delta(A, g) := \inf_{h\in H} f_A(h)$. Since $O(\mathbb{R}^n)$ is a compact subset of $\mathbb{R}^{n^2}$ and H is its closed subset, H is compact. On the contrary, suppose that $hA \neq Ag$ for every $h \in H$. The compactness of H implies that $\Delta(A, g) \neq 0$. Let $\lambda \in \left(0, \frac{\Delta(A,g)}{2n\|A\|_2}\right)$ and $x = \sum_{i=1}^{n} \lambda^{i-1} e_i$. Since A preserves $\prec_G$ and $\prec_H$, Theorem 2.1 implies that $hAx = Agx$ for some $h \in H$. Hence

$$\sum_{i=1}^{n} \lambda^{i-1}(hA - Ag)e_i = 0. \tag{3}$$

Since $hA \neq Ag$, there exists i such that $(hA-Ag)e_i \neq 0$. Let i be the first integer with this property. By equation (3), $(hA - Ag)e_i = \lambda \sum_{j=i+1}^{n} \lambda^{j-i-1}(hA - Ag)e_j$. So $\Delta(A, g) \leq \|(hA - Ag)e_i\|_2 \leq \lambda \sum_{j=i+1}^{n} \lambda^{j-i-1}\|(hA - Ag)e_j\|_2 \leq 2n\lambda\|A\|_2$. Then $\lambda \geq \dfrac{\Delta(A, g)}{2n\|A\|_2}$, a contradiction.

□

3. acu-majorization on $\mathbb{R}^3$

In the following, we introduce an uncountable subgroup of $O(\mathbb{R}^3)$ and define a group majorization on $\mathbb{R}^3$. Then by using Theorem 2.2, we will characterize its linear preservers. To do this, we need to define circulant permutation. An operator $S : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined by $S(x_1, \dots, x_n)^t = (x_n, x_1, \dots, x_{n-1})^t$ is called the shift operator. The circulant matrix associated to $x = (x_1 \ \dots \ x_n)^t$ is the $n \times n$ matrix whose the k^{th} column is given by $S^{k-1}x$. Let P_1 be the circulant permutation matrix associated to e_2 and $P_i = P_1^i$. For $n = 3$, we have

$$P_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \ P_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \ P_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The set $\{P_1, P_2, P_3\}$ is denoted by $\mathcal{C}_3$. The following theorem provides the eigenvalues of circulant matrices.

Theorem 3.1. *[4, Theorem 3.2.2] Let $p_\gamma(x) = \gamma_1 x + \gamma_2 x^2 + \cdots + \gamma_n x^n$ and C be the circulant matrix $p_\gamma(P_1)$. Then $C = F^* diag\left(p_\gamma(1), p_\gamma(\omega), \ldots, p_\gamma(\omega^{n-1})\right) F$, where $\omega = e^{\frac{2\pi i}{n}}$ and F is the Vandermonde matrix associated to $\left(1, \omega, \omega^2, \ldots, \omega^{n-1}\right)$.*

Let $x \in \mathbb{R}^3$ be not a scalar multiple of $e = (1,1,1)^t$. It is easy to show that the set $\{P_1x, P_2x, P_3x\}$ is affinely independent. So the affine space generated by P_1x, P_2x, P_3x is as same as the affine space generated by the orbit of x under the permutation matrices. In the other words, For $x \in \mathbb{R}^3$,

$$\text{aff}\left(\{Px : P \in \mathbb{P}_3\}\right) = \left\{\sum_{i=1}^{3} \alpha_i P_i x : \sum_{i=1}^{3} \alpha_i = 1, P_i \in \mathcal{C}_3\right\}. \tag{4}$$

Theorem 3.2. *The set $\left\{\sum_{i=1}^3 \alpha_i P_i : \sum_{i=1}^3 \alpha_i = 1, \sum_{i=1}^3 \alpha_i^2 = 1, P_i \in \mathcal{C}_3\right\}$ is a closed subgroup of $O(\mathbb{R}^3)$.*

