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COMMUTATORS-BASED GRAPH IN POLYGROUP

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Abstract. In this paper, first, we study commutators of a polygroup. Then for a finite polygroup P and a fixed element $g \in P$, we introduce the g-graph Δ_P^g. In addition, with some additional conditions, we see that it is connected and the diameter is at most 3. Then, we investigate isomorphic graphs. Specially, we obtain a new isomorphic graph derived from an isomorphic graph and two non-commutative isomorphic polygroups. Also, we show that two polygroups with isomorphic graphs preserve nilpotency.

Keywords: Polygroup, Related graph, Commutator.
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1. Introduction

Graph theory has been used to prove fundamental results in other areas of mathematics. Also, it has some applications in computer science, networking, etc. Today, modeling real-world problems based on graphs is one of the most important solutions to problems. Pure modeling without any comprehensive information about the data cannot be a proper modeling. When a set is equipped with algebraic and hyperalgebra operations or hyperoperations, it can find a regular structure or hyperstructure and act as a purposeful system. Therefore, the modeling that is based on an algebraic hyperstructure can be accurate and comprehensive and examine problem-solving in a real way. Many researches have been done in the field of graphs and algebraic structures, which have many applications. Knowledge of graphs can innovate new ideas for studying other algebraic structures such as groups and polygroups. For some extensive papers on assigning a graph to a ring, group, etc you can see [1, 8, 12, 13]. In [14], translating some properties of polygroups into graph-theoretic language, help us to study nilpotent polygroups. The theory of algebraic hyperstructures, that is a well-established branch in algebraic theory was first introduced by Marty [11]. Since then many researchers have worked on algebraic hyperstructures and developed them. The application of hyperstructures has been studied in geometry, automata, probabilities, and so on (see [5, 9]). Polygroups as a

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certain subclass of hypergroups are studied in [6]. Polygroup theory extends some well-known group results and introduces new topics such as Nilpotent and Engel polygroups (see [2, 15]). Polygroups have been discussed by Corsini (see [3, 4]), Ameri [2], Comer [5], Davvaz [6] and so on. Now, we are going to introduce a new graph called g- graph, for a fined element $g \in P$. A main graph related to a polygroup is a non-commuting graph. Here, we check the conditions that the g-graph coincides with the non-commuting graph. In addition, we obtained some results on nilpotent polygroups. Also, we obtain a necessary and sufficient condition between the isomorphic graph of polygroups and groups.

2. Preliminaries

We recall some basic definitions which are proposed by the pioneers of this subject. A *hypergroupoid* is a non-empty set H with a *hyperoperation* "$\circ$" defined on H, that is, a mapping of $H \times H$ into the family of non-empty subsets of H. If $(x, y) \in H \times H$, it's image under $\circ$ is denoted by $x \circ y$. If A, B are non-empty subsets of H, then $A \circ B = \bigcup\{x \circ y | x \in A, y \in B\}$. For any given $x \in H$, we identify $x \circ A$ for $\{x\} \circ A$ and $A \circ x$ for $A \circ \{x\}$. Generally, the singleton a is identified with its member a. The structure $(H, \circ)$ is called a *semihypergroup* if for all $a, b, c \in H$, $a \circ (b \circ c) = (a \circ b) \circ c$. A semihypergroup $(H, \circ)$ is a *hypergroup*, if for any given $x \in H$, $x \circ H = H \circ x = H$, for all $x \in H$.

Definition 2.1. [6] A polygroup is a hypergroup $\langle P, \cdot, e, ^{-1} \rangle$, where $e \in P$ and $^{-1}$, is an unitary operation on P and for all $x, y, z \in P$, the following axioms hold:
(i) $e \cdot x = x \cdot e = x$;
(ii) $x \in y \cdot z \Leftrightarrow y \in x \cdot z^{-1} \Leftrightarrow z \in y^{-1} \cdot x$.

Definition 2.2. [6] A non-empty subset K of a polygroup P is a *subpolygroup* of P if for any $a, b \in K$ we have $ab \subseteq K$ and $a^{-1} \in K$. Also, a subpolygroup N of P is *normal* if for any $a \in P$, $a^{-1}Na \subseteq N$.

A polygroup is called commutative if for any $x, y \in P$, $x \cdot y = y \cdot x$. Let $(P, \circ)$ and $(P, \star)$ be two polygroups. A map $f : (P, \circ) \to (P, \star)$ is called a *homomorphism*, if for any $a, b \in P$, $f(a \circ b) \subseteq f(a) \star f(b)$. Also, f is a *good homomorphism*, if for any $a, b \in P$, $f(a \circ b) = f(a) \star f(b)$. In addition, $(P_1 \times P_2, \bigcirc)$, where $\bigcirc$ is defined as follows, is a polygroup (see [6]):

$$(x_1, y_1) \bigcirc (x_2, y_2) = \{(x, y) \mid x \in x_1 \circ x_2, \text{ and } y \in y_1 \star y_2\}.$$

For an equivalence relation $\rho \subseteq P \times P$ and non-empty subsets A and B of P, it is defined $A\overline{\overline{\rho}}B \Longleftrightarrow a\rho b$, which, $a \in A$ and $b \in B$ are arbitrary. The relation ρ is called strongly regular if for all $x, y, a \in P$,

$$x\overline{\overline{\rho}}y \Longrightarrow (a \circ x)\ \overline{\overline{\rho}}\ (a \circ y) \text{ and } (x \circ a)\ \overline{\overline{\rho}}\ (y \circ a).$$

Theorem 2.3. [6] *If $(H, \cdot)$ is a hypergroup and ρ is a strongly regular relation on H, then H/ρ is a group under the operation $\rho(x) \otimes \rho(y) = \rho(z)$, for all $z \in x \cdot y$.*

For all $n \geq 1$, we define the relation β_n on a semihypergroup H as follows:

$$a\beta_n b \Longleftrightarrow \exists(x_1, ..., x_n) \in H, \text{ such that} \{a, b\} \subseteq \prod_{i=1}^{n} x_i$$

and $\beta = \bigcup_{n \geq 1} \beta_n$, where $\beta_1 = \{(x, x) \mid x \in H\}$, is the diagonal relation on H. This relation was introduced by Koskas [10]. Suppose that β^* is the transitive closure of β, the relation β^* is a strongly regular relation. Also we have:

Theorem 2.4. *[7] If H is hypergroup, then $\beta = \beta^*$.*

The kernel of the canonical map $\varphi : P \longrightarrow P/\beta^*$ is called the *core* (or heart) of P and it is denoted by ω_P (or ω). It is easy to prove that $\omega_P = \beta^*(e)$ and $\beta^*(x)^{-1} = \beta^*(x^{-1})$ for all $x \in P$ (see [6]).

Theorem 2.5. *[6] Assume P_1 and P_2 are two polygroups. Then $\omega_{P_1 \times P_2} = \omega_{P_1} \times \omega_{P_2}$.*

From now on, let $(P, \cdot)$ is a polygroup, $n \in \mathbb{N}$ and for any $x \in P$, we use $\bar{x}$ and xy instead of $\beta^*(x)$ and $x \cdot y$, respectively.

Theorem 2.6. *[14] For any $x, y \in P$, $\overline{x} \otimes \overline{y} = \overline{y} \otimes \overline{x^\alpha}$ if and only if $xy\omega = yx^\alpha\omega$.*

In Theorem 2.6, take α be the identity map i. Then we get the following corollary.

Corollary 2.7. *Let $x, y \in P$. Then $xyw = yxw$ if and only if $\bar{x} \otimes \bar{y} = \bar{y} \otimes \bar{x}$.*

Definition 2.8. [14]

(i) For an automorphism α the set $\zeta^\alpha(P)$ is called α-*center* of P and is defined as follows:

$$\zeta^\alpha(P) = \{x \in P | xy\omega = yx^\alpha\omega \text{ for any } y \in P\}.$$

(ii) Center of P, denoted by $Z(P)$, is defined by $\{x \in P | xy\omega = yx\omega \ for \ all \ y \in P\}$.

(iii) For $x \in P$ the *centralizer* x in P is defined by $C_P(x) = \{y \in P \mid yx\omega = xy\omega\}$.

In [15], we see that $Z(P)$ is a normal subpolygroup of P and $C_P(x)$ is a subpolygroup of P.
Assume $Z_0(P) = \omega$, and for $n \geq 2$,

$$Z_n(P) = \{x \in P \ xyZ_{n-1}(P) = yxZ_{n-1}(P) \ for \ any \ y \in P\}.$$

It is proved that $Z_n(P)$ is a normal subpolygroup of P.

Definition 2.9. [15] A polygroup P is called nilpotent if and only if for some $n \in \mathbb{N}$, $Z_n(P) = P$.

Remark 2.10. Clearly $Z_n(P) = \{x \in P\ [x, y] \subseteq \omega\ for\ any\ y \in Z_{n-1}(P)\}$.

Theorem 2.11. *[6] Let $(G, .)$ be a group. Then $(P_G, \circ, e, ^{-1})$ is a polygroup, where $P_G = G \cup \{a\}$, $a \notin G$ and $\circ$ is defined as follows:*

(i) $a \circ a = e$,
(ii) $e \circ x = x \circ e = x$, $\forall x \in G$,
(iii) $a \circ x = x \circ a = x$, $\forall x \in G - \{e, a\}$,
(iv) $x \circ y = x.y$, $\forall (x, y) \in G^2; y \neq x^{-1}$,
(v) $x \circ x^{-1} = x^{-1} \circ x = \{e, a\}$, $\forall x \in G - \{e, a\}$.

Clearly, $Z(P_G) = Z(G) \cup \{e, a\}$ and $\omega_{P_G} = \{e, a\}$ and $|P_G| = |G| + 1$.

We recall from [17], that for a group $(G, \cdot)$ the *commutator* of $x, y \in G$ is $[x, y] = x^{-1} \cdot y^{-1} \cdot x \cdot y$, and inductively, $[x, y_1, y_2, ..., y_n] = [[x, y_1, y_2, ..., y_{n-1}], y_n]$, for any $y_1, y_2, ..., y_n \in G$. Also, G is called *nilpotent* if the upper central series $Z_n(G)$ terminates at finite steps, where

$$Z_n(G) = \{x \in G; [x, y_1, y_2, ..., y_n] = e \text{ for any } y_1, y_2, ..., y_n \in G\ \}.$$

In a graph a *path* ρ is a sequence $v_0 e_1 v_1 ... e_k v_k$ whose terms are alternately distinct vertices and distinct edges, such that the ends of e_i are v_{i-1} and v_i for any i, $1 \leq i \leq k$. In this case, ρ is called a path between v_0 and v_k and the number k is called the *length* of ρ. If v_0 and v_k are adjacent by an edge e_{k+1}, then $\rho \cup \{e_{k+1}\}$ is called a *cycle*. The *length* of a cycle define the number of its edges. The length of the shortest cycle in a graph Δ is called *girth* of Γ and denoted by $girth(\Delta)$. If v and w are vertices of Δ, then $d(v, w)$ denotes the length of the shortest path between v and w. The largest distance between all pairs of the vertices of Δ is called the *diameter* of Δ, and is denoted by $diam(\Delta)$. A graph is *connected* if there is a path between each pair of the vertices of Δ.