Proof. Let $G = \left\{\sum_{i=1}^3 \alpha_i P_i : \sum_{i=1}^3 \alpha_i = 1, \sum_{i=1}^3 \alpha_i^2 = 1, P_i \in \mathcal{C}_3\right\}$ and $g_1 = \sum_{i=1}^3 \alpha_i P_i, g_2 = \sum_{i=1}^3 \beta_i P_i$ be arbitrary elements of G. So

$$\begin{aligned} g_1 g_2 = & (\alpha_1\beta_3 + \alpha_2\beta_2 + \alpha_3\beta_1)P_1 + (\alpha_1\beta_1 + \alpha_2\beta_3 + \alpha_3\beta_2)P_2 \\ & + (\alpha_1\beta_2 + \alpha_2\beta_1 + \alpha_3\beta_3)P_3 = \gamma_1 P_1 + \gamma_2 P_2 + \gamma_3 P_3, \end{aligned}$$

and than $\gamma_1 + \gamma_2 + \gamma_3 = \left(\sum_{i=1}^3 \alpha_i\right)\left(\sum_{i=1}^3 \beta_i\right) = 1$. On the other hand,

$$\sum_{i=1}^{3} \gamma_i^2 = (\sum_{i=1}^{3} \alpha_i^2)(\sum_{i=1}^{3} \beta_i^2) + 2(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3)(\beta_1\beta_2 + \beta_1\beta_3 + \beta_2\beta_3).$$

By the definition of G, It is clear that $\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 = 0$. These show that $\sum_{i=1}^3 \gamma_i^2 = 1$ and then $g_1 g_2 \in G$.
Since the inverse of the matrix $g = \sum_{i=1}^3 \alpha_i P_i$ is $g^t = \alpha_2 P_1 + \alpha_1 P_2 + \alpha_3 P_3 \in G$, we know that G is a subgroup of $O(\mathbb{R}^3)$.
Let $\varphi : \mathbb{R}^3 \longrightarrow M_3$ be defined by $\varphi(\alpha_1, \alpha_2, \alpha_3) = \sum_{i=1}^3 \alpha_i P_i$. The linear map φ is bounded and then it is continuous. The set $\{(\alpha_1, \alpha_2, \alpha_3) : \sum_{i=1}^3 \alpha_i = 1, \sum_{i=1}^3 \alpha_i^2 = 1\}$ is a closed subset of $\mathbb{R}^3$ and this implies that G is a closed subset of M_3.

□

The set $\left\{\sum_{i=1}^3 \alpha_i P_i : \sum_{i=1}^3 \alpha_i = 1, \sum_{i=1}^3 \alpha_i^2 = 1, P_i \in \mathcal{C}_3\right\}$ is called the affine circulant unitary group and is denoted by ACU. The majorization induced by ACU is called the affine circulant unitary majorization or in short acu-majorization.

Definition 3.3. For $x, y \in \mathbb{R}^3$, x is said to be acu-majorized by y, denoted by $\prec_{acu}$, if $x \in \text{conv}\{gx : g \in ACU\}$.

With this argument, it can be shown that G is a subgroup of $O(\mathbb{R}^p)$ for every prime number p, so it is possible to define acu-majorization on $\mathbb{R}^p$. But for

non-prime numbers, it is not possible to prove that $g_1 g_2 \in G$ with the above method.
We know that $\{x : x \prec y\} \subsetneqq \{x : x \prec_{acu} y\}$. For example, the vector $x = \left(\frac{1}{3} - \frac{1}{\sqrt{3}}, \frac{1}{3} + \frac{1}{\sqrt{3}}, \frac{1}{3}\right)$ is acu-majorized by $(1, 0, 0)$, but x is not majorized by $(1, 0, 0)$. In Figure 1, we check the difference between majorization and acu-majorization. To understand this, we used equation (4).

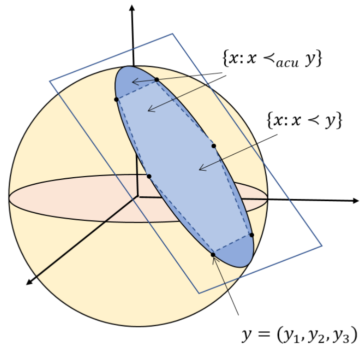

FIGURE 1. majorization and acu-majorization

In the other words, $x \prec_{acu} y$ if and only if $x_1 + x_2 + x_3 = y_1 + y_2 + y_3$ and $x_1^2 + x_2^2 + x_3^2 \leq y_1^2 + y_2^2 + y_3^2$. In the following, by using Theorem 2.2, we will characterize the linear preservers of acu-majorization.