3. Commutators of a polygroup

In this section, we introduce the notion of commutator of elements in polygroups and investigate the properties of commutators of polygroup.

The commutator of two elements $x, y \in P$ is defined by $[x, y] = \{t \in P; t \in x^{-1}y^{-1}xy\}$. If $z \in P$ and $A \subseteq P$, then $[A, z] = \{t; t \in A^{-1}z^{-1}Ay\}$. Therefore

$$[[x, y], z] = \{t; t \in [x, y]^{-1}z^{-1}[x, y]z\}.$$

And inductively, we define $[x, y_1, y_2, \ldots, y_n] = [[x, y_1, y_2, \ldots, y_{n-1}], y_n]$. Also, $A^x = \{t; t \in x^{-1}Ax\}$ (see [15]).

Theorem 3.1. *Consider $x, y, z, a, g \in P$ and $A \subseteq P$. Then*

(i) $[xy, z] \subseteq [x, z]^y[y, z]$. *Moreover, if $y = z$, then $[xy, y] = [x, y]^y$. Also, $[Ay, y] = [A, y]^y$.*

(ii) $[A,B]^{-1} = [B,A]$.
(iii) *If* $a \in A$, *then* $[a,y] \subseteq [A,y]$.
(iv) $[x^{-1},y] \subseteq [x,y]^{-x^{-1}}$.
(v) $[x,yz] \subseteq [x,z][x,y]^x$. *Also,* $[x,y^{-1}] \subseteq [y,x]^{-y^{-1}}$.

Proof. (i) Let $t \in [xy,z] = \{t \in (xy)^{-1}z^{-1}xyz = y^{-1}x^{-1}z^{-1}xyz\}$. Then

$$t \in y^{-1}x^{-1}z^{-1}xeyz \subseteq y^{-1}x^{-1}z^{-1}xzez^{-1}yz \subseteq y^{-1}x^{-1}z^{-1}xzyy^{-1}z^{-1}yz.$$

Therefore, $t \in mn$ for some $m \in [x,z]^y$ and $n \in [y,z]$.
Thus, $[xy,z] \subseteq [x,z]^y[y,z]$.
Moreover,

$$t \in [x,y]^y \Leftrightarrow t \in y^{-1}(x^{-1}y^{-1}xy)y = (xy)^{-1}y^{-1}(xy)y$$
$$\Leftrightarrow t \in [xy,y].$$

Hence, $[x,y]^y = [xy,y]$. Also, we have

$$\begin{aligned} t \in [Ay,y] &\Leftrightarrow t \in (Ay)^{-1}y^{-1}(Ay)y \\ &\Leftrightarrow t \in y^{-1}A^{-1}y^{-1}Ayy \Leftrightarrow t \in y^{-1}[A\cdot y]y \\ &\Leftrightarrow t \in [A\cdot y]^y. \end{aligned}$$

Consequently, $[Ay,y] = [A,y]^y$.
Parts (ii) and (iii) are clear.

(iv) Assume $t \in [x^{-1},y]$. Then

$$t \in (xy^{-1}x^{-1}ye) \subseteq xy^{-1}x^{-1}yxx^{-1},$$

and so $t \in [x,y]^{-x^{-1}}$.

(v) By part (i) and (ii) we have

$$\begin{aligned} [x,yz] &= [yz,x]^{-1} \subseteq ([y,x]^z[z,x])^{-1} \\ &= [z,x]^{-1}[y,x]^{-z} \\ &= [x,z][x,y]^z. \end{aligned}$$

Also, by parts (i), (ii) and (iv), $[x,y^{-1}] = ([y^{-1},x])^{-1} \subseteq [y,x]^{-y^{-1}}$. Therefore, $[x,y^{-1}] \subseteq [y,x]^{-y^{-1}}$. This complete the proof □

In what follows, we examined that the equalities existing in group theory do not exist in polygroups, in general.

Example 3.2. *Let* $P = \{e,a,b,c\}$ *be a polygroup by the Table 1.*
Then $[ea,e] \neq [e,e]^a[a,e]$*, since*

$$[ea,e] = [a,e] = ba = \{e,a,b\} \text{and } [e,e]^a[a,e] = (ba)(ba) = \{e,a,b,e\}.$$

Also, $[b^{-1},a] \neq [b,a]^{-b^{-1}}$*, since* $[b^{-1},a] = bbaa = ba = \{e,a,b\}$ *and* $[b,a]^{-b^{-1}} = bbaaba = \{e,a,b,c\}$.

Theorem 3.3. *Let* P *and* H *be two polygroups. Then*
(i) $Z(P\times H) = Z(P)\times Z(H)$.

Table 1. Polygroup $(P, \cdot)$.

$\cdot$	e	a	b	c
e	e	a	b	c
a	a	a	P	c
b	b	$\{e, a, b\}$	b	$\{b, c\}$
c	c	$\{a, c\}$	c	P

(ii) $(P \times H) - Z(P \times H) = (P - Z(P)) \times (H - Z(H))$.

Proof. (i) Let $(x, a) \in Z(P \times H)$. Then $(x, a)(y, b)\omega_{P\times H} = (y, b)(x, a)\omega_{P\times H}$ for any $x, y \in P$ and $a, b \in H$. It follows by $\omega_{P\times H} = \omega_P \times \omega_H$ that $xy\omega_P = yx\omega_P$ and $ab\omega_H = ba\omega_H$ and so $x \in Z(P)$ and $a \in Z(H)$, Consequently, $(x, a) \in Z(P) \times Z(H)$ i.e, $Z(P \times H) \subseteq Z(P) \times Z(H)$. In a similar way, $Z(P) \times Z(H) \subseteq Z(P \times H)$. Therefore, $Z(P \times H) = Z(P) \times Z(H)$.

(ii) Assume $(x, a) \in (P \times H) - Z(P \times H)$. By (i), we have $(x, a) \in (P \times H) - (Z(P) \times Z(H))$ and so $x \in P - Z(H)$ and $a \in H - Z(H)$. Therefore, $(x, a) \in (P - Z(P)) \times (H - Z(H))$ i.e, $(P \times H) - Z(P \times H) \subseteq (P - Z(P)) \times (H - Z(H))$. Similarly,

$$(P \times H) - Z(P \times H) \supseteq (P - Z(P)) \times (H - Z(P)).$$

Consequently, $(P \times H) - Z(P \times H) = (P - Z(P)) \times (H - Z(P))$. $\square$

Theorem 3.4. *Let G be a group. Then G is a nilpotent if and only if P_G is nilpotent.*

Proof. $(\Rightarrow)$ Assume G is a nilpotent group of class n and $x, y_1, \ldots, y_n$ are arbitrary elements of G. Then $Z_n(G) = G$. Thus by Theorem 2.11, $[x, y_1, \ldots, y_n] \subseteq \omega$. On the other hand, for $a \in P_G$, we have $[x, a] \subseteq \omega$. Consequently, P_G is a nilpotent polygroup.
$(\Leftarrow)$ Assume P_G is a nilpotent of class n. Then for any $x, y_1, \ldots, y_n \in G \subseteq P_G$, we have $[x, y_1, \ldots, y_n] \subseteq \omega$. By Theorem 2.11 and $x = \bar{x}$ and $y_1 = \bar{y}_1, \ldots, y_n = \bar{y}_n$, we have

$$[x, y_1, \ldots, y_n] = [x, \bar{y}_1, \ldots, \bar{y}_n] = \bar{e} = \{e, a\}.$$

Moreover, $[x, y_1, \ldots, y_n] = e$ or a. But $[x, y_1, \ldots, y_n] = a$, is a contradiction. Thus $[x, y_1, \ldots, y_n] = e$ and so $Z_n(G) = G$, i.e G is a nilpotent group. $\square$

4. The g-graph of a polygroup

In [14], a graph Γ_P^α is associated to a polygroup P, whose vertices are elements of $P \setminus \zeta^\alpha(P)$ and x connected to y by edge in case $xy\omega \neq yx^\alpha\omega$ or $yx\omega \neq xy^\alpha\omega$.
Now, if α is the identity automorphism i, then we call Γ_P^i the non-commuting

graph and denoted by $\triangle_P$. In addition, if x connected to y by edge in case $xy\omega = yx\omega$, then we call it the *commuting graph and denote by* τ_P.

Definition 4.1. For any polygroup P and a fixed element $g \in P$, the *g-graph* of P is the graph with vertex set $P - Z(P)$ and two distinct vertices x and y join by an edge if $[x,y] \nsubseteq gw$ and $g^{-1}w$ or $[y,x] \nsubseteq g\omega$ and $g^{-1}\omega$.

We denote by $V(P)$, the set of vertices of Δ_P^g. Also, we use $\{x,y\}$ if there exist an edge between two vertices x and y.

Example 4.2. *Since each polygroup of order less than 4 is commutative, so its graph is empty. Also, the graph of non-commutative polygroup ($P = \{e,a,b,c\},\cdot$), where $\cdot$ is defined on P by the Table 2, is empty since, $Z(P) = P$.*

Table 2. Polygroup $(P,\cdot)$.

.	e	a	b	c
e	e	a	b	c
a	a	a	P	c
b	b	$\{e,a,b\}$	b	$\{b,c\}$
c	c	$\{a,c\}$	c	P

Example 4.3. *Let $P = \{e,a,b,c,d,f,g\}$. We consider the proper non-commutative polygroup $(P,\cdot,e,-1)$ ([6]), where $\cdot$ is defined on P as Table 3.*

Table 3. Polygroup $(P,\cdot)$.

.	e	a	b	c	d	f	g
e	e	a	b	c	d	f	g
a	a	e	b	c	d	f	g
b	b	b	$\{e,a\}$	g	f	d	c
c	c	c	f	$\{e,a\}$	g	b	d
d	d	d	g	f	$\{e,a\}$	c	b
f	f	f	c	d	b	g	$\{e,a\}$
g	g	g	d	b	c	$\{e,a\}$	f

First, note that $\omega = Z(P) = \{e,a\}$. $b\omega = b^{-1}\omega = \{b\}$. In addition, $[g,d] = g^{-1}d^{-1}gd = bc = g$. Similarly,

$$[b,f] = [b,d] = [g,c] = [c,f] = [g,d] = g,\ [b,g] = [f,d][b,c] = [c,d] = f,\ [f,g] = \{e,a\}.$$

Therefore, Δ_P^b is a graph with $V(P) = \{b,c,d,f,g\}$ and $\{b,c\}$ is an edge, because $[b,c] = f \nsubseteq b\omega = \{b\}$. In a same way, for any $x,y \in V(P)$, $\{x,y\}$ is an edge. Consequently, Δ_P^b is a connected graph.