Theorem 3.4. *Let A be a linear operator on $\mathbb{R}^3$. Then A preserves acu-majorization if and only if $AP_1 = P_j A$ for some $P_j \in \mathcal{C}_3$.*

Proof. Assume that A preserves acu-majorization. Since $P_1 \in ACU$, Theorem 2.2 implies that there exists $G \in ACU$ such that $GA = AP_1$. If the first column of A is a, then $A = \left(a|Ga|G^2 a\right)$ and $G^3 a = a$. Since $G^3 \in ACU$, there are $\beta_1, \beta_2, \beta_3$ such that

$$G^3 = \begin{pmatrix} \beta_3 & \beta_2 & \beta_1 \\ \beta_1 & \beta_3 & \beta_2 \\ \beta_2 & \beta_1 & \beta_3 \end{pmatrix}.$$

Therefore

$$G^3 - I = \begin{pmatrix} -\beta_1 - \beta_2 & \beta_2 & \beta_1 \\ \beta_1 & -\beta_1 - \beta_2 & \beta_2 \\ \beta_2 & \beta_1 & -\beta_1 - \beta_2 \end{pmatrix},$$

because $\beta_1 + \beta_2 + \beta_3 = 1$. If $\beta_1 \neq 0$ or $\beta_2 \neq 0$, then the rank of the matrix $G^3 - I$ is 2. So $G^3 a = a$ implies that the vector a is a scalar multiple of $e = (1, 1, 1)^t$.

Hence $G^3 = I$ or $a = (\gamma, \gamma, \gamma)^t$. In case $a = (\gamma, \gamma, \gamma)^t$, we have $A = J_3 = ee^t$ and the assertion hold. Now assume that $G^3 = I$. Since $G \in ACU$,

$$G = \begin{pmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ \alpha_1 & \alpha_3 & \alpha_2 \\ \alpha_2 & \alpha_1 & \alpha_3 \end{pmatrix}$$

and Theorem 3.1 implies that the eigenvalues of G are $\lambda_1 = p_\gamma(\omega^2) = \alpha_1\omega^2 + \alpha_2\omega + \alpha_3, \lambda_2 = p_\gamma(\omega) = \alpha_1\omega + \alpha_2\omega^2 + \alpha_3, \lambda_3 = p_\gamma(1) = 1$, where $\omega = e^{\frac{2\pi i}{3}}$. Since $G^3 = I$, the eigenvalues of G must be the 3th roots of unity. If $\lambda_1 = \lambda_2 = \lambda_3 = 1$, then $G = P_3 = I$. Otherwise $\lambda_1 = \omega, \lambda_2 = \omega^2$ or $\lambda_1 = \omega^2, \lambda_2 = \omega$.
Case 1: Let $\lambda_1 = \omega, \lambda_2 = \omega^2$. So

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 &= 1 \\ \alpha_1\omega^2 + \alpha_2\omega + \alpha_3 &= \omega \\ \alpha_1\omega + \alpha_2\omega^2 + \alpha_3 &= \omega^2. \end{aligned}$$

This linear equation system is equal to

$$\begin{pmatrix} 1 & 1 & 1 \\ \omega^2 & \omega & \omega^3 \\ \omega^4 & \omega^2 & \omega^6 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 1 \\ \omega \\ \omega^2 \end{pmatrix}. \tag{5}$$

We know that the determinant of the Vandermonde matrix of equation (5) is $(\omega^3 - \omega^2)(\omega - \omega^2)(\omega^3 - \omega)$. So the Vandermonde matrix is invertible and the unique solution of (5) is $(\alpha_1, \alpha_2, \alpha_3)^t = (0, 1, 0)^t$. This means that $G = P_2$.
Case 2: Assume that $\lambda_1 = \omega^2, \lambda_2 = \omega$. Then the induced linear system is

$$\begin{pmatrix} 1 & 1 & 1 \\ \omega^2 & \omega & \omega^3 \\ \omega^4 & \omega^2 & \omega^6 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 1 \\ \omega^2 \\ \omega \end{pmatrix}.$$

By the same argument as in above, we have $(\alpha_1, \alpha_2, \alpha_3)^t = (1, 0, 0)^t$ and this implies that $G = P_1$.
Therefore, G must be in the set $\{P_1, P_2, P_3\}$ and the direct proof is complete.
Conversely, Since $AP_1 = P_jA$, for every $G = \alpha_1P_1 + \alpha_2P_2 + \alpha_3P_3$, we have

$$\begin{aligned} AG = \alpha_1AP_1 + \alpha_2AP_2 + \alpha_3AP_3 &= \alpha_1P_jA + \alpha_2P_j^2A + \alpha_3P_j^3A \\ &= (\alpha_1P_j + \alpha_2P_j^2 + \alpha_3P_j^3)A. \end{aligned}$$

Since $\{P_j, P_j^2, P_j^3\} = \mathcal{C}_3$ or $P_j = I_3$, we know that $\widehat{G} = \alpha_1P_j + \alpha_2P_j^2 + \alpha_3P_j^3$ is included in ACU. By Theorem 2.2, the matrix A preserves $\prec_{acu}$.