Example 4.4. *Let P be a polygroup as Example 4.3. Then $f\omega = \{f\}$ and $f^{-1}\omega = \{g\}$. Therefore, Δ_P^f is a graph with $V(P) = \{b, c, d, f, g\}$. Note that $[f, g] = \{e, a\} \nsubseteq f\omega, f^{-1}\omega$, i.e, $\{f, g\}$ is an edge. But $\{b, c\}$ is not an edge, because $[b, c] = f \nsubseteq f^{-1}\omega = \{g\}$ and $[b, c] = f \subseteq f\omega = \{f\}$. Similarly,*

$$\{f, d\},\ \{f, c\},\ \{f, b\},\ \{g, c\},\ \{g, b\},\ \{g, d\},\ \{d, c\},\ \{b, d\},\ \{b, c\}$$

are not edges. The diagram of Δ_P^f is shown as Figure 1.

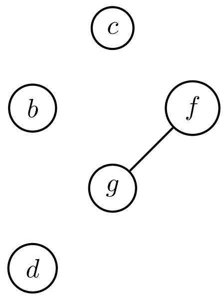

FIGURE 1. Graph Δ_P^f

Remark 4.5. From now on, $K(P) = \{[x, y], x, y \in P\}$.

Remark 4.6. If $g \in \omega$ and $\{x, y\}$ be an edge in Δ_P^g, then $[x, y] \nsubseteq g\omega = \omega$, and so $xy\omega \neq yx\omega$. Therefore, Δ_P^g coincides with Δ_P. In addition, if for any $x, y \in P$, we have $[x, y] \nsubseteq g\omega$ and $g^{-1}\omega$, then Δ_P^g is a connected graph. Hence, in this paper, we assume $g \notin \omega$ and there are $x, y \in P$ such that $[x, y] \subseteq g\omega, g^{-1}\omega$.

Definition 4.7 (Spanning subgraph)**.** A graph τ' is called spanning subgraph of τ. We denote by $\tau' \subseteq \tau$, if

(i) τ' is a subgraph of τ.
(ii) $V(\tau') = V(\tau)$.

Example 4.8. *Suppose Δ_P^f and Δ_P^b are as Examples 4.3 and 4.4. Clearly, $\Delta_P^f \subseteq \Delta_P^b$.*

Lemma 4.9. *Let $x, y \in V$. Then*

(i) *$[x, y] \subseteq \omega$ implies that $[x, y] \nsubseteq g\omega$.*
(ii) *$\tau_P \subseteq \Delta_P^g$.*

Proof. (i) If $[x, y] \subseteq \omega$ and $[x, y] \subseteq g\omega$, then $\bar{e} = [\bar{x}, \bar{y}] = \bar{g}$ and so $g \in \omega$, a contradiction. Therefore, $[x, y] \nsubseteq g\omega$.

(ii) Clearly, the vertex set of τ_P and Δ_P^g is $P - Z(P)$. Now, if $\{x, y\}$ is an edge in τ_P, then $xy\omega = yx\omega$, i.e $[x, y] \subseteq \omega$. It follows from (i), that $\{x, y\}$ is an edge in Δ_P^g. Consequently, $\tau_P \subseteq \Delta_P^g$.

□

Lemma 4.10. *Assume $K(P) = \{\omega, g\omega\}$ or $\{\omega, g\omega, g^{-1}\omega\}$. Then Δ_P^g is equal to τ_P.*

Proof. Obviously, the vertex set of Δ_P^g and τ_P is $P - Z(P)$. Now, let $\{x, y\}$ is an edge is Δ_P^g. Then $[x, y] \nsubseteq g\omega$. Using hypotheses and $[x, y] \in K(P)$, we get $[x, y] = \omega$ i.e, $\{x, y\}$ is an edge in τ_P. Similarly, consider $K(P) = \{\omega, g\omega, g^{-1}\omega\}$ and $\{x, y\}$ be an edge in Δ_P^g. Now, since $[x, y] \in K(P)$ by $[x, y] \nsubseteq g\omega$ and $g^{-1}\omega$, we conclude that $[x, y] = \omega$. Moreover, $\{x, y\}$ is an edge in τ_P. Consequently, Δ_P^g is the commuting graph τ_P. □

Theorem 4.11. *Consider g is a non-central element of P.*

(*i*) *If $g^2 \subseteq \omega$, then* $\mathrm{diam}(\Delta_P^g) = 2$.

(*ii*) *If $g^2 \nsubseteq \omega$ and $g^3 \nsubseteq \omega$, then* $\mathrm{diam}(\Delta_P^g) \leq 3$.

Proof. (*i*) Take $x \in P - Z(P)$ such that $x \neq g$. If $[x, g] \subseteq g\omega$, then $[\bar{x}, \bar{g}] = \bar{g}$. It implies that $\bar{g} = \bar{e}$ and so $g \in \omega$, a contradiction. Therefore,

$$[x, g] \nsubseteq g\omega. \tag{1}$$

Also, by $g^2 \subseteq \omega$, we get $\bar{g}^2 = \bar{e}$, i.e, $\bar{g}^{-1} = \bar{g}$. Now, if $[x, g] \subseteq g^{-1}\omega$, then $[\bar{x}, \bar{g}] = \bar{g}^{-1} = \bar{g}$. It implies that $\bar{g} = \bar{e}$, i.e, $g \in \omega$, a contradiction. Consequently,

$$[x, g] \nsubseteq g^{-1}\omega. \tag{2}$$

By (1) and (2), we have $\{x, g\}$ is an edge in Δ_P^g and $\mathrm{diam}(\Delta_P^g) \leq 2$. On the other hand by Remark 4.6, there exist $x, y \in P$ such that $[x, y] \subseteq g\omega$, $g^{-1}\omega$. Therefore, $d(x, y) \geq 2$. Consequently, $\mathrm{diam}(\Delta_P^g) = 2$.

(*ii*) We consider two cases, (1) $[x, g] \nsubseteq g^{-1}\omega$, (2) $[x, g] \subseteq g^{-1}\omega$.

For (1), we have $\{x, g\}$ is an edge in Δ_P^g.

For (2), by (1), we get

$$\begin{aligned} [x, g^2] &\subseteq [x, g][x, g]^g \subseteq [x, g]g^{-1}[x, g]g \\ &\subseteq [x, g]g^{-1}\omega[x, g]g\omega \\ &\subseteq g^{-1}\omega g^{-1}\omega g^{-1}\omega g\omega. \end{aligned} \tag{3}$$

Since $g^{-1}g \subseteq \omega$, we get that $g^{-1}\omega = g\omega = \omega$. Then by (3), $[x, g^2] \subseteq g^{-2}\omega$. It follows by hypotheses that $[x, g^2] \subseteq g^{-2}\omega \nsubseteq g\omega$, and $g^{-1}\omega$. Now, for any $z \in g^2$, $[x, z] \subseteq [x, g^2] \nsubseteq g\omega$ and $g^{-1}\omega$ and so $\{x, z\}$ is an edge. Therefore, for any $x \in P - Z(P)$, we have $\{x, g\}$ or $\{x, z\}$ is an edge, where $z \in g^2$. Moreover, for any $x, y \in V$ if x, y, join with g or z, then $d(x, y) \leq 2$. Otherwise $\{x, g\}$ and $\{y, z\}$ or $\{y, g\}$ and $\{x, z\}$ are edges and so by $[g, g^2] = g^{-3}g^3 \subseteq \omega$ and Lemma 4.9, we have $[g, g^2] \nsubseteq g\omega$, $g^{-1}\omega$. Therefore, $d(x, y) \leq 3$. Hence, $\mathrm{diam}(\Delta_P^g) \leq 3$. □

Theorem 4.12. *Let $|Z(P)| = 1$, $g \in P$, $g \notin \omega$ and $|C_P(g)| \neq 3$. Then Δ_P^g is connected.*

Proof. First, consider $g^3 \nsubseteq \omega$. Then by Theorem 4.11 Δ_P^g is connected. Now, if $g^3 \subseteq \omega$, then $g^2 \nsubseteq \omega$. Since $|C_P(g)| \neq 3$, we conclude that $|C_P(g)| > 3$. Therefore there exist $a \in C_P(g)$ such that $a \neq e$, g and g^{-1}. We can take $x_0 \in P - Z(P)$ such that $[x_0, g^2] \subseteq g\omega$ and $[x_0, g] \subseteq g^{-1}\omega$. By Lemma 3.1, $a \in C_P(g)$ and

$$g\omega = a^{-1}ag\omega = a^{-1}ga\omega = a^{-1}ga\omega = g\omega,$$

we have

$$[x_0, ga] \subseteq [x_0, a][x_0, g]^a \subseteq [x_0, a]g^{-a}\omega$$

and

$$\begin{aligned}[x_0, g^2a] &\subseteq [x_0, a][x_0, g^2]^a \subseteq [x_0, a](g^a\omega)\\ &= [x_0, a]g\omega = [x_0, a]g^{-2}\omega.\end{aligned}$$

Since $a \in C_P(g)$ we have $[a, g] \subseteq \omega$ and so by Lemma 4.9, $[a, g] \nsubseteq g^{-1}\omega$, $g\omega$. Then $\{a, g\}$ is an edge. Now, if $[x_0, a] \nsubseteq g\omega$, $g^{-1}\omega$, then $\{a, x_0\}$ and $\{a, g\}$ are edges. Thus, Δ_P^g is connected. Otherwise, $[x_0, a] \subseteq g\omega$ or $[x_0, a] \subseteq g^{-1}\omega$ But $[x_0, a] \subseteq g\omega$, implies that

$$[x_0, ga] \subseteq [x_0, a]g^{-a}\omega \subseteq g\omega g^{-a}\omega = [g, a]\omega \subseteq \omega \tag{4}$$

We consider two cases (i) $ga \subseteq Z(P)$, (ii) $ga \nsubseteq Z(P)$.
For (i), by $|Z(P)| = 1$, we get $ga = \{e\}$ and so $g^{-1} = a$, a contradiction.
For (ii), we get $ga \subseteq P - Z(P)$ and so for any $t \in ga$, by Theorem 3.1, and hypotheses, we get

$$[x_0, g^2a] \subseteq [x_0, a]g^{-2}\omega \subseteq g^{-3}\omega \subseteq \omega. \tag{5}$$

Again we consider two cases (i) $g^2a \subseteq Z(P)$, (ii) $g^2a \nsubseteq Z(P)$.
For (i), by hypotheses, we have $a = g^{-2} = g$, a contradiction.
For (ii), we have $g^2a \subseteq P - Z(P)$ and so for any $t \in g^2a$ by (5), we obtain $\{t, x_0\}$ and $\{t, g\}$ are edges, i.e, Δ_P^g is connected. □

Remark 4.13. The converse of Theorem 4.12, is not true, because by Example 4.3, we have a connected graph that $|Z(P)| \neq 1$ and $C_P(b) = 3$.