□

Now, we are able to characterize linear preservers of acu-majorization on $\mathbb{R}^3$. More details about the following corollary are available in [6].

Corollary 3.5. *An operator $A : \mathbb{R}^3 \to \mathbb{R}^3$ preserves $\prec_{acu}$ if and only if $A = \big(a|P_ja|P_j^2a\big)$ for some $P_j \in \mathcal{C}_3$ and $a \in \mathbb{R}^3$.*

4. linear preservers of acu-matrix majorization

In this section, we talk about matrix majorization and define a class of group majorization on $M_{n,m}$. We will find an equivalent condition for linear preservers of group majorization on $M_{n,m}$. By using that, we will characterize all linear preservers of acu-matrix majorization. The concept of matrix majorization is defined as multivariate majorization, see [2].

Definition 4.1. For $X, Y \in M_{n,m}$, we say that X is multivariate majorized by Y and write $X \prec_m Y$ if there exists doubly stochastic matrix $D \in M_n$ such that $X = DY$.

The theory of group majorization can be extended for matrices and a class of group matrix majorization can be defined as follows.

Definition 4.2. For $X, Y \in M_{n,m}$, X is said to be multivariate group majorized by Y (written as $X \prec_{mg} Y$), if $X = \sum_{i=1}^{k} c_i g_i Y$ where $g_i \in G$, $c_i \geq 0$, $\sum_{i=1}^{k} c_i = 1$ and G is a subgroup of $O(\mathbb{R}^n)$.

We need some preliminaries to study linear preservers of multivariate group majorization. For every $A = (a_{ij}) \in M_{n,m}$, we associate the vector $\mathrm{vec}(A) \in \mathbb{R}^{nm}$ defined by $\mathrm{vec}(A) = [a_{11}, \ldots, a_{n1}, a_{12}, \ldots, a_{n2}, \ldots, a_{1m}, \ldots, a_{nm}]^t$.

Let $\mathcal{B} = \{E_{11}, \ldots, E_{n1}, E_{12}, \ldots, E_{n2}, \ldots, E_{1m}, \ldots, E_{nm}\}$ be the standard basis of $M_{n,m}$ and $[T]_{\mathcal{B}}$ be the representation of T with respect to $\mathcal{B}$. Then

$$[T]_{\mathcal{B}} = \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1m} \\ B_{21} & B_{22} & \cdots & B_{2m} \\ \vdots & \vdots & & \vdots \\ B_{m1} & B_{m2} & \cdots & B_{mm} \end{pmatrix}, \tag{6}$$

where each $B_{ij} \in M_n$ and $\mathrm{vec}(T(X)) = [T]_{\mathcal{B}}(\mathrm{vec}(X))$. Let $A \in M_{n,m}$, $X \in M_{m,p}$, $B \in M_{p,q}$ and $C \in M_{n,q}$. By [5, Lemma 4.3.1], $AXB = C$ if and only if

$$\mathrm{vec}(C) = \mathrm{vec}(AXB) = (B^t \otimes A)\mathrm{vec}(X). \tag{7}$$

An equivalent condition for matrix representations of linear preservers of multivariate group majorization is presented in the following theorem.

Theorem 4.3. *Let G be a closed subgroup of $O(\mathbb{R}^n)$, $T : M_{n,m} \to M_{n,m}$ be a linear operator and $[T]_{\mathcal{B}}$ be as (6). Then T preserves multivariate group majorization if and only if for every $g \in G$ there exists a matrix $\widehat{g} \in G$ such that $\widehat{g}B_{ij} = B_{ij}g$ for each $i, j = 1, \ldots, m$.*

Proof. Let $I \otimes G = \{I_m \otimes g : g \in G\}$. Since G is a closed subgroup of $O(\mathbb{R}^n)$, we know that $I \otimes G$ is the closed subgroup of $O(\mathbb{R}^{mn})$. It is easy to see that $X \sim_{mg} Y$ if and only if $\mathrm{vec}(X) \sim_{I\otimes G} \mathrm{vec}(Y)$. By the hypothesis and equation (7), we know that $[T]_{\mathcal{B}}$ preserves $\sim_{I\otimes G}$. Therefore, Theorem 2.2 implies that for every $I_m \otimes g$ there exists $I_m \otimes \widehat{g}$ such that $[T]_{\mathcal{B}}(I_m \otimes g) = (I_m \otimes \widehat{g})[T]_{\mathcal{B}}$. By equation (6), $\widehat{g}B_{ij} = B_{ij}g$ for each $i, j = 1, \ldots, m$. Reversing the above process gives us the other side of the proof. $\square$

In the following, we want to extend the concept of acu-majorization to matrices such as Definition 4.2.