Theorem 4.14. *Assume P is a non-commutative simple polygroup and for any $x \in P$, $|C_P(x)| \geq 3$. Then Δ_P^g has no isolated vertices.*

Proof. Take $x \in P - Z(P)$. If $x^2 \nsubseteq \omega$, then $[x, x^{-1}] \subseteq \omega$ and so by Lemma 4.9, $\{x, x^{-1}\}$ is an edge. If $x^2 \nsubseteq \omega$ from $\mid C_P(x) \mid \geq 3$, there exist $t \in C_P(x)$ such that $t \neq e, x$. Since $[t, x] \subseteq \omega$ by Lemma 4.9, we have $\{t, x\}$ is an edge. Therefore Δ_P^g has no isolated vertices. □

Definition 4.15. Let P and H be two polygroups. Then Δ_P^g and Δ_H^h are said to be isomorphic, we denote by $\Delta_P^g \simeq \Delta_H^h$, if there is a bijection map $\varphi : P - Z(P) \to H - Z(H)$ preserving edges, i.e, for any $x, y \in P - Z(P)$, $\{x, y\}$ is an edge if and only if $\{\varphi(x), \varphi(y)\}$ is an edge.

Example 4.16. *Assume P is as Example 4.3 and the map $\varphi : P - Z(P) \to P - Z(P)$ is defined by $\varphi(b) = c$, $\varphi(c) = b$ and otherwise $\varphi(x) = x$. It is easy to see that $\Delta_P^b \simeq \Delta_P^c$.*

Lemma 4.17. *Let $\varphi : P \to H$ be an isomorphism of polygroups. Then*

(*i*) $\varphi(\omega_P) = \omega_H$.
(*ii*) $\Delta_P^g \simeq \Delta_H^h$, *where $h = \varphi(g)$.*

Proof. (*i*) Take $x \in \omega_H$. Then $\bar{x} = \bar{e}_H$, i.e, $\{x, e_H\} \subseteq \prod_{i=1}^{n} z_i$ for some $z_i \in H$, $1 \leq i \leq n$. Since e_H and $x \in \prod_{i=1}^{n} z_i$, we have $\varphi^{-1}(e_H)$ and $\varphi^{-1}(x) \in \prod_{i=1}^{n} \varphi^{-1}(z_i)$ and so $\{\varphi^{-1}(x), \varphi^{-1}(e_H)\} \in \prod_{i=1}^{n} \varphi^{-1}(z_i)$. This implies that $\varphi^{-1}(x) \in \omega_P$, i.e, $x \in \varphi(\omega_P)$. Similarly, $\varphi(\omega_P) \subseteq \omega_H$. Therefore, $\varphi(\omega_P) = \omega_H$.
(*ii*) It is clear by part (*i*). $\square$

Lemma 4.18. *Let A and B be two polygroups and $f : A \to B$ be an isomorphism. Then for any $a_1, a_2 \in A$, we have $[a_1, a_2] \nsubseteq \omega_A$ if and only if $[f(a_1), f(a_2)] \nsubseteq \omega_B$.*

Proof. It is clear by Lemmas 2.7 and 4.17. $\square$

Theorem 4.19. *Let A and B be two non-commutative isomorphic polygroups. If $\Delta_P^g \overset{\varphi}{\simeq} \Delta_H^h$, then $\Delta_{P\times A}^{(g,a)} \simeq \Delta_{H\times B}^{(h,b)}$, where $h = \varphi(g)$ and $b = f(a)$ for bijection map $f : A \to B$.*

Proof. Let $\varphi : P - Z(P) \to H - Z(H)$ be a graph isomorphism. Then by Lemma 4.17 (*i*), for any $x_1, x_2 \in P - Z(P)$, $[x_1, x_2] \nsubseteq g\omega_P$ if and only if $[\varphi(x_1), \varphi(x_2)] \nsubseteq \varphi(g)\omega_H$. Since A, B are isomorphic, we have a bijection map $f : A \to B$. Now, for $x_1, x_2 \in P$ and $a_1, a_2 \in A$ by Lemma 4.17 (*i*) and 4.18, we have $\{(x_1, a_1), (x_2, a_2)\}$ is an edge in $\Delta_{P\times A}^{(g,a)}$ if and only if $[(x_1, a_1), (x_2, a_2)] \nsubseteq (g, a)\omega_{P\times A}$ if and only if $([x_1, x_2], [a_1, a_2]) \nsubseteq (g, a)\omega_{P\times A}$ if and only if $[x_1, x_2] \nsubseteq g\omega_P$ and $[a_1, a_2] \nsubseteq a\omega_A$ if and only if $[\varphi(x_1), \varphi(x_2)] \nsubseteq \varphi(g)\omega_H$ and $[f(a_1), f(a_2)] \nsubseteq f(a)\omega_B$ if and only if $[\varphi(x_1), \varphi(x_2)] \nsubseteq h\omega_H$ and $[f(a_1), f(a_2)] \nsubseteq b\omega_B$ if and only if $\{\varphi(x_1), \varphi(x_2)\}$ is an edge in Δ_H^h and $\{f(a_1), f(a_2)\}$ is an edge in Δ_B^b. Consequently, by Theorem 3.3 $\Delta_{P\times A}^{(g,a)} \overset{(\varphi,f)}{\simeq} \Delta_{H\times B}^{(h,b)}$. $\square$

Example 4.20. *Let Δ_P^b and Δ_P^c be as Examples 4.4 and 4.16. Then $\Delta_P^b \simeq \Delta_P^c$ and $\Delta_{P\times P}^{(b,f)} \simeq \Delta_{P\times P}^{(c,f)}$.*

Theorem 4.21. *Assume $P_H = H \cup \{b\}$ and $P_G = G \cup \{a\}$, $g \notin \omega_{P_G}$ and $f(g) \notin \omega_{P_H}$. Then $\Delta_{P_G}^g \overset{f}{\simeq} \Delta_{P_H}^{f(g)}$ if and only if $\Delta_G^g \overset{f}{\simeq} \Delta_H^{f(g)}$*

Proof. ($\Rightarrow$) Let $g_1, g_2 \in G$. $Z(G)$ and $[g_1, g_2] \neq g$. Then by Theorem 2.11 $[g_1, g_2] \nsubseteq g\omega_{P_G} = \{g\}$. Since $\Delta_{P_G}^g \overset{f}{\simeq} \Delta_{P_H}^{f(g)}$, there exists a bijection $f : P_G - Z(P_G) \to P_H - Z(P_H)$ and $[f(g_1), f(g_2)] \nsubseteq f(g)\omega_{P_H} = \{f(g)\}$. It follows that $[f(g_1), f(g_2)] \neq f(g)$. Therefore, $\Delta_G^g \overset{f}{\simeq} \Delta_H^{f(g)}$, where $f\mid_{G-Z(G)}: G - Z(G) \to H - Z(H)$, is a bijection.
($\Leftarrow$) Since $\Delta_G^g \overset{f}{\simeq} \Delta_H^{f(g)}$, there is a bijection $f : G - Z(G) \to H - Z(H)$. Then $f : P_G - Z(P_G) \to P_H - Z(P_H)$ is a bijection. If $x, y \in P_G - Z(P_G)$ and $[x, y] \nsubseteq g\omega_{P_G} = \{g\}$, then $[x, y] \neq g$. Since $\Delta_G^g \overset{f}{\simeq} \Delta_H^{f(g)}$, we conclude that $[f(x), f(y)] \neq f(g)$ and so $[f(x), f(y)] \nsubseteq f(g)\omega_{P_H} = f$. Consequently, $\Delta_{P_G}^g \overset{f}{\simeq} \Delta_{P_H}^{f(g)}$. □

Definition 4.22. [6] A polygroup $(P, \circ)$ is called a polygroup of *exponent n* ($n \in \mathbb{N}$) if for each non-trivial element x of P we have $\underbrace{x \circ x \circ ... \circ x}_{n} = e$.

Theorem 4.23. *Every polygroup of exponent 3 is a group. In addition, if P and H are two finite non-abelian polygroups of exponent 3 and $\Delta_P^g \simeq \Delta_H^h$, for some non-identity element $h \in H$. Then $|Z(P)| = |Z(H)|$.*

Proof. Let P be a polygroup of exponent 3 and $b \in P$ be an arbitrary element. Since $bbb = e$, then $bx = e$ for some $x \in bb$. Thus $b^{-1} = x$, which implies

$$bb^{-1} = e \text{ for any } b \in P. \ (*)$$

Let $x \in yz$. By Definition 2.1, we have $z \in y^{-1}x$. Thus by (*) $x \in yz \subseteq yy^{-1}x = x$. Therefore, $x = yz$ and so every polygroups of exponent 3 is a group.

In addition, consider P and H are two polygroups of exponent 3 and $\Delta_P^g \simeq \Delta_H^h$. Then $|Z(P)| = |Z(H)|$. □

Theorem 4.24. *[16] Assume G is a finite nilpotent non-abelian group of add order and Δ_G^g has no vertex adjacent to all other vertices. If $\Delta_G^g \simeq \Delta_H^h$, then H is nilpotent*

Theorem 4.25. *Assume G is a finite non-abelian group of add order and Δ_G^g has no vertex adjacent to all other vertices. If $\Delta_{P_G}^g \overset{f}{\simeq} \Delta_{P_H}^{f(g)}$ and P_G is nilpotent, then P_H is nilpotent.*

Proof. Since $\Delta^g_{P_G} \overset{f}{\simeq} \Delta^{f(g)}_{P_H}$ by 4.21, we conclude that $\Delta^g_G \overset{f}{\simeq} \Delta^{f(g)}_H$. Also, by Theorem 3.4, G is nilpotent. It follows by Theorem 4.24, that H is nilpotent. Consequently, by Theorem 3.4, P_H is nilpotent. □

Corollary 4.26. *Let G and H be two non-abelian finite groups such that $\Delta^g_{P_G} \overset{f}{\simeq} \Delta^{f(g)}_{P_H}$, for some non-identity element $h \in H$. Then $|P_G| = |P_H|$.*

Proof. Since $\Delta^g_{P_G} \overset{f}{\simeq} \Delta^{f(g)}_{P_H}$, by Theorem 4.21, we get $\Delta^g_G \simeq \Delta^{f(g)}_H$. Thus by Theorem 4.23, $|G| = |H|$. Consequently $|P_G| = |P_H|$. □

5. Conclusion

In this paper, some results on the commutator of polygroups were obtained. Moreover, a relation between nilpotent polygroups and groups was obtained. Then the notion of g-graph was introduced. Also, some properties of this graph, such as diameter and connectivity are stated. Moreover, isomorphic g-graphs were investigated. This paper would be useful for studying nilpotent polygroups. In addition, you can obtain some results on Engel and solvable polygroups by the relation between these structures. Also, by this method, you can study other algebraic structures such as hypergroups, hyperrings, and so on.

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