Definition 4.4. For $X, Y \in M_{3,m}$, we say that Y acu-matrix majorizes X if $X = DY$ for some $D = \sum_{i=1}^{n} \alpha_i g_i$ where $\alpha_i \geq 0, \sum_{i=1}^{n} \alpha_i = 1$, $g_i \in ACU$.

Now, we will characterize the linear preservers of acu-matrix majorization by using Theorem 4.3.

Theorem 4.5. *Let $T : M_{3,m} \to M_{3,m}$ be an operator. T preserves acu-matrix majorization if and only if there exist $1 \leq j \leq 3$ and $A, B, C \in M_m$ such that $T(X) = P_1 R_j X A + P_2 R_j X B + P_3 R_j X C$ where $R_j = [e_1 | P_j e_1 | P_j^2 e_1]$.*

Proof. Let $[T]_{\mathcal{B}}$ be the block matrix as (6). By Theorem 3.4 and Theorem 4.3, there exists a circulant permutation $P_j \in \mathcal{C}_3$ such that $B_{ts} P_1 = P_j B_{ts}$ for every $t, s \in \{1, \ldots, m\}$. Then $B_{ts} = (v_{ts} | P_j v_{ts} | P_j^2 v_{ts})$ for some $v_{ts} \in \mathbb{R}^3$. It means that $B_{ts} = a_{ts} R_j + b_{ts} P_1 R_j + c_{ts} P_2 R_j$ where $v_{ts} = (a_{ts}, b_{ts}, c_{ts})^t$. By choosing $A = (a_{ts}), B = (b_{ts}), C = (c_{ts})$, we have $[T]_{\mathcal{B}} = A \otimes R_j + B \otimes P_1 R_j + C \otimes P_2 R_j$. Now, equation (7) implies that

$$T(X) = P_1 R_j X B^t + P_2 R_j X C^t + P_3 R_j X A^t.$$

Conversely, let $T(X) = P_1 R_j X A + P_2 R_j X B + P_3 R_j X C$ and $X \prec_{acu} Y$. So there exists $D = \sum_{i=1}^{n} \alpha_i g_i$ such that $\alpha_i \geq 0, \sum_{i=1}^{n} \alpha_i = 1$, $g_i \in ACU$ and $X = DY$. If $g = \beta_1 P_1 + \beta_2 P_2 + \beta_3 P_3 \in ACU$, then $R_j g = \beta_1 P_j R_j + \beta_2 P_j^2 R_j + \beta_3 P_j^3 R_j$. So $R_j g = g' R_j$ for some $g' \in ACU$, because $\{P_j, P_j^2, P_j^3\} = \mathcal{C}_3$ or $P_j = I_3$. It implies that $R_j D = D' R_j$ where $D' = \sum_{i=1}^{n} \alpha_i g_i'$. On the other hand, it is clear that $P_t g' = g' P_t$ for every $t = 1, 2, 3$ and then $P_t D' = D' P_t$. Therefore, $T(DY) = D' T(Y)$ and $T(X) \prec_{acu} T(Y)$. $\square$

The following example is presented as a linear preserver of acu-matrix majorization on $M_{3,2}$.

Example 4.6. *Let $m = 2$, $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 2 \\ 5 & 6 \end{pmatrix}$. For $j = 2$, we have $R_j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. By Theorem 4.5, the operator T is* *defined as*

$$T\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{pmatrix}\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 5 & 6 \end{pmatrix},$$

preservers acu-matrix majorization.

5. Conclusion

Let q be a prime number. With an argument similar to the proof of Theorem 3.2, it can be shown that the set

$$G = \left\{ \sum_{i=1}^{q} \alpha_i P_i : \sum_{i=1}^{q} \alpha_i = 1, \sum_{i=1}^{q} \alpha_i^2 = 1, P_i \in \mathcal{C}_q \right\}$$

is a subgroup of $O(\mathbb{R}^p)$. Using this, the concept of acu-majorization can be extended to $\mathbb{R}^p$. But in general, this argument cannot be shown that G is a subgroup of $O(\mathbb{R}^n)$. This concept can be extended to $\mathbb{R}^n$ as an affine unitary majorization. For $x, y \in \mathbb{R}^n$, x is said to be affine unitary majorized by y, if $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$ and $\|x\|_2 \leq \|y\|_2$. As a future work, it can be checked that this definition is a group majorization on $\mathbb{R}^n$ and checked its linear preservers.

